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Research Article

On Complete Convergence for Arrays of Rowwise ρ-Mixing Random Variables and Its Applications

Xing-cai Zhou[1, 2] and Jin-guan Lin[1]

[1] *Department of Mathematics, Southeast University, Nanjing 210096, China*
[2] *Department of Mathematics and Computer Science, Tongling University, Tongling, Anhui 244000, China*

Correspondence should be addressed to Jin-guan Lin, jglin@seu.edu.cn

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We give out a general method to prove the complete convergence for arrays of rowwise ρ-mixing random variables and to present some results on complete convergence under some suitable conditions. Some results generalize previous known results for rowwise independent random variables.

1. Introduction

Let $\{\Omega, \mathcal{F}, P\}$ be a probability space, and let $\{X_n; n \geq 1\}$ be a sequence of random variables defined on this space.

Definition 1.1. The sequence $\{X_n; n \geq 1\}$ is said to be ρ-mixing if

$$\rho(n) = \sup_{k \geq 1} \sup_{X \in L^2(\mathcal{F}_1^k),\, Y \in L^2(\mathcal{F}_{n+k}^\infty)} \left\{ \frac{|EXY - EXEY|}{\sqrt{E(X-EX)^2 E(Y-EY)^2}} \right\} \longrightarrow 0 \qquad (1.1)$$

as $n \to \infty$, where $\mathcal{F}_m^n$ denotes the σ-field generated by $\{X_i; m \leq i \leq n\}$.

The ρ-mixing random variables were first introduced by Kolmogorov and Rozanov [1]. The limiting behavior of ρ-mixing random variables is very rich, for example, these in the study by Ibragimov [2], Peligrad [3], and Bradley [4] for central limit theorem; Peligrad [5] and Shao [6, 7] for weak invariance principle; Shao [8] for complete convergence; Shao

[9] for almost sure invariance principle; Peligrad [10], Shao [11] and Liang and Yang [12] for convergence rate; Shao [11], for the maximal inequality, and so forth.

For arrays of rowwise independent random variables, complete convergence has been extensively investigated (see, e.g., Hu et al. [13], Sung et al. [14], and Kruglov et al. [15]). Recently, complete convergence for arrays of rowwise dependent random variables has been considered. We refer to Kuczmaszewska [16] for ρ-mixing and $\tilde{\rho}$-mixing sequences, Kuczmaszewska [17] for negatively associated sequence, and Baek and Park [18] for negatively dependent sequence. In the paper, we study the complete convergence for arrays of rowwise ρ-mixing sequence under some suitable conditions using the techniques of Kuczmaszewska [16, 17]. We consider the case of complete convergence of maximum weighted sums, which is different from Kuczmaszewska [16]. Some results also generalize some previous known results for rowwise independent random variables.

Now, we present a few definitions needed in the coming part of this paper.

Definition 1.2. An array $\{X_{ni}; i \geq 1, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a constant C, such that

$$P\{|X_{ni}| > x\} \leq CP\{C|X| > x\} \tag{1.2}$$

for all $x \geq 0$, $i \geq 1$ and $n \geq 1$.

Definition 1.3. A real-valued function $l(x)$, positive and measurable on $[A, \infty)$ for some $A > 0$, is said to be slowly varying if

$$\lim_{x \to \infty} \frac{l(\lambda x)}{l(x)} = 1 \quad \text{for each } \lambda > 0. \tag{1.3}$$

Throughout the sequel, C will represent a positive constant although its value may change from one appearance to the next; $[x]$ indicates the maximum integer not larger than x; $I[B]$ denotes the indicator function of the set B.

The following lemmas will be useful in our study.

Lemma 1.4 (Shao [11]). *Let $\{X_n; n \geq 1\}$ be a sequence of ρ-mixing random variables with $EX_i = 0$ and $E|X_i|^q < \infty$ for some $q \geq 2$. Then there exists a positive constant $K = K(q, \rho(\cdot))$ depending only on q and $\rho(\cdot)$ such that for any $n \geq 1$*

$$\begin{aligned} E\max_{1\leq i\leq n}\left|\sum_{j=1}^{i} X_j\right|^q \leq K\Bigg(& n^{2/q}\exp\left(K\sum_{i=0}^{[\log n]}\rho\left(2^i\right)\right)\max_{1\leq i\leq n}\left(E|X_i|^2\right)^{q/2} \\ & + n\exp\left(K\sum_{i=0}^{[\log n]}\rho^{2/q}\left(2^i\right)\right)\max_{1\leq i\leq n}E|X_i|^q\Bigg). \end{aligned} \tag{1.4}$$

Lemma 1.5 (Sung [19]). *Let $\{X_n; n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. For any $\alpha > 0$ and $b > 0$, the following statement holds:*

$$E|X_n|^{\alpha} I[|X_n| \leq b] \leq C\left\{E|X|^{\alpha} I[|X| \leq b] + b^{\alpha} P\{|X| > b\}\right\}. \tag{1.5}$$

Lemma 1.6 (Zhou [20]). *If $l(x) > 0$ is a slowly varying function as $x \to \infty$, then*

(i) $\sum_{n=1}^{m} n^s l(n) \le C m^{s+1} l(m)$ *for* $s > -1$,

(ii) $\sum_{n=m}^{\infty} n^s l(n) \le C m^{s+1} l(m)$ *for* $s < -1$.

This paper is organized as follows. In Section 2, we give the main result and its proof. A few applications of the main result are provided in Section 3.

2. Main Result and Its Proof

This paper studies arrays of rowwise ρ-mixing sequence. Let $\rho_n(i)$ be the mixing coefficient defined in Definition 1.1 for the nth row of an array $\{X_{ni}; i \ge 1, n \ge 1\}$, that is, for the sequence $X_{n1}, X_{n2}, \ldots, n \ge 1$.

Now, we state our main result.

Theorem 2.1. *Let $\{X_{ni}; i \ge 1, n \ge 1\}$ be an array of rowwise ρ-mixing random variables satisfying $\sup_n \sum_{i=1}^{\infty} \rho_n^{2/q}(2^i) < \infty$ for some $q \ge 2$, and let $\{a_{ni}; i \ge 1, n \ge 1\}$ be an array of real numbers. Let $\{b_n; n \ge 1\}$ be an increasing sequence of positive integers, and let $\{c_n; n \ge 1\}$ be a sequence of positive real numbers. If for some $0 < t < 2$ and any $\varepsilon > 0$ the following conditions are fulfilled:*

(a) $\sum_{n=1}^{\infty} c_n \sum_{i=1}^{b_n} P\{|a_{ni}X_{ni}| \ge \varepsilon b_n^{1/t}\} < \infty$,

(b) $\sum_{n=1}^{\infty} c_n b_n^{-q/t+1} \max_{1\le i\le b_n} |a_{ni}|^q E|X_{ni}|^q I[|a_{ni}X_{ni}| < \varepsilon b_n^{1/t}] < \infty$,

(c) $\sum_{n=1}^{\infty} c_n b_n^{-q/t+q/2} (\max_{1\le i\le b_n} |a_{ni}|^2 E|X_{ni}|^2 I[|a_{ni}X_{ni}| < \varepsilon b_n^{1/t}])^{q/2} < \infty$,

then

$$\sum_{n=1}^{\infty} c_n P\left\{\max_{1\le i\le b_n}\left|\sum_{j=1}^{i}\left(a_{nj}X_{nj} - a_{nj}EX_{nj}I\left[\left|a_{nj}X_{nj}\right| < \varepsilon b_n^{1/t}\right]\right)\right| > \varepsilon b_n^{1/t}\right\} < \infty. \tag{2.1}$$

Remark 2.2. Theorem 2.1 extends some results of Kuczmaszewska [17] to the case of arrays of rowwise ρ-mixing sequence and generalizes the results of Kuczmaszewska [16] to the case of maximum weighted sums.

Remark 2.3. Theorem 2.1 firstly gives the condition of the mixing coefficient, so the conditions (a)–(c) do not contain the mixing coefficient. Thus, the conditions (a)–(c) are obviously simpler than the conditions (i)–(iii) in Theorem 2.1 of Kuczmaszewska [16]. Our conditions are also different from those of Theorem 2.1 in the study by Kuczmaszewska [17]: $q \ge 2$ is only required in Theorem 2.1, not $q > 2$ in Theorem 2.1 of Kuczmaszewska [17]; the powers of b_n in (b) and (c) of Theorem 2.1 are $-q/t+1$ and $-q/t+q/2$, respectively, not $-q/t$ in Theorem 2.1 of Kuczmaszewska [17].

Now, we give the proof of Theorem 2.1.

Proof. The conclusion of the theorem is obvious if $\sum_{n=1}^{\infty} c_n$ is convergent. Therefore, we will consider that only $\sum_{n=1}^{\infty} c_n$ is divergent. Let

$$Y_{nj} = a_{nj}X_{nj}I\left[\left|a_{nj}X_{nj}\right| < \varepsilon b_n^{1/t}\right], \qquad T_{ni} = \sum_{j=1}^{i} Y_{nj}, \qquad S_{ni} = \sum_{j=1}^{i} a_{nj}X_{nj},$$
$$A = \bigcap_{i=1}^{b_n}\{a_{ni}X_{ni} = Y_{ni}\}, \qquad B = \bigcup_{i=1}^{b_n}\{a_{ni}X_{ni} \neq Y_{ni}\}. \tag{2.2}$$

Note that

$$\begin{aligned} P\left\{\max_{1\le i\le b_n}|S_{ni} - ET_{ni}| > \varepsilon b_n^{1/t}\right\} &= P\left\{\left\{\max_{1\le i\le b_n}|S_{ni} - ET_{ni}| > \varepsilon b_n^{1/t}\right\} \bigcap A\right\} \\ &\quad + P\left\{\left\{\max_{1\le i\le b_n}|S_{ni} - ET_{ni}| > \varepsilon b_n^{1/t}\right\} \bigcap B\right\} \\ &\le P\left\{\max_{1\le i\le b_n}|T_{ni} - ET_{ni}| > \varepsilon b_n^{1/t}\right\} + \sum_{i=1}^{b_n} P\left\{|a_{ni}X_{ni}| > \varepsilon b_n^{1/t}\right\}. \end{aligned} \tag{2.3}$$

By (a) it is enough to prove that for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} c_n P\left\{\max_{1\le i\le b_n}|T_{ni} - ET_{ni}| > \varepsilon b_n^{1/t}\right\} < \infty. \tag{2.4}$$

By Markov inequality and Lemma 1.4, and note that the assumption $\sup_n \sum_{i=1}^{\infty} \rho_n^{2/q}(2^i) < \infty$ for some $q \ge 2$, we get

$$\begin{aligned} P\left\{\max_{1\le i\le b_n}|T_{ni} - ET_{ni}| > \varepsilon b_n^{1/t}\right\} &\le C b_n^{-q/t} E\max_{1\le i\le b_n}|T_{ni} - ET_{ni}|^q \\ &\le C b_n^{-q/t}\left\{ b_n \exp\left(K\sum_{i=0}^{[\log b_n]} \rho_n^{2/q}\left(2^i\right)\right)\max_{1\le i\le b_n} E|a_{ni}X_{ni}|^q \right. \\ &\quad \times I\left[|a_{ni}X_{ni}| < \varepsilon b_n^{1/t}\right] + K\exp\left(K\sum_{i=1}^{[\log b_n]}\rho_n\left(2^i\right)\right) \end{aligned}$$

$$\times\left(b_n \max_{1\le i\le b_n} E|a_{ni}X_{ni}|^2 I[|a_{ni}X_{ni}| < \varepsilon b_n^{1/t}]\right)^{q/2}\Bigg\}$$
$$\le Cb_n^{-q/t+1}\max_{1\le i\le b_n}|a_{ni}|^q E|X_{ni}|^q I\left[|a_{ni}X_{ni}| < \varepsilon b_n^{1/t}\right]$$
$$+ Cb_n^{-q/t+q/2}\left(\max_{1\le i\le b_n}|a_{ni}|^2 E|X_{ni}|^2 I[|a_{ni}X_{ni}| < \varepsilon b_n^{1/t}]\right)^{q/2}. \tag{2.5}$$

From (b), (c), and (2.5), we see that (2.4) holds. □

3. Applications

Theorem 3.1. *Let $\{X_{ni}; i \ge 1, n \ge 1\}$ be an array of rowwise ρ-mixing random variables satisfying $\sup_n \sum_{i=1}^{\infty} \rho_n^{2/q}(2^i) < \infty$ for some $q \ge 2$, $EX_{ni} = 0$, and $E|X_{ni}|^p < \infty$ for all $n \ge 1$, $i \ge 1$, and $1 \le p \le 2$. Let $\{a_{ni}; i \ge 1, n \ge 1\}$ be an array of real numbers satisfying the condition*

$$\max_{1\le i\le n}|a_{ni}|^p E|X_{ni}|^p = O\left(n^{\nu-1}\right), \quad \text{as } n \longrightarrow \infty, \tag{3.1}$$

for some $0 < \nu < 2/q$. Then for any $\varepsilon > 0$ and $\alpha p \ge 1$

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left\{\max_{1\le i\le n}\left|\sum_{j=1}^{i} a_{nj}X_{nj}\right| > \varepsilon n^{\alpha}\right\} < \infty. \tag{3.2}$$

Proof. Put $c_n = n^{\alpha p-2}$, $b_n = n$, and $1/t = \alpha$ in Theorem 2.1. By (3.1), we get

$$\sum_{n=1}^{\infty} c_n \sum_{i=1}^{b_n} P\left\{|a_{ni}X_{ni}| \ge \varepsilon b_n^{1/t}\right\}$$
$$\le C\sum_{n=1}^{\infty} n^{\alpha p-2}\sum_{i=1}^{n} n^{-\alpha p}|a_{ni}|^p E|X_{ni}|^p \le C\sum_{n=1}^{\infty} n^{-1}\max_{1\le i\le n}|a_{ni}|^p E|X_{ni}|^p \le C\sum_{n=1}^{\infty} n^{-2+\nu} < \infty,$$
$$\sum_{n=1}^{\infty} c_n b_n^{-q/t+1}\max_{1\le i\le b_n} E|a_{ni}X_{ni}|^q I\left[|a_{ni}X_{ni}| < \varepsilon b_n^{1/t}\right]$$
$$\le C\sum_{n=1}^{\infty} n^{\alpha p-2} n^{-\alpha q+1} n^{\alpha(q-p)}\max_{1\le i\le n}|a_{ni}|^p E|X_{ni}|^p \le C\sum_{n=1}^{\infty} n^{-2+\nu} < \infty,$$
$$\sum_{n=1}^{\infty} c_n b_n^{-q/t+q/2}\left(\max_{1\le i\le b_n} E|a_{ni}X_{ni}|^2 I\left[|a_{ni}X_{ni}| < \varepsilon b_n^{1/t}\right]\right)^{q/2}$$

$$\le C\sum_{n=1}^{\infty} n^{\alpha p-2} n^{-\alpha q+q/2} n^{\alpha(2-p)q/2}\left(\max_{1\le i\le n}|a_{ni}|^p E|X_{ni}|^p\right)^{q/2}$$
$$\le C\sum_{n=1}^{\infty} n^{\alpha p(1-q/2)+(\nu q/2-1)-1} < \infty \tag{3.3}$$

following from $\nu q/2-1<0$. By the assumption $EX_{ni}=0$ for $n\ge 1$, $i\ge 1$ and by (3.1), we have

$$\begin{aligned}
n^{-\alpha}\max_{1\le i\le n}&\left|\sum_{j=1}^{i} a_{nj} EX_{nj} I\left[\left|a_{nj}X_{nj}\right|<\varepsilon n^{\alpha}\right]\right| \\
&\le Cn^{-\alpha}\sum_{j=1}^{n}\left|a_{nj} EX_{nj} I\left[\left|a_{nj}X_{nj}\right|<\varepsilon n^{\alpha}\right]\right| \\
&\le Cn^{-\alpha}\sum_{j=1}^{n}\left|a_{nj} EX_{nj} I\left[\left|a_{nj}X_{nj}\right|\ge\varepsilon n^{\alpha}\right]\right| \\
&\le Cn^{-\alpha p}\sum_{j=1}^{n}\left|a_{nj}\right|^p E\left|X_{nj}\right|^p \\
&\le Cn^{-\alpha p+1}\max_{1\le j\le n}|a_{nj}|^p E|X_{nj}|^p \le Cn^{-\alpha p+\nu}\longrightarrow 0, \quad \text{as } n\longrightarrow\infty,
\end{aligned} \tag{3.4}$$

because $\nu<1$ and $\alpha p\ge 1$. Thus, we complete the proof of the theorem. □

Theorem 3.2. *Let $\{X_{ni}; i\ge 1, n\ge 1\}$ be an array of rowwise ρ-mixing random variables satisfying $\sup_n\sum_{i=1}^{\infty}\rho_n^{2/q}(2^i)<\infty$ for some $q\ge 2$, $EX_{ni}=0$, and $E|X_{ni}|^p<\infty$ for all $n\ge 1, i\ge 1$, and $1\le p\le 2$. Let the random variables in each row be stochastically dominated by a random variable X, such that $E|X|^p<\infty$, and let $\{a_{ni}; i\ge 1, n\ge 1\}$ be an array of real numbers satisfying the condition*

$$\max_{1\le i\le n}|a_{ni}|^p = O\left(n^{\nu-1}\right), \quad \text{as } n\longrightarrow\infty, \tag{3.5}$$

for some $0<\nu<2/q$. Then for any $\varepsilon>0$ and $\alpha p\ge 1$ (3.2) holds.

Theorem 3.3. *Let $\{X_{ni}, n\ge 1, i\ge 1\}$ be an array of rowwise ρ-mixing random variables satisfying $\sup_n\sum_{i=1}^{\infty}\rho_n^{2/q}(2^i)<\infty$ for some $q\ge 2$ and $EX_{ni}=0$ for all $n\ge 1, i\ge 1$. Let the random variables in each row be stochastically dominated by a random variable X, and let $\{a_{ni}; i\ge 1, n\ge 1\}$ be an array of real numbers. If for some $0<t<2$, $\nu>1/2$*

$$\sup_{i\ge 1}|a_{ni}| = O\left(n^{1/t-\nu}\right), \qquad E|X|^{1+2/\nu}<\infty, \tag{3.6}$$

then for any $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P\left\{\max_{1\le i\le n}\left|\sum_{j=1}^{i} a_{nj}X_{nj}\right| > \varepsilon n^{1/t}\right\} < \infty. \tag{3.7}$$

Proof. Take $c_n = 1$ and $b_n = n$ for $n \ge 1$. Then we see that (a) and (b) are satisfied. Indeed, taking $q \ge \max(2, 1 + 2/\nu)$, by Lemma 1.5 and (3.6), we get

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n \sum_{i=1}^{b_n} P\left\{|a_{ni}X_{ni}| \ge \varepsilon b_n^{1/t}\right\} \\
&\quad = \sum_{n=1}^{\infty}\sum_{i=1}^{n} P\left\{|a_{ni}X_{ni}| \ge \varepsilon n^{1/t}\right\} \le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} P\{|X| \ge Cn^{\nu}\} \\
&\quad = C\sum_{n=1}^{\infty} n \sum_{k=n}^{\infty} P\{Ck^{\nu} \le |X| < C(k+1)^{\nu}\} \\
&\quad \le C\sum_{k=1}^{\infty} k^2 P\{Ck^{\nu} \le |X| < C(k+1)^{\nu}\} \le CE|X|^{2/\nu} < \infty, \\
&\sum_{n=1}^{\infty} c_n b_n^{-q/t+1} \max_{1\le i\le b_n} |a_{ni}|^q E|X_{ni}|^q I\left[|a_{ni}X_{ni}| < \varepsilon b_n^{1/t}\right] \\
&\quad \le C\sum_{n=1}^{\infty} n^{-q/t+1} \max_{1\le i\le n} |a_{ni}|^q \left\{E|X|^q I\left[|a_{ni}X| < \varepsilon n^{1/t}\right] + \frac{n^{q/t}}{|a_{ni}|^q} P\left\{|a_{ni}X| \ge \varepsilon n^{1/t}\right\}\right\} \\
&\quad \le C\sum_{n=1}^{\infty} n^{-(1+2/\nu)/t+1} \max_{1\le i\le n} |a_{ni}|^{1+2/\nu} E|X|^{1+2/\nu} + C\sum_{n=1}^{\infty} n \max_{1\le i\le n} P\left\{|a_{ni}X| \ge \varepsilon n^{1/t}\right\} \\
&\quad \le C\sum_{n=1}^{\infty} n^{-(1+2/\nu)/t+1} \left(\sup_{i\ge 1}|a_{ni}|\right)^{1+2/\nu} E|X|^{1+2/\nu} + C\sum_{n=1}^{\infty} nP\{|X| \ge Cn^{\nu}\} \\
&\quad \le C\sum_{n=1}^{\infty} n^{-\nu-1} E|X|^{1+2/\nu} + C\sum_{n=1}^{\infty} n^{-\nu-1} E|X|^{1+2/\nu} \le C\sum_{n=1}^{\infty} n^{-\nu-1} < \infty
\end{aligned} \tag{3.8}$$

In order to prove that (c) holds, we consider the following two cases.

If $\nu > 2$, by Lemma 1.5, C_r inequality, and (3.6), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n b_n^{-q/t+q/2} \left(\max_{1\le i\le b_n} |a_{ni}|^2 E|X_{ni}|^2 I\left[|a_{ni}X_{ni}| < \varepsilon b_n^{1/t}\right]\right)^{q/2} \\
&\quad \le C\sum_{n=1}^{\infty} n^{-q/t+q/2} \left(\max_{1\le i\le n} |a_{ni}|^2 E|X|^2 I\left[|a_{ni}X_{ni}| < \varepsilon n^{1/t}\right]\right)^{q/2}
\end{aligned}$$

$$
\begin{aligned}
&+ C\sum_{n=1}^{\infty} n^{q/2}\left(\max_{1\le i\le n} P\left\{|a_{ni}X| \ge \varepsilon n^{1/t}\right\}\right)^{q/2} \\
&\le C\sum_{n=1}^{\infty} n^{-q/t+q/2} n^{(1/t)(1-2/\nu)(q/2)}\left(\max_{1\le i\le n}|a_{ni}|^{1+2/\nu} E|X|^{1+2/\nu}\right)^{q/2} \\
&+ C\sum_{n=1}^{\infty} n^{q/2}(P\{|X| \ge Cn^{\nu}\})^{q/2} \\
&\le C\sum_{n=1}^{\infty} n^{-q/t+q/2+(1/t)(1-2/\nu)(q/2)}\left(\sup_{i\ge 1}|a_{ni}|\right)^{(1+2/\nu)(q/2)}\left(E|X|^{1+2/\nu}\right)^{q/2} \\
&+ C\sum_{n=1}^{\infty} n^{q/2} n^{-(1+2/\nu)(\nu q/2)}\left(E|X|^{1+2/\nu}\right)^{q/2} \\
&\le C\sum_{n=1}^{\infty} n^{-(\nu+1)(q/2)}\left(E|X|^{1+2/\nu}\right)^{q/2} < \infty.
\end{aligned}
\tag{3.9}
$$

If $1/2 < \nu \le 2$, take $q > 2/(2\nu - 1)$. We have that $(2\nu - 1)q/2 > 1$. Note that in this case $E|X|^2 < \infty$. We have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} c_n b_n^{-q/t+q/2}\left(\max_{1\le i\le b_n}|a_{ni}|^2 E|X_{ni}|^2 I\left[|a_{ni}X_{ni}| < \varepsilon b_n^{1/t}\right]\right)^{q/2} \\
&\le C\sum_{n=1}^{\infty} n^{-q/t+q/2}\left(\max_{1\le i\le n}|a_{ni}|^2 E|X|^2 I\left[|a_{ni}X_{ni}| < \varepsilon n^{1/t}\right]\right)^{q/2} + C\sum_{n=1}^{\infty} n^{q/2}(P\{|X| \ge Cn^{\nu}\})^{q/2} \\
&\le C\sum_{n=1}^{\infty} n^{-q/t+q/2}\left(\sup_{i\ge 1}|a_{ni}|\right)^{q}\left(E|X|^2\right)^{q/2} + C\sum_{n=1}^{\infty} n^{q/2-\nu q}\left(E|X|^2\right)^{q/2} \\
&\le C\sum_{n=1}^{\infty} n^{-(2\nu-1)(q/2)}\left(E|X|^2\right)^{q/2} < \infty.
\end{aligned}
\tag{3.10}
$$

The proof will be completed if we show that

$$
n^{-1/t}\max_{1\le i\le n}\left|\sum_{j=1}^{i} a_{nj} EX_{nj} I\left[\left|a_{nj}X_{ni}\right| < \varepsilon n^{1/t}\right]\right| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.11}
$$

Indeed, by Lemma 1.5, we have

$$
\begin{aligned}
n^{-1/t}\max_{1\le i\le n}\left|\sum_{j=1}^{i}a_{nj}EX_{nj}I\left[\left|a_{nj}X_{ni}\right|<\varepsilon n^{1/t}\right]\right| &\le Cn^{-1/t}\sum_{j=1}^{n}\left|a_{nj}\right|E|X|+C\sum_{j=1}^{n}P\left\{\left|a_{nj}X\right|\ge\varepsilon n^{1/t}\right\}\\
&\le Cn^{-\nu}E|X|+CnP\{|X|\ge\varepsilon n^{\nu}\}\\
&\le Cn^{-\nu}E|X|+Cn^{-(\nu+1)}E|X|^{1+2/\nu}\longrightarrow 0,\quad \text{as } n\longrightarrow\infty.
\end{aligned}
\tag{3.12}
$$

□

Theorem 3.4. *Let* $\{X_{ni}; i\ge 1, n\ge 1\}$ *be an array of rowwise* ρ*-mixing random variables satisfying* $\sup_n\sum_{i=1}^{\infty}\rho_n^{2/q}(2^i)<\infty$ *for some* $q\ge 2$, *and let* $\{a_{ni}; i\ge 1, n\ge 1\}$ *be an array of real numbers. Let* $l(x)>0$ *be a slowly varying function as* $x\to\infty$. *If for some* $0<t<2$ *and real number* λ, *and any* $\varepsilon>0$ *the following conditions are fulfilled:*

(A) $\sum_{n=1}^{\infty}n^{\lambda}l(n)\sum_{i=1}^{n}P\{|a_{ni}X_{ni}|\ge\varepsilon n^{1/t}\}<\infty$,

(B) $\sum_{n=1}^{\infty}n^{\lambda-q/t+1}l(n)\max_{1\le i\le n}E|a_{ni}X_{ni}|^{q}I[|a_{ni}X_{ni}|<\varepsilon n^{1/t}]<\infty$,

(C) $\sum_{n=1}^{\infty}n^{\lambda-q/t+q/2}l(n)(\max_{1\le i\le n}|a_{ni}|^{2}E|X_{ni}|^{2}I[|a_{ni}X_{ni}|<\varepsilon n^{1/t}])^{q/2}<\infty$,

then

$$
\sum_{n=1}^{\infty}n^{\lambda}l(n)P\left\{\max_{1\le i\le n}\left|\sum_{j=1}^{i}\left(a_{nj}X_{nj}-a_{nj}EX_{nj}I\left[\left|a_{nj}X_{nj}\right|<\varepsilon n^{1/t}\right]\right)\right|>\varepsilon n^{1/t}\right\}<\infty. \tag{3.13}
$$

Proof. Let $c_n=n^{\lambda}l(n)$ and $b_n=n$. Using Theorem 2.1, we obtain (3.13) easily. □

Theorem 3.5. *Let* $\{X_{ni}; i\ge 1, n\ge 1\}$ *be an array of rowwise* ρ*-mixing identically distributed random variables satisfying* $\sum_{i=1}^{\infty}\rho_n^{2/q}(2^i)<\infty$ *for some* $q\ge 2$ *and* $EX_{11}=0$. *Let* $l(x)>0$ *be a slowly varying function as* $x\to\infty$. *If for* $\alpha>1/2$, $\alpha p>1$, *and* $0<t<2$

$$
E|X_{11}|^{\alpha pt}l\left(|X_{11}|^{t}\right)<\infty, \tag{3.14}
$$

then

$$
\sum_{n=1}^{\infty}n^{\alpha p-2}l(n)P\left\{\max_{1\le i\le n}\left|\sum_{j=1}^{i}X_{nj}\right|>\varepsilon n^{1/t}\right\}<\infty. \tag{3.15}
$$

Proof. Put $\lambda=\alpha p-2$ and $a_{ni}=1$ for $n\ge 1, i\ge 1$ in Theorem 3.4. To prove (3.15), it is enough to note that under the assumptions of Theorem 3.4, the conditions (A)–(C) of Theorem 3.4 hold.

By Lemma 1.6, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{\alpha p-1} l(n) P\left\{|X_{11}| > \varepsilon n^{1/t}\right\} &= \sum_{n=1}^{\infty} n^{\alpha p-1} l(n) \sum_{m=n}^{\infty} P\left\{\varepsilon m^{1/t} < |X_{11}| \leq \varepsilon (m+1)^{1/t}\right\} \\
&\leq C \sum_{m=1}^{\infty} P\left\{\varepsilon m^{1/t} < |X_{11}| \leq \varepsilon (m+1)^{1/t}\right\} \sum_{n=1}^{m} n^{\alpha p-1} l(n) \\
&\leq C \sum_{m=1}^{\infty} m^{\alpha p} l(m) P\left\{\varepsilon m^{1/t} < |X_{11}| \leq \varepsilon (m+1)^{1/t}\right\} \\
&\leq C E|X_{11}|^{\alpha p t} l\left(|X_{11}|^{t}\right) < \infty,
\end{aligned} \tag{3.16}$$

which proves that condition (A) is satisfied.

Taking $q > \max(2, \alpha p t)$, we have $\alpha p - q/t < 0$. By Lemma 1.6, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-1-(q/t)} l(n) E|X_{11}|^{q} I\left[|X_{11}| \leq \varepsilon n^{1/t}\right] \\
&\quad = \sum_{n=1}^{\infty} n^{\alpha p-1-(q/t)} l(n) \sum_{m=1}^{n} E|X_{11}|^{q} I\left[\varepsilon (m-1)^{1/t} \leq |X_{11}| < \varepsilon m^{1/t}\right] \\
&\quad \leq C \sum_{m=1}^{\infty} E|X_{11}|^{q} I\left[\varepsilon (m-1)^{1/t} \leq |X_{11}| < \varepsilon m^{1/t}\right] \sum_{n=m}^{\infty} n^{\alpha p-1-(q/t)} l(n) \\
&\quad \leq C \sum_{m=1}^{\infty} m^{\alpha p-(q/t)} l(m) E|X_{11}|^{q} I\left[\varepsilon (m-1)^{1/t} \leq |X_{11}| < \varepsilon m^{1/t}\right] \\
&\quad \leq C E|X_{11}|^{\alpha p t} l\left(|X_{11}|^{t}\right) < \infty,
\end{aligned} \tag{3.17}$$

which proves that (B) holds.

In order to prove that (C) holds, we consider the following two cases.

If $\alpha p t < 2$, take $q > 2$. We have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-q/t+q/2} l(n) \left(E|X_{11}|^{2} I\left[|X_{11}| < \varepsilon n^{1/t}\right]\right)^{q/2} \\
&\quad \leq C \sum_{n=1}^{\infty} n^{\alpha p-2-q/t+q/2} l(n) n^{q/t-\alpha p q/2} \left(E|X_{11}|^{\alpha p t} I\left[|X_{11}| < \varepsilon n^{1/t}\right]\right)^{q/2} \\
&\quad \leq C \sum_{n=1}^{\infty} n^{(\alpha p-1)(1-q/2)-1} l(n) < \infty.
\end{aligned} \tag{3.18}$$

If $\alpha pt \geq 2$, take $q > \max(2, 2t(\alpha p - 1)/(2 - t))$. We have $\alpha p - q/t + q/2 < 1$. Note that in this case $E|X_{11}|^2 < \infty$. We obtain

$$\sum_{n=1}^{\infty} n^{\alpha p-2-q/t+q/2} l(n) \left(E|X_{11}|^2 I\left[|X_{11}| < \varepsilon n^{1/t}\right]\right)^{q/2} \leq C \sum_{n=1}^{\infty} n^{\alpha p-2-q/t+q/2} l(n) < \infty. \tag{3.19}$$

The proof will be completed if we show that

$$n^{-1/t} \max_{1\leq i\leq n} \left| \sum_{j=1}^{i} E X_{nj} I\left[|X_{11}| < \varepsilon n^{1/t}\right]\right| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.20}$$

If $\alpha pt < 1$, then

$$n^{-1/t} \max_{1\leq i\leq n} \left| \sum_{j=1}^{i} E X_{nj} I\left[|X_{11}| < \varepsilon n^{1/t}\right]\right| \leq C n^{1-\alpha p} E|X_{11}|^{\alpha pt} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.21}$$

If $\alpha pt \geq 1$, note that $EX_{11} = 0$, then

$$\begin{aligned} &n^{-1/t} \max_{1\leq i\leq n} \left| \sum_{j=1}^{i} E X_{nj} I\left[|X_{11}| < \varepsilon n^{1/t}\right]\right| \\ &\quad\leq n^{-1/t+1} \left| E X_{11} I\left[|X_{11}| \geq \varepsilon n^{1/t}\right]\right| \leq C n^{1-\alpha p} E|X_{11}|^{\alpha pt} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.22}$$

We complete the proof of the theorem. □

Noting that for typical slowly varying functions, $l(x) = 1$ and $l(x) = \log x$, we can get the simpler formulas in the above theorems.

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