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Research Article

Relative Isoperimetric Inequality for Minimal Submanifolds in a Riemannian Manifold

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Let Σ be a domain on an m-dimensional minimal submanifold in the outside of a convex set C in $\mathbb{S}^n$ or $\mathbb{H}^n$. The modified volume $M(\Sigma)$ is introduced by Choe and Gulliver (1992) and we prove a sharp modified relative isoperimetric inequality for the domain Σ, $(1/2)m^m\omega_m M(\Sigma)^{m-1} \leq \text{Volume}(\partial\Sigma - \partial C)^m$, where ω_m is the volume of the unit ball of $\mathbb{R}^m$. For any domain Σ on a minimal surface in the outside convex set C in an n-dimensional Riemannian manifold, we prove a weak relative isoperimetric inequality $\pi\text{Area}(\Sigma) \leq \text{Length}(\partial\Sigma - \partial C)^2 + K\text{Area}(\Sigma)^2$, where K is an upper bound of sectional curvature of the Riemannian manifold.

1. Introduction

Let ∂D be the simple closed curve of a domain D in a two-dimensional space form with constant curvature K. Then the well-known sharp isoperimetric inequality is the following:

$$4\pi\text{Area}(\text{D}) \leq \text{Length}(\partial D)^2 + K\text{Area}(D)^2, \tag{1.1}$$

where equality holds if and only if D is a geodesic disk (see [1]).

An immediate consequence of this inequality is that if H is a closed half-space of a two-dimensional space form with constant curvature K and D is a domain in H with $\partial D \cap \partial H \neq \emptyset$, then

$$2\pi\text{Area}(D) \leq \text{Length}(\partial D - \partial H)^2 + K\text{Area}(D)^2, \tag{1.2}$$

where equality holds if and only if D is a totally geodesic half-disk with the geodesic part of its boundary contained in ∂H. This follows the original isoperimetric inequality after extending the domain by mirror symmetry with respect to ∂H.

Motivated by this, one arises natural questions as follows.

If C is a convex set in an n-dimensional space form with constant curvature K and D is a minimal surface in the outside of C with $\partial D \cap \partial C \neq \emptyset$, does D satisfy inequality

$$2\pi \operatorname{Area}(D) \leq \operatorname{Length}(\partial D - \partial C)^2 + K \operatorname{Area}(D)^2? \tag{1.3}$$

How about an m-dimensional minimal submanifold case?

Equation (1.3) is called the *relative* isoperimetric inequality, C is called the supporting set of D, and $\operatorname{Area}(\partial D - \partial C)$ is called the relative area of ∂D. A partial result is obtained by Kim [2], when part of the boundary $\partial D - \partial C$ of the domain $D$is *radially connected* from a point $p \in \partial D \cap \partial C$, that is, $\{r(q) = \operatorname{dist}(p, q) \mid q \in \partial D - \partial C\}$ is a connected interval. And there are some partial results on the higher-dimensional submanifold case (see [3, 4]). In case of $\mathbb{S}^n$, the problem remains open, even in the two-dimensional case (see [5]).

In this paper, we obtain two different type relative isoperimetric inequalities. First, using the modified volume introduced by Choe and Gulliver [6], we have a *modified* relative isoperimetric inequality in $\mathbb{S}^n$ or $\mathbb{H}^n$ without the curvature correct term:

$$\frac{1}{2} m^m \omega_m M_p(\Sigma)^{m-1} \leq \operatorname{Volume}(\partial\Sigma - \partial C)^m, \tag{1.4}$$

where ω_m is the volume of a unit ball of $\mathbb{R}^m$, and Σ is a domain of an m-dimensional submanifold. In Theorem 2.11, (1.4) holds for $\partial\Sigma - \partial C$ lies on a geodesic sphere of $\mathbb{S}^n$ or $\mathbb{H}^n$. In Theorem 2.3, (1.4) holds for $m = 2$, $\Sigma \subset \mathbb{S}^n_+$ and $\partial\Sigma - \partial C$ is radially connected for a point $p \in \partial\Sigma \cap \partial C$.

Second, in Section 3 we obtain an inequality on usual volume for any minimal surface of a Riemannian manifold with sectional curvature bounded above by a constant K:

$$\pi \operatorname{Area}(\Sigma) \leq \operatorname{Length}(\partial\Sigma - \partial C)^2 + K \operatorname{Area}(\Sigma)^2. \tag{1.5}$$

But we cannot find a minimal surface which satisfies the equality. That is why we call (1.5) a *weak* relative isoperimetric inequality.

2. Modified Relative Isoperimetric Inequalities in a Space Form

We review the *modified volume* in $\mathbb{S}^n$ and $\mathbb{H}^n$ with constant sectional curvature 1 and -1, respectively. Let p be a point in the n-dimensional sphere $\mathbb{S}^n$ and let $r(x)$ be the distance from p to x in $\mathbb{S}^n$.

Definition 2.1 (modified volume in $\mathbb{S}^n$). Given that Σ is an m-dimensional submanifold in $\mathbb{S}^n$, the modified volume of Σ with center at p is defined by

$$M_p(\Sigma) \equiv \int_\Sigma \cos r. \tag{2.1}$$

Embed $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ with p being the north pole $(0,\ldots,0,1)$. For a domain, the geometric meaning of the modified volume of $\Sigma^n \subset \mathbb{S}^n$ is the Euclidean volume of the orthogonal projection of Σ into the $x_{n+1} = 0$ counting orientation. Clearly, we have in $\mathbb{S}^n$

$$M_p(\Sigma) \leq \mathrm{Volume}(\Sigma), \tag{2.2}$$

where Volume(Σ) is the usual volume of Σ.

Similarly, let p be a point in the n-dimensional hyperbolic space $\mathbb{H}^n$ and let $r(x)$ be the distance from p to x in $\mathbb{H}^n$.

Definition 2.2 (modified volume in $\mathbb{H}^n$). Given that Σ is an m-dimensional submanifold in $\mathbb{H}^n$, the modified volume of Σ with center at p is defined by

$$M_p(\Sigma) \equiv \int_\Sigma \cosh r. \tag{2.3}$$

Embed $\mathbb{H}^n$ isometrically onto the hyperboloid $\mathscr{H} = \{(x_1,\ldots,x_{n+1}) \mid x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1, x_{n+1} > 0\}$ in $\mathbb{R}^{n+1}$ with the Minkowski metric $ds^2 = x_1^2 + \cdots + x_n^2 - x_{n+1}^2$ such that p is the point $(0,\ldots,0,1) \in \mathscr{H}$. Then for a domain, the modified volume equals the Euclidean volume of the projection of Σ^n onto the hyperplane $x_{n+1} = 0$. Clearly, we have in $\mathbb{H}^n$

$$M_p(\Sigma) \geq \mathrm{Volume}(\Sigma). \tag{2.4}$$

More precisely, see Choe and Gulliver's paper (see [6]).

Theorem 2.3. *Let C be a closed convex set in $\mathbb{S}^n$. Assume that Σ is a compact minimal surface in the outside C such that Σ is orthogonal to ∂C along $\Gamma_C := \partial\Sigma \cap \partial C$. And $r(x)$ is the distance from $p \in \Gamma_C$ to $x \in \mathbb{S}^n$ and $r(x) \leq \pi/2$ on Σ. If $\Gamma := \partial\Sigma - \Gamma_C$ is radially connected from the point $p \in \Gamma_C$, that is, $\{r(q) = \mathrm{dist}(p,q) \mid q \in \Gamma\}$ is a connected interval, then one has*

$$2\pi M_p(\Sigma) \leq \mathrm{Length}(\Gamma)^2. \tag{2.5}$$

Equality holds if and only if Σ is a totally geodesic half-disk with Γ being a geodesic half-circle.

If $\partial\Sigma - \partial C$ is connected, then it is trivially radially connected from $p \in \partial\Sigma \cap \partial C$. If $\partial\Sigma - \partial C$ has two components, then using the same argument as [2, Corollary 1] we obtain the following.

Corollary 2.4. *Let Σ be a compact minimal surface satisfying the same assumptions as in Theorem 2.3 except the radially connectedness. Then the modified relative isoperimetric inequality in Theorem 2.3 holds if Γ is connected or Γ has two components that are connected by a component Υ of $\partial\Sigma \cap \partial C$.*

Before giving lemmas for proving Theorem 2.3, we define a *cone*. Given an $(m-1)$-dimensional submanifold Ω of $\mathbb{S}^n$ or $\mathbb{H}^n$ and a point p in $\mathbb{S}^n$ or $\mathbb{H}^n$, the m-dimensional cone $p ⋇ \Omega$ with center at p is defined by the set of all minimizing geodesics from p to a point of Ω.

Lemma 2.5 (see [6, Lemma 4]). (a) *If $\Sigma \subset \mathbb{S}^n$ is an m-dimensional minimal submanifold or a cone, and r is the distance in $\mathbb{S}^n$ from a fixed point, then*

$$\Delta \cos r = -m \cos r, \tag{2.6}$$

where Δ is the Laplacian on the submanifold Σ and, in case Σ is a cone, r is the distance from the center of Σ.

(b) *Suppose that $\Sigma \subset \mathbb{H}^n$ is an m-dimensional minimal submanifold or a cone. Then*

$$\Delta \cosh r = m \cosh r. \tag{2.7}$$

Here again, in case of a cone Σ, r is the distance from the center of Σ.

Proposition 2.6. *Let C be a closed convex set in $\mathbb{S}^n$ or $\mathbb{H}^n$. Assume that Σ is an m-dimensional minimal submanifold in the outside C such that Σ is orthogonal to ∂C along $\Gamma_C = \partial\Sigma \cap \partial C$ and $r(x)$ is the distance from $p \in \Gamma_C$ to $x \in \mathbb{S}^n$ or $\mathbb{H}^n$. Let $\Gamma = \partial\Sigma - \Gamma_C$. In case of $\Sigma \subset \mathbb{S}^n$, one assumes that $r(x) \le \pi/2$ for all $x \in \Sigma$. Then one has*

$$M_p(\Sigma) \le M_p(p \divideontimes \Gamma). \tag{2.8}$$

Proof. Let ν and η be the unit conomals to $\partial\Sigma$ on Σ and $p \divideontimes \partial\Sigma$, respectively. By Lemma 2.5, we have

$$M_p(\Sigma) = -\frac{1}{m}\int_\Sigma \Delta \cos r = \frac{1}{m}\int_\Gamma \sin r \cdot \frac{\partial r}{\partial \nu} + \frac{1}{m}\int_{\Gamma_C} \sin r \cdot \frac{\partial r}{\partial \nu}. \tag{2.9}$$

The η makes the smallest angle with ∇r, that is, the unit normal vector to $\partial\Sigma$ that lies in the two-dimensional plane spanned by ∇r and the tangent line of $\partial\Sigma$ such that $\partial r/\partial\eta \ge 0$. Clearly $\partial r/\partial\nu \le \partial r/\partial\eta = \sqrt{1 - \langle \nabla r, \tau\rangle^2}$, where τ is a unit tangent to $\partial\Sigma$. Since C is a convex set, $p \in \Gamma_C$, and $\Gamma_C \subset \partial C$, we see that $\overline{\nabla} r(x)$ points outward of C for every $x \in \Gamma_C$, where $\overline{\nabla} r$ is the gradient in the $\mathbb{S}^n$. From the orthogonality condition, $\nu(x)$ is a unit normal toward inside C along the Γ_C. So we have

$$\frac{\partial r}{\partial \nu}(x) = \left\langle \overline{\nabla} r(x), \nu(x) \right\rangle \le 0, \tag{2.10}$$

for every $x \in \Gamma_C$, and

$$M_p(\Sigma) \le \frac{1}{m}\int_\Gamma \sin r \cdot \frac{\partial r}{\partial \nu} \le \frac{1}{m}\int_\Gamma \sin r \cdot \frac{\partial r}{\partial \eta} = -\frac{1}{m}\int_{p \divideontimes \Gamma} \Delta \cos r = M_p(p \divideontimes \Gamma). \tag{2.11}$$

The similar proof holds for $\Sigma \subset \mathbb{H}^n$. □

Lemma 2.7 (see [6, Lemma 6]). *Let $G(x)$ be Green's function of $\mathbb{S}^n$ ($\mathbb{H}^n$, resp.), whose derivative is $\sin^{1-m} x$ for $0 < x < \pi$ ($\sinh^{1-m} x$ for $0 < x < \infty$, resp.). If Σ is an m-dimensional minimal submanifold of $\mathbb{S}^n$ ($\mathbb{H}^n$, resp.), then $G \circ r$ is subharmonic on $\Sigma - \{\pm p\} \subset \mathbb{S}^n$ ($\Sigma - \{p\} \subset \mathbb{H}^n$, resp.).*

Now we estimate the angle of Γ viewed from a point $p \in \Gamma_C$. Recall the definition of the angle viewed from a point. For an $(m-1)$-dimensional rectifiable set Ω in $\mathbb{S}^n$ and a point $p \in \mathbb{S}^n$ such that $\operatorname{dist}(p,q) < \pi$ for all $q \in \Omega$, the $(m-1)$-dimensional angle $A^{m-1}(\Omega, p)$ of Ω viewed from p is defined by

$$A^{m-1}(\Omega,p) = \sin^{1-m} t \cdot \text{Volume}\left[p ※ \Omega \cap S(p,t)\right], \tag{2.12}$$

where $S(p,t)$ is the geodesic sphere of radius $t < \operatorname{dist}(p,\Omega)$ centered at p, and the volume is measured counting multiplicity. Clearly, the angle does not depend on t. There is obviously an analogous definition for the angle of $\Omega \subset \mathbb{H}^n$ viewed from $p \in \mathbb{H}^n$.

Note that

$$A^{m-1}(\Omega,p) = m\omega_m \Theta^m(p ※ \Omega, p), \tag{2.13}$$

where $\Theta^m(p ※ \Omega, p)$ is the m-dimensional density of $p ※ \Omega$ at p.

Proposition 2.8. *Let Σ be an m-dimensional minimal submanifold satisfying the same assumptions as in Proposition 2.6. Then for any $p \in \partial C$,*

$$A^{m-1}(\Gamma,p) \geq \frac{m\omega_m}{2}. \tag{2.14}$$

Equality holds if and only if Σ is totally geodesic and star shaped with respect to p.

Proof. Let $B^n(p,t) \subset \mathbb{S}^n$ be the geodesic ball centered at p, radius $t < \operatorname{dist}(p,\Gamma)$, $\Sigma_t = \Sigma - B^n(p,t)$, and $S_t = \Sigma \cap S(p,t)$. By Lemma 2.7, we have subharmonic $G(r)$, where r is distance from p. Hence

$$0 \leq \int_{\Sigma_t} \Delta G(r) = \int_{\Sigma_t} \operatorname{div}\left(\sin^{1-m} r \nabla r\right) = \int_{\partial\Sigma_t} \sin^{1-m} r \cdot \frac{\partial r}{\partial \nu} + \int_{\partial\Sigma} \sin^{1-m} r \cdot \frac{\partial r}{\partial \nu}, \tag{2.15}$$

where ν is outward unit conormal along the boundary.

Since, near p, Σ can be identified with totally geodesic half-sphere and $\partial r/\partial\nu \to -1$and $\sin t/t \to 1$ on $\partial\Sigma_t$ as $t \to 0$, we have

$$\begin{aligned} \frac{m\omega_m}{2} &= \lim_{t\to 0}\int_{\partial\Sigma} \sin^{1-m} r \cdot \frac{\partial r}{\partial \nu} \leq \int_{\partial\Sigma-\partial C} \sin^{1-m} r \cdot \frac{\partial r}{\partial \nu} \\ &\leq \int_{\partial\Sigma-\partial C} \sin^{1-m} r \cdot \frac{\partial r}{\partial \eta} = A^{m-1}(\Gamma,p), \end{aligned} \tag{2.16}$$

where η is the unit conormal of the cone and the same argument holds as in Proposition 2.6.

Equality holds if and only if $\Delta G(r) = 0$, $\Theta(\Sigma,p) = 1/2$, and $\nu = \eta$, that is, Σ is a star-shaped minimal cone with density at the center equal to $1/2$. Since $\mathbb{S}^{m-1}$ is the only $(m-1)$-dimensional minimal submanifold in $\mathbb{S}^n$ with volume $m\omega_m$, this completes the proof for $\Sigma \subset \mathbb{S}^n$. A similar proof holds for $\Sigma \subset \mathbb{H}^n$. □

Proof of Theorem 2.3. We approach to the proof by comparison between Σ and the cone $p \divideontimes \Gamma$. Since the cone $p \divideontimes \Gamma$ is locally developable on a totally geodesic sphere $\mathbb{S}^2$, we reduce the proof to the proof of Theorem 1 in [7] by doubling argument.

For each geodesic sphere $S(p,t)$ centered at p and radius t, one has a local isometry between the curve $S(p,t) \cap (p \divideontimes \Gamma)$ and a great circle on $S(p,t)$. Hence we can develop $p \divideontimes \Gamma$ on a great sphere $\mathbb{S}^2 \subset \mathbb{S}^n$; one can find a curve γ (not necessarily closed) in $\mathbb{S}^2$ and a local isometry from $p \divideontimes \Gamma$ into $\overline{p} \divideontimes \gamma$, where $\overline{p}$ is the north pole of $\mathbb{S}^2$. Clearly we have $M_p(p \divideontimes \Gamma) = M_{\overline{p}}(\overline{p} \divideontimes \gamma)$, $\operatorname{Length}(\Gamma) = \operatorname{Length}(\gamma)$, and $A^1(\Gamma, p) = A^1(\gamma, \overline{p})$. Moreover, if we let q_1 and q_2 be the endpoints of γ, then $\operatorname{dist}(\overline{p}, q_1) = \operatorname{dist}(\overline{p}, q_2)$.

We write $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_N$, where Γ_j, $j = 1, \ldots, N$, is a connected component of Γ. Note that $\partial\Gamma_j$ may be empty or not.

If $\partial\Gamma_j = \emptyset$, then, after cutting $p \divideontimes \Gamma_j$ along an appropriate geodesic and developing it onto the great sphere $\mathbb{S}^2$, $p \divideontimes \Gamma_j$ may be identified with the cone $\overline{p} \divideontimes \gamma_j$ in $\mathbb{S}^2$, where γ_j is a curve in $\mathbb{S}^2$ given in terms of the polar coordinates by $\rho = \gamma_j(\theta)$ satisfying $0 \le \theta \le \overline{\theta}_j = A^1(\Gamma_j, p)$ and $\gamma_j(0) = \gamma_j(\overline{\theta}_j) = \operatorname{dist}(p, \Gamma_j) = \operatorname{dist}(\overline{p}, \gamma_j)$. Here, θ is the angle parameter of the cone. Now we define the doubling $\widetilde{\gamma}_j$ of γ_j by the doubling parametrization as follows:

$$\rho = \widetilde{\gamma}_j(\theta) = \begin{cases} \gamma_j(\theta), & 0 \le \theta \le \overline{\theta}_j, \\ \gamma_j\left(\theta - \overline{\theta}_j\right), & \overline{\theta}_j \le \theta \le 2\overline{\theta}_j. \end{cases} \tag{2.17}$$

If $\partial\Gamma_j \ne \emptyset$, then, after developing it onto $\mathbb{S}^2$, $p \divideontimes \Gamma_j$ is identified with $\overline{p} \divideontimes \gamma_j$ in $\mathbb{S}^2$, where γ_j is a curve in $\mathbb{S}^2$ given in terms of the polar coordinates by $\rho = \gamma_j(\theta)$ defined on $0 \le \theta \le \overline{\theta}_j = A^1(\Gamma_j, p)$, where θ is the angle parameter of the cone. Choose $\theta_{j_0} \in [0, \overline{\theta}_j]$ such that $\gamma_j(\theta_{j_0}) = \operatorname{dist}(p, \Gamma_j) = \operatorname{dist}(\overline{p}, \gamma_j)$. Then we define the doubling $\widetilde{\gamma}_j$ of γ_j as follows:

$$\rho = \widetilde{\gamma}_j(\theta) = \begin{cases} \gamma_j\left(\theta + \theta_{j_0}\right), & 0 \le \theta \le \overline{\theta}_j - \theta_{j_0}, \\ \gamma_j\left(-\theta + 2\overline{\theta}_j - \theta_{j_0}\right), & \overline{\theta}_j - \theta_{j_0} \le \theta \le 2\overline{\theta}_j - \theta_{j_0}, \\ \gamma_j\left(\theta - 2\overline{\theta}_j + \theta_{j_0}\right), & 2\overline{\theta}_j - \theta_{j_0} \le \theta \le 2\overline{\theta}_j. \end{cases} \tag{2.18}$$

In both cases, we have the following equalities:

$$\begin{gathered} M_{\overline{p}}(\overline{p} \divideontimes \widetilde{\gamma}_j) = 2M_p(p \divideontimes \Gamma_j), \\ \operatorname{Length}(\widetilde{\gamma}_j) = 2\operatorname{Length}(\Gamma_j), \\ A^1(\widetilde{\gamma}_j, \overline{p}) = 2A^1(\Gamma_j, p), \\ \widetilde{\gamma}_j(0) = \widetilde{\gamma}_j\left(2\overline{\theta}_j\right) = \operatorname{dist}(\overline{p}, \widetilde{\gamma}_j) = \operatorname{dist}(p, \Gamma_j). \end{gathered} \tag{2.19}$$

Now let us define $\tilde{\gamma} = \tilde{\gamma}_1 \cup \cdots \cup \tilde{\gamma}_N$. Because of doubling process, we have

$$\begin{aligned} M_{\overline{p}}(\overline{p} \divideontimes \tilde{\gamma}) &= 2M_p(p \divideontimes \Gamma), \\ \mathrm{Length}(\tilde{\gamma}) &= 2\,\mathrm{Length}(\Gamma), \\ A^1(\overline{p} \divideontimes \tilde{\gamma}) &= 2A^1(\Gamma, p). \end{aligned} \tag{2.20}$$

By Proposition 2.8, $A^1(\overline{p} \divideontimes \tilde{\gamma}) \geq 2\pi$ and then $\tilde{\gamma}$ has a self-intersection point. By the geometric meaning of the modified volume, $M_{\overline{p}}(\overline{p} \divideontimes \tilde{\gamma})$ equals the Euclidean area of the standard projection of $\overline{p} \divideontimes \tilde{\gamma}$ onto the plane containing the equator of $\mathbb{S}^2$. Let $\tilde{\gamma}'$ be the image of $\tilde{\gamma}$ under the projection and 0 the origin of the plane. Then $M_{\overline{p}}(\overline{p} \divideontimes \tilde{\gamma}) = \mathrm{Area}(0 \divideontimes \tilde{\gamma}')$, $A^1(\tilde{\gamma}, \overline{p}) = A^1(\tilde{\gamma}', 0)$, but $\mathrm{Length}(\tilde{\gamma}) \geq \mathrm{Length}(\tilde{\gamma}')$. The last inequality arises from the fact that the projection is a length-shrinking map. Moreover, let q_1' and q_2' be the endpoints of $\tilde{\gamma}'$, then $\mathrm{dist}(0, q_1') = \mathrm{dist}(0, q_2')$. So we can apply [7, Lemma 1] and conclude the following sharp isoperimetric inequality:

$$4\pi\,\mathrm{Area}(0 \divideontimes \tilde{\gamma}') \leq \mathrm{Length}(\tilde{\gamma}')^2, \tag{2.21}$$

and equality holds if and only if $\tilde{\gamma}'$ is the boundary of a circle. So we have

$$2\pi M_p(p \divideontimes \Gamma) \leq \mathrm{Length}(\Gamma)^2. \tag{2.22}$$

By Proposition 2.6, we finally get

$$2\pi M_p(\Sigma) \leq \mathrm{Length}(\Gamma)^2, \tag{2.23}$$

and equality holds if and only if $\mathrm{Length}(\tilde{\gamma}') = 2\,\mathrm{Length}(\Gamma)$ and $A^1(\Gamma, p) = \pi$, or equivalently, Σ is a totally geodesic half-disk centered at p. □

Proposition 2.9. *Let C be a closed convex set in $\mathbb{S}^n$ ($\mathbb{H}^n$, resp.) and let Σ be an m-dimensional compact minimal submanifold of $\mathbb{S}^n$ ($\mathbb{H}^n$, resp.) satisfying that Σ is orthogonal to ∂C along $\partial\Sigma \cap \partial C$. Define $r(x) = \mathrm{dist}(p, x)$ for $p \in \partial\Sigma \cap \partial C$. Then $\sin^{-m} r \cdot M_p(\Sigma \cap B^n(p, r))$ ($\sinh^{-m} r \cdot M_p(\Sigma \cap B^n(p, r))$, resp.) is a monotonically nondecreasing function of r for $0 < r < \min(\pi/2, \mathrm{dist}(p, \partial\Sigma))$ ($0 < r < \mathrm{dist}(p, \partial\Sigma)$, resp.), where $B^n(p, r)$ is the n-dimensional geodesic ball of radius r centered p in $\mathbb{S}^n$ ($\mathbb{H}^n$, resp.).*

Proof. Let $\Sigma_r = B^n(p, r) \subset \mathbb{S}^n$. C is a convex set, $r \leq \pi/2$, and the same argument holds as in Proposition 2.6; we have

$$\begin{aligned} M_p(\Sigma_r) &= -\frac{1}{m}\int_{\Sigma_r} \Delta \cos r = \frac{1}{m}\int_{\partial\Sigma_r} \sin r \frac{\partial r}{\partial \nu} \\ &\leq \frac{1}{m}\int_{\partial\Sigma_r - \partial C} \sin r \frac{\partial r}{\partial \nu} \leq \frac{1}{m} \sin r \int_{\partial\Sigma_r - \partial C} |\nabla r|. \end{aligned} \tag{2.24}$$

Denote the volume forms on Σ and $\partial\Sigma_t$ by dv and $d\Sigma_t$, respectively. Then we have

$$dv = \frac{1}{|\nabla r|} d\Sigma_r dr. \tag{2.25}$$

It follows that

$$\frac{d}{dr}\int_{\Sigma_r} \cos r |\nabla r|^2 dv = \frac{d}{dr}\int_0^r \int_{\partial\Sigma_r - \partial C} \cos r |\nabla r| d\Sigma_r \, dr = \cos r \int_{\partial\Sigma_r - \partial C} |\nabla r|. \tag{2.26}$$

Hence we have

$$\begin{aligned} M_p(\Sigma_r) &\le \frac{1}{m}\frac{\sin r}{\cos r}\cos r \int_{\partial\Sigma_r - \partial C} |\nabla r| = \frac{1}{m}\frac{\sin r}{\cos r}\frac{d}{dr}\int_0^r \int_{\partial\Sigma_r - \partial C} \cos r |\nabla r|^2 \\ &\le \frac{1}{m}\frac{\sin r}{\cos r}\frac{d}{dr}\int_0^r \int_{\partial\Sigma_r - \partial C} \cos r = \frac{1}{m}\frac{\sin r}{\cos r}\frac{d}{dr} M_p(\Sigma_r). \end{aligned} \tag{2.27}$$

In the above inequality we used the fact that $r \le \pi/2$ and $|\nabla r| \le 1$ on Σ. Therefore we obtain

$$\frac{d}{dr}\log\left[\sin^{-m} r \cdot M_p(\Sigma_r)\right] \ge 0. \tag{2.28}$$

This completes the proof for $\Sigma \subset \mathbb{S}^n$.

The similar argument applies to $\Sigma \subset \mathbb{H}^n$. □

Remark 2.10. The classical monotonicity of a minimal submanifold in the Euclidean or hyperbolic space can be found in [6, 8, 9].

In case $\partial\Sigma - \partial C$ lies on a geodesic sphere, Σ automatically satisfies the radially connectivity. In this case, the relative isoperimetric inequality can be extended to the hyperbolic space, and to a minimal submanifold case (not necessarily a minimal surface case). More precisely, we have the following theorem.

Theorem 2.11. *Let C be a closed convex set in $\mathbb{S}^n$ or $\mathbb{H}^n$ and let Σ be an m-dimensional compact minimal submanifold of $\mathbb{S}^n$ or $\mathbb{H}^n$ satisfying that Σ is orthogonal to ∂C along $\partial\Sigma \cap \partial C$. Assume that $\partial\Sigma - \partial C$ lies on a geodesic sphere centered at a point $p \in \partial\Sigma \cap \partial C$ and that r is the distance in $\mathbb{S}^n$ or $\mathbb{H}^n$ from $p \in \partial\Sigma \cap \partial C$. Furthermore, in case of $\Sigma \subset \mathbb{S}^n$, assume that $r \le \pi/2$. Then*

$$\frac{1}{2} m^m \omega_m M_p(\Sigma)^{m-1} \le \mathrm{Volume}(\partial\Sigma - \partial C)^m, \tag{2.29}$$

where equality holds if and only if Σ is an m-dimensional totally geodesic half-ball centered at p.

Proof. Assume that Σ is a minimal submanifold in $\mathbb{H}^n$ with $\partial\Sigma - \partial C$ lying on a geodesic sphere. Let R be the radius of the geodesic sphere. Since C is a convex set and the same argument holds as in Proposition 2.6, then

$$M_p(\Sigma) = \frac{1}{m}\int_{\Sigma} \Delta \cosh r \le \frac{1}{m}\int_{\partial\Sigma-\partial C} \sinh r \cdot \frac{\partial r}{\partial \nu} = \frac{1}{m} \sinh R \cdot \mathrm{Volume}(\partial\Sigma - \partial C). \tag{2.30}$$

By Proposition 2.9, $\lim_{r\to 0}\sinh^{-m} r \cdot M_p(\Sigma_r) = \omega_m/2$. Hence

$$M_p(\Sigma) \le \frac{1}{m}\left(\frac{\omega_m}{2}\right)^{-1/m} M_p(\Sigma)^{1/m}\mathrm{Volume}(\partial\Sigma - \partial C) \tag{2.31}$$

and so the desired inequality follows. Equality holds if and only if Σ is a cone with density at the center equal to $1/2$, or equivalently, $\partial\Sigma \cap \partial C$ is a totally geodesic half-ball in $\mathbb{H}^n$. A similar proof holds for $\Sigma \subset \mathbb{S}^n$. □

3. Weak Relative Isoperimetric Inequalities in a Riemannian Manifold

The results in Section 2 are sharp but those require some extra assumptions on their boundary. And the results are concerned with the modified volume. In this section, by contrast, we obtain weak relative isoperimetric inequality which holds for any minimal surface and holds for the usual volume.

From now on, we denote Area(Σ) as A and Length$(\partial\Sigma - \partial C)$ as L for simplicity.

Theorem 3.1. *Let C be a closed convex set in a complete simply connected Riemannian manifold $W(K)$ of sectional curvature bounded above by a constant K. Assume that Σ^2 is a compact minimal surface in the outside of C such that Σ^2 is orthogonal to ∂C along $\partial\Sigma \cap \partial C$. In case of $K > 0$, one assumes that* $\mathrm{diam}(\Sigma) \le \pi/2\sqrt{K}$. *Then one has*

$$\pi A \le L^2 + KA^2. \tag{3.1}$$

To prove the above theorem, we begin with the following lemmas on the Laplacian on functions of distance.

Lemma 3.2. *Let Σ be an m-dimensional compact minimal submanifold in a simply connected Riemannian manifold $W(K)$ of sectional curvature bounded above by a constant K. Define $r(x) =$* $\mathrm{dist}(p, x)$ *for a fixed point $p \in W(K)$.*

(a) *If $K = 0$, then one has $\Delta r \ge (1/r)(m - |\nabla r|^2)$ on Σ.*

(b) *If $K = -k^2 < 0$, then one has $\Delta r \ge k(m - |\nabla r|^2) \coth kr$ on Σ.*

(c) *If $K = k^2 > 0$, then one has $\Delta r \ge k(m - |\nabla r|^2)\cot kr$ on Σ.*

Proof. See [6, 10] for the proof. □

Lemma 3.3. *Let Σ^2 be a compact minimal surface satisfying the same assumptions as in Theorem 3.1. Define $r(x) = \mathrm{dist}(p, x)$ for any $p \in \partial\Sigma \cap \partial C$.*

(a) *If $K = 0$, then one has on Σ*

(1) $\Delta \log r \geq \pi\delta_p$.

(b) *If $K = -k^2 < 0$, then one has on Σ*

(2) $\Delta \log(\sinh kr/(1 + \cosh kr)) \geq \pi\delta_p$,

(3) $\Delta \log \sinh kr \geq \pi\delta_p - K$.

(c) *If $K = k^2 > 0$, then one has on Σ*

(4) $\Delta \log(\sin kr/(1 + \cos kr)) \geq \pi\delta_p$ *if* $r \leq \pi/2k$,

(5) $\Delta \log \sin kr \geq \pi\delta_p - K$ *if* $r \leq \pi/2k$.

Proof. For $K = k^2$, using Lemma 3.2(c), we have

$$\begin{aligned}\Delta \log \frac{\sin kr}{1 + \cos kr} &= \operatorname{div} \frac{k}{\sin kr}\nabla r = -\frac{k^2 \cos kr}{\sin^2 kr}|\nabla r|^2 + \frac{k}{\sin kr}\Delta r \\ &\geq \frac{2k^2 \cos kr}{\sin^2 kr}\left(1 - |\nabla r|^2\right) \geq 0.\end{aligned} \tag{3.2}$$

Near p, Σ can be identified with totally geodesic half-sphere with constant sectional curvature K. Since $f(r) = \log(\sin kr/(1 + \cos kr))$ is a fundamental solution of Laplacian on $\mathbb{S}^2(K)$, $\Delta f(r) = 2\pi\delta_p$. Hence we obtain (4).

Next for (5) we compute

$$\begin{aligned}\Delta \log \sin kr &= \operatorname{div}\left(\frac{k \cos kr}{\sin kr}\nabla r\right) = -k^2 \csc^2 kr |\nabla r|^2 + k \cot kr \Delta r \\ &\geq k^2 \csc^2 kr\left[2\cos^2 kr - \left(1 + \cos^2 kr\right)|\nabla r|^2\right] \geq -k^2.\end{aligned} \tag{3.3}$$

Note that

$$\lim_{r \to 0} \frac{(d/dr) \log \sin kr}{(d/dr) \log(\sin kr/(1 + \cos kr))} = 1, \tag{3.4}$$

which proves (5) by (4).

For $K = 0$, using Lemma 3.2(a) and $f(r) = \log r$ being a fundamental solution of Laplacian on $\mathbb{R}^2$, we get (1).

For $K = -k^2$, using Lemma 3.2(b) and $f(r) = \log(\sinh kr/(1 + \cosh kr))$ being a fundamental solution of Laplacian on $\mathbb{H}^2(K)$, we get (2). By similar argument as before, we get (3). □

Proof of Theorem 3.1. (i) $K = -k^2 < 0$. Let $r(x)$ be the distance from $p \in \partial\Sigma \cap \partial C$ to $x \in \Sigma$. Integrating Lemma 3.3(3) over Σ for the point $p \in \Sigma \cap \partial C$, we get

$$\pi - KA \le \int_{\Sigma} \Delta \log \sinh kr \le \int_{\partial\Sigma - \partial C} k \coth kr \cdot \frac{\partial r}{\partial \nu} + \int_{\partial\Sigma \cap \partial C} k \coth kr \cdot \frac{\partial r}{\partial \nu}. \tag{3.5}$$

Since the same argument holds as in Proposition 2.6, we have

$$\pi - KA \le \int_{\partial\Sigma - \partial C} k \coth kr. \tag{3.6}$$

This inequality holds for all $x \in \Sigma$, and we can integrate it over Σ and apply Fubini's theorem to obtain

$$\pi A - KA^2 \le \int_{\Sigma}\int_{\partial\Sigma - \partial C} k \coth kr = \int_{\partial\Sigma - \partial C}\int_{\Sigma} k \coth kr. \tag{3.7}$$

By Lemma 3.2(b) and convexity of C, we get

$$\begin{aligned} \pi A - KA^2 &\le \int_{\partial\Sigma - \partial C}\int_{\Sigma} \Delta r = \int_{\partial\Sigma - \partial C}\int_{\partial\Sigma - \partial C} \frac{\partial r}{\partial \nu} + \int_{\partial\Sigma - \partial C}\int_{\partial\Sigma \cap \partial C} \frac{\partial r}{\partial \nu} \\ &\le \int_{\partial\Sigma - \partial C}\int_{\partial\Sigma - \partial C} \frac{\partial r}{\partial \nu} \le \text{Length}(\partial\Sigma - \partial C)^2. \end{aligned} \tag{3.8}$$

(ii) $K = 0$. Integrate Lemma 3.3(1) twice and apply Lemma 3.2(a) as in (i).
(iii) $K = k^2$. Integrate Lemma 3.3(5) twice and apply Lemma 3.2(c) as in (i).
This completes the proof. □

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