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*Research Article*

# On Certain Multivalent Starlike or Convex Functions with Negative Coefficients

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By means of a differential operator, we introduce and investigate some new subclasses of $p$-valently analytic functions with negative coefficients, which are starlike or convex of complex order. Relevant connections of the definitions and results presented in this paper with those obtained in several earlier works on the subject are also pointed out.

## 1. Introduction

Let $\mathcal{A}_m(p)$ denote the class of functions of the following form:

$$f(z) = z^p - \sum_{k=p+m}^{\infty} a_k z^k \quad (a_k \geq 0;\ m, p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are analytic and multivalent in the open unit dics $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

Let $f^{(q)}$ denote the $q$th-order ordinary differential operator for a function $f \in \mathcal{A}_m(p)$, that is,

$$f^{(q)}(z) = \frac{p!}{(p-q)!} z^{p-q} - \sum_{k=p+m}^{\infty} \frac{k!}{(k-q)!} a_k z^{k-q}, \tag{1.2}$$

where $p > q$; $p \in \mathbb{N}$; $q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $z \in \mathcal{U}$.

Next, we define the differential operator $D^n f^{(q)}$ as

$$D^n f^{(q)}(z) = \frac{p!}{(p-q)!} z^{p-q} - \sum_{k=p+m}^{\infty} \frac{k!}{(k-q)!} \left(\frac{k-q}{p-q}\right)^n a_k z^{k-q} \quad (m \in \mathbb{N};\ z \in \mathcal{U}). \tag{1.3}$$

In view of (1.3), it is clear that

$$\begin{gathered} D^0 f^{(q)}(z) = f^{(q)}(z), \qquad D^1 f^{(q)}(z) = D f^{(q)}(z) = \frac{1}{p-q} z \left( f^{(q)}(z) \right)', \\ D^n f^{(q)}(z) = D^{n-1}\left( D f^{(q)}(z) \right). \end{gathered} \tag{1.4}$$

If we take $p = 1$ and $q = 0$ for $D^n f^{(q)}$, then $D^n f^{(q)}$ become the differential operator defined by Sălăgean [1].

Finally, in terms of a differential operator $D^n f^{(q)}$ defined by (1.3) above, let $E^n_{m,p}(q)$ denote the subclass of $\mathcal{A}_m(p)$ consisting of functions $f$ which satisfy the following inequality:

$$E^n_{m,p}(q) = \left\{ f \in \mathcal{A}_m(p) : \frac{D^n f^{(q)}(z)}{z^{p-q}} \neq 0,\ (z \in \mathbb{C} - \{0\}),\ f(z) = z^p - \sum_{k=p+m}^{\infty} a_k z^k,\ a_k \geq 0 \right\}, \tag{1.5}$$

where $k \in \mathbb{N}$, $n \in \mathbb{N}_0$, $k > n$, $m \in \mathbb{N}$; $(k-q)/(p-q) \geq p - q - n - 1 \geq 0$; $z \in \mathcal{U}$.

For $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, and $\gamma \in \mathbb{C} - \{0\}$, we define the next subclasses of $E^n_{m,p}(q)$.

$$\begin{aligned} E^n_{m,p}(q,\gamma) &= \left\{ f \in E^n_{m,p}(q) : \operatorname{Re}\left\{ 1 + \frac{1}{\gamma} \left( \frac{D^{n+1} f^{(q)}(z)}{D^n f^{(q)}(z)} - p + q + n \right) \right\} > 0,\ (z \in \mathcal{U}) \right\}, \\ N^n_{m,p}(q,\gamma) &= \left\{ f \in E^n_{m,p}(q) : \sum_{k=p+m}^{\infty} \frac{k!}{(k-q)!} \left(\frac{k-q}{p-q}\right)^n \left[ \left( \frac{k-q}{p-q} - p + q + n \right) \frac{\operatorname{Re}\gamma}{|\gamma|} + |\gamma| \right] a_k \right. \\ &\qquad \left. \leq \frac{p!}{(p-q)!} \left( |\gamma| + (-p+q+n+1) \frac{\operatorname{Re}\gamma}{|\gamma|} \right) \right\}, \\ K^n_{m,p}(q,\gamma) &= \left\{ f \in E^n_{m,p}(q) : \sum_{k=p+m}^{\infty} \frac{k!}{(k-q)!} \left(\frac{k-q}{p-q}\right)^n \left[ \frac{k-q}{p-q} - p + q + n + |\gamma| \right] a_k \right. \\ &\qquad \left. \leq \frac{p!}{(p-q)!} \left( |\gamma| - p + q + n + 1 \right) \right\}, \end{aligned} \tag{1.6}$$

where $\gamma \in \mathbb{C} - \{0\}$; $m \in \mathbb{N}$; $(k-q)/(p-q) \geq p - q - n - 1 \geq 0$; $z \in \mathcal{U}$.

*Remark 1.1.* (1) $E^0_{m,1}(0,\gamma) = S^*(b)$ was studied by Nasr and Aouf [2] (also see Bulboacă et al. [3]).

(2) $E^0_{m,1}(0,1-\alpha) = T_\alpha(m)$ and $E^1_{m,1}(0,1-\alpha) = C_\alpha(m)$, $\alpha \in [0,1)$ were introduced by Srivastava et al. [4].

(3) $E^0_{1,1}(0,1-\alpha) = T^*(\alpha)$ and $E^1_{1,1}(0,1-\alpha) = C(\alpha)$, $\alpha \in [0,1)$ were introduced by Silverman [5].

(4) $K^0_{m,1}(0,\gamma) = O^0_m(\gamma)$ and $K^1_{m,1}(0,\gamma) = O^1_m(\gamma)$ were introduced by Parvathan and Ponnusanny [6, pages 163-164].

(5) For $p = 1$ and $q = 0$, the classes $E^n_{m,p}(q,\gamma)$, $N^n_{m,p}(q,\gamma)$, and $K^n_{m,p}(q,\gamma)$ are closely related with $T_{n,m}(\gamma)$, $O_{n,m}(\gamma)$, and $P_{n,m}(\gamma)$ which are defined by Owa and Sălăgean in [7].

In this paper we give relationships between the classes of $E^n_{m,p}(q,\gamma)$, $N^n_{m,p}(q,\gamma)$, and $K^n_{m,p}(q,\gamma)$. In the particular case when $m \in \mathbb{N}$ and $n = 0$, $p = 1$, and $q = 0$, we obtain the same results as in [8].

## 2. Main Results

Our main results are contained in

**Theorem 2.1.** *Let $m \in \mathbb{N}$, $n \in \mathbb{N}_0$ and let $\gamma \in \mathbb{C} - \{0\}$; then*

(1) $K^n_{m,p}(q,\gamma) \subseteq E^n_{m,p}(q,\gamma)$;

(2) $E^n_{m,p}(q,\gamma) \subseteq N^n_{m,p}(q,\gamma)$;

(3) *if* $\gamma \in (0,\infty)$, *then*

$$K^n_{m,p}(q,\gamma) = E^n_{m,p}(q,\gamma) = N^n_{m,p}(q,\gamma); \tag{2.1}$$

(4) *if* $\gamma \in (-\infty,0)$, *then* $N^n_{m,p}(q,\gamma) \not\subseteq E^n_{m,p}(q,\gamma)$;

(5) *if* $\gamma \in (-\infty,0)$, *then* $E^n_{m,p}(q,\gamma) \not\subseteq K^n_{m,p}(q,\gamma)$.

*Proof.* (1) Let $f \in K^n_{m,p}(q,\gamma)$. We prove that

$$\left| \frac{D^{n+1} f^{(q)}(z)}{D^n f^{(q)}(z)} - p + q + n \right| < |\gamma|, \quad (z \in \mathcal{U}). \tag{2.2}$$

If $f$ has the series expansion

$$f(z) = z^p - \sum_{k=p+m}^{\infty} a_k z^k, \quad a_k \geq 0, \tag{2.3}$$

then

$$\left|\frac{D^{n+1}f^{(q)}(z)}{D^n f^{(q)}(z)}-p+q+n\right|-|\gamma|$$
$$\leq \frac{-(p!/(p-q)!)(1-p+q+n)}{p!/(p-q)!-\sum_{k=p+m}^{\infty}(k!/(k-q)!)((k-q)/(p-q))^n a_k|z|^{k-p}}$$
$$+\frac{\sum_{k=p+m}^{\infty}(k!/(k-q)!)((k-q)/(p-q))^n a_k((k-q)/(p-q)-p+q+n)|z|^{k-p}}{p!/(p-q)!-\sum_{k=p+m}^{\infty}(k!/(k-q)!)((k-q)/(p-q))^n a_k|z|^{k-p}}-|\gamma|. \tag{2.4}$$

We use the fact that $D^n f^{(q)}(z)/z^{p-q}\neq 0$ for $z\in\mathcal{U}-\{0\}$ and $\lim_{z\to o}[D^n f^{(q)}(z)/z^{p-q}]=p!/(p-q)!$; these imply

$$\frac{p!}{(p-q)!}-\sum_{k=p+m}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k-q}{p-q}\right)^n a_k|z|^{k-p}>0 \tag{2.5}$$

for $z\in\mathcal{U}$.

From (2.4) and (2.5), we deduce

$$\left|\frac{D^{n+1}f^{(q)}(z)}{D^n f^{(q)}(z)}-p+q+n\right|-|\gamma|$$
$$<\frac{-(p!/(p-q)!)(1-p+q+n+|\gamma|)}{p!/(p-q)!-\sum_{k=p+m}^{\infty}k!/(k-q)!((k-q)/(p-q))^n a_k}.$$
$$+\frac{\sum_{k=p+m}^{\infty}(k!/(k-q)!)((k-q)/(p-q))^n a_k[((k-q)/(p-q)-p+q+n)+|\gamma|]}{p!/(p-q)!-\sum_{k=p+m}^{\infty}(k!/(k-q)!)((k-q)/(p-q))^n a_k}. \tag{2.6}$$

By using the definition of $K_{m,p}^n(q,\gamma)$ from this last inequality we, obtain (2.2) which implies

$$\operatorname{Re}\left\{\frac{1}{\gamma}\left(\frac{D^{n+1}f^{(q)}(z)}{D^n f^{(q)}(z)}-p+q+n\right)\right\}>-1 \quad (z\in\mathcal{U}), \tag{2.7}$$

hence $f\in E_{m,p}^n(q,\gamma)$.

(2) Let $f$ be in $E^{n}_{m,p}(q,\gamma)$. Then (2.7) holds and, by using (2.3), this is equivalent to

$$\operatorname{Re}\left\{\frac{1}{\gamma}\left(\frac{(p!/(p-q)!)z^{p-q}-\sum_{k=p+m}^{\infty}(k!/(k-q)!)((k-q)/(p-q))^{n+1}a_k z^{k-q}}{(p!/(p-q)!)z^{p-q}-\sum_{k=p+m}^{\infty}(k!/(k-q)!)((k-q)/(p-q))^{n}a_k z^{k-q}}-p+q+n\right)\right\} > -1 \quad (z\in\mathcal{U}). \tag{2.8}$$

For $z=t\in[0,1)$ if $t\to 1^{-}$, from (2.8) we obtain

$$\left\{\left(\frac{-(p!/(p-q)!)+\sum_{k=p+m}^{\infty}(k!/(k-q)!)((k-q)/(p-q))^{n+1}a_k}{p!/(p-q)!-\sum_{k=p+m}^{\infty}(k!/(k-q)!)((k-q)/(p-q))^{n}a_k}\right)\right\}\leq\frac{|\gamma|^2}{\operatorname{Re}\gamma}-p+q+n \tag{2.9}$$

which is equivalent to

$$\sum_{k=p+m}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k-q}{p-q}\right)^{n}\left[\frac{k-q}{p-q}+\frac{|\gamma|^2}{\operatorname{Re}\gamma}-p+q+n\right]a_k\leq\frac{p!}{(p-q)!}\left(\frac{|\gamma|^2}{\operatorname{Re}\gamma}-p+q+n+1\right). \tag{2.10}$$

Then multiplying the relation last inequality with $\operatorname{Re}\gamma/|\gamma|$, we obtain $f\in N^{n}_{m,p}(q,\gamma)$.

(3) if $\gamma$ is a real positive number, then the definitions of $N^{n}_{m,p}(q,\gamma)$ and $K^{n}_{m,p}(q,\gamma)$ are equivalent, hence $N^{n}_{m,p}(q,\gamma)=K^{n}_{m,p}(q,\gamma)$. By using (1) and (2) from this theorem, we obtain (3).

(4) We have the following two cases.

*Case 1.* $\gamma\in[p-q-n-1-m/(p-q),0)$.

Let $f_{m,\alpha}(p,q,n;z)$ be defined by

$$f_{m,\alpha}(p,q,n;z)=z^{p}-\alpha\left(\frac{p+m-q}{p-q}\right)^{-n}\frac{p!}{(p+m)!}\frac{(p+m-q)!}{(p-q)!}z^{p+m} \tag{2.11}$$

and let $\alpha>0$. We have

$$\begin{aligned}
&\sum_{k=p+m}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k-q}{p-q}\right)^{n}\left[\left(\frac{k-q}{p-q}-p+q+n\right)\frac{\operatorname{Re}\gamma}{|\gamma|}+|\gamma|\right]a_k\\
&\quad\leq\frac{(p+m)!}{(p+m-q)!}\left(\frac{p+m-q}{p-q}\right)^{n}\left\{\left(\frac{p+m-q}{p-q}-p+q+n\right)\frac{\operatorname{Re}\gamma}{|\gamma|}+|\gamma|\right\}\\
&\qquad\times\alpha\left(\frac{p+m-q}{p-q}\right)^{-n}\frac{p!}{(p+m)!}\frac{(p+m-q)!}{(p-q)!}\\
&\quad=\alpha\frac{p!}{(p-q)!}\left\{\left(\frac{p+m-q}{p-q}-p+q+n\right)\frac{\gamma}{-\gamma}-\gamma\right\}
\end{aligned} \tag{2.12}$$

or

$$\sum_{k=p+m}^{\infty} \frac{k!}{(k-q)!}\left(\frac{k-q}{p-q}\right)^{n}\left\{\left(\frac{k-q}{p-q}-p+q+n\right)\frac{\operatorname{Re}\gamma}{|\gamma|}+|\gamma|\right\}a_k$$
$$\leq -\alpha\frac{p!}{(p-q)!}\left(-p+q+n+1+\frac{m}{p-q}+\gamma\right)\leq 0 \tag{2.13}$$
$$< \frac{p!}{(p-q)!}\left((-p+q+n+1)\frac{\operatorname{Re}\gamma}{|\gamma|}+|\gamma|\right),$$

and then $f_{m,\alpha}(p,q,n;z)\in N_{m,p}^{n}(q,\gamma)$ (see the definition of $N_{m,p}^{n}(q,\gamma)$).

Let now

$$F(z)=1+\frac{1}{\gamma}\left(\frac{D^{n+1}f_{m,\alpha}^{(q)}(p,q,n;z)}{D^{n}f_{m,\alpha}^{(q)}(p,q,n;z)}-p+q+n\right)\quad(z\in\mathcal{U}). \tag{2.14}$$

Then, by a simple computation and by using the fact that

$$\begin{aligned}
f_{m,\alpha}^{(q)}(p,q,n;z) &= f_{m,\alpha}^{(q)}(z)\\
&= \frac{p!}{(p-q)!}z^{p-q}-\alpha\frac{(p+m)!}{(p+m-q)!}\left(\frac{p+m-q}{p-q}\right)^{-n}\frac{p!}{(p+m)!}\frac{(p+m-q)!}{(p-q)!}z^{p+m-q}\\
&= \frac{p!}{(p-q)!}z^{p-q}-\alpha\frac{p!}{(p-q)!}\left(\frac{p+m-q}{p-q}\right)^{-n}z^{p+m-q}.\\
D^{n}f_{m,\alpha}^{(q)}(z) &= \frac{p!}{(p-q)!}z^{p-q}-\alpha\frac{p!}{(p-q)!}\left(\frac{p+m-q}{p-q}\right)^{-n}\left(\frac{p+m-q}{p-q}\right)^{n}z^{p+m-q}\\
&= \frac{p!}{(p-q)!}z^{p-q}(1-\alpha z^{m}),\\
D^{n+1}f_{m,\alpha}^{(q)}(z) &= \frac{p!}{(p-q)!}z^{p-q}\left(1-\alpha\frac{p+m-q}{p-q}z^{m}\right),
\end{aligned} \tag{2.15}$$

we obtain

$$\begin{aligned} F(z) &= 1+\frac{1}{\gamma}\left(\frac{D^{n+1}f_{m,\alpha}^{(q)}(z)}{D^n f_{m,\alpha}^{(q)}(z)}-p+q+n\right) \\ &= 1+\frac{1}{\gamma}\left(\frac{(p!/(p-q)!)z^{p-q}(1-\alpha((p+m-q)/(p-q))z^m)}{(p!/(p-q)!)z^{p-q}(1-\alpha z^m)}-p+q+n\right) \\ &= 1+\frac{-p+q+n+1-\alpha z^m(-p+q+n+(p+m-q)/(p-q))}{\gamma(1-\alpha z^m)} \\ &= 1+\frac{(a-\alpha b\zeta)}{\gamma(1-\alpha\zeta)} = 1+\varphi(\zeta), \end{aligned} \tag{2.16}$$

where $\zeta = z^m$, $a = -p+q+n+1$, $b = -p+q+n+1+m/(p-q)$, and

$$\varphi(\zeta) = \frac{a-\alpha b\zeta}{\gamma(1-\alpha\zeta)} \ . \tag{2.17}$$

For $\alpha > 1$ we, have $\varphi(U) = \mathbb{C}_\infty - D(c,d)$, where $D$ is the disc with the center

$$c = \frac{\alpha^2 b - a}{\gamma(\alpha^2-1)} \tag{2.18}$$

and the radius

$$d = \frac{\alpha(b-a)}{\gamma(1-\alpha^2)}. \tag{2.19}$$

We have $F(\mathcal{U}) = \mathbb{C}_\infty - D(c+1,d)$ where $D(c,d) = \{w : |w-c| < d\}$ and we deduce that $\operatorname{Re} F(z) > 0$ for all $z \in \mathcal{U}$ does not hold.

We have obtained that for $\alpha > 1$, $f_{m,\alpha} \in N_{m,p}^n(q,\gamma)$, but $f_{m,\alpha} \notin E_{m,p}^n(q,\gamma)$ and in this case $N_{m,p}^n(q,\gamma) \not\subseteq E_{m,p}^n(q,\gamma)$.

*Case 2.* $\gamma \in (-\infty, p-q-n-1-m/(p-q))$.

We consider the function $f_{m,\alpha}$ defined by (2.11) for $\alpha \in (1, (-p+q+n+1+\gamma)/(-p+q+n+1+m/(p-q)))$. In this case, the inequality (2.13) holds too and this implies that $f_{m,\alpha} \in N_{m,p}^n(q,\gamma)$.

We also obtain that $f \notin E_{m,p}^n(q,\gamma)$ like in Case 1.

(5) Let $f = f_{m,\alpha}$ be given by (2.11), where $\alpha > (|\gamma| - p + q + n + 1)/(|\gamma| - p + q + n + 1 + m/(p-q))$ and $|\gamma| - p + q + n + 1 + m/(p-q) > 0$. Then

$$\begin{aligned}
&\sum_{k=p+m}^{\infty} \frac{k!}{(k-q)!}\left(\frac{k-q}{p-q}\right)^{n}\left[\frac{k-q}{p-q} - p + q + n + |\gamma|\right] a_k \\
&\quad = \frac{(p+m)!}{(p+m-q)!}\left(\frac{p+m-q}{p-q}\right)^{n}\left[\frac{p+m-q}{p-q} - p + q + n + |\gamma|\right] \\
&\qquad \times \alpha\left(\frac{p+m-q}{p-q}\right)^{-n}\frac{p!}{(p+m)!}\cdot\frac{(p+m-q)!}{(p-q)!} \\
&\quad = \alpha\frac{p!}{(p-q)!}\left[\left(\frac{p+m-q}{p-q} - p + q + n\right) + |\gamma|\right] \\
&\quad > \frac{p!}{(p-q)!}\left[|\gamma| + (-p + q + n + 1)\right]
\end{aligned} \tag{2.20}$$

which implies that

$$f_{m,\alpha} \notin K^{n}_{m,p}(q,\gamma) \quad \text{for } m \in \mathbb{N},\ n \in \mathbb{N}_0,\ \gamma \in (-\infty, 0). \tag{2.21}$$

We have

$$F(z) = 1 + \frac{1}{\gamma}\left(\frac{D^{n+1} f^{(q)}_{m,\alpha}(z)}{D^{n} f^{(q)}_{m,\alpha}(z)} - p + q + n\right) = 1 + \frac{a - \alpha b\zeta}{\gamma(1-\alpha\zeta)} = 1 + \varphi(\zeta), \tag{2.22}$$

where $\varphi$ is given by (2.17).

From $\varphi(\mathcal{U}) = D(c,d)$ where $c$ and $d$ are given by (2.18) and (2.19), we obtain

$$\operatorname{Re} F(z) \geq 1 + \frac{\alpha b + a}{\gamma(\alpha+1)}. \tag{2.23}$$

If $\gamma \in (-\infty, p - q - n - 1 - m/(p-q))$ and $\alpha \in ((|\gamma| - p + q + n + 1)/(|\gamma| - p + q + n + 1 + m/(p-q)), 1)$, then

$$\frac{\alpha(\gamma + b) + \gamma + a}{\gamma(\alpha+1)} > 0, \tag{2.24}$$

and if

$$\gamma \in \left(p-q-n-1-\frac{m}{p-q},0\right),$$
$$\alpha \in \left(\frac{|\gamma|-p+q+n+1}{|\gamma|-p+q+n+1+(m/(p-q))},\frac{|\gamma|-p+q+n+1}{|-p+q+n+1+(m/(p-q))-|\gamma||}\right)\cap(0,1), \tag{2.25}$$

then (2.24) also holds. By combining (2.24) with (2.23) and the definition of $E^n_{m,p}(q,\gamma)$, we obtain that

$$f_{m,\alpha} \in E^n_{m,p}(q,\gamma) \quad \text{for } \alpha \in \left(\frac{|\gamma|-p+q+n+1}{|\gamma|-p+q+n+1+m/(p-q)},\right.$$
$$\left.\frac{|\gamma|-p+q+n+1}{|-p+q+n+1+m/(p-q)-|\gamma||}\right)\cap(0,1),\ \gamma\in(-\infty,0). \tag{2.26}$$

□

## Appendix

In this paper, we discuss the class $E^n_{m,p}(q,\gamma)$ of analytic functions with negative coefficients. Let us consider the functions $f$ given by

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \tag{A.1}$$

which are analytic in $\mathcal{U}$. For such a function $f$, we say that $f \in G^n_{1,p}(q,\gamma)$ if it satisfies

$$\operatorname{Re}\left(1+\frac{1}{\gamma}\left(\frac{D^{n+1}f^{(q)}(z)}{D^n f^{(q)}(z)}-p+q+n\right)\right)>0 \quad (z\in\mathcal{U}) \tag{A.2}$$

for some complex number $\gamma$ with $0<\operatorname{Re}(1/\gamma)<1/(p-q-n-1)$.

If we define the function $F$ for $f \in G^n_{1,p}(q,\gamma)$ by

$$F(z) = \frac{1+(1/\gamma)\left(D^{n+1}f^{(q)}(z)/D^n f^{(q)}(z)-p+q+n\right)-i(1-p+q+n)\operatorname{Im}(1/\gamma)}{1+(1-p+q+n)\operatorname{Re}(1/\gamma)}, \tag{A.3}$$

then we know that $F$ is analytic in $\mathcal{U}$, $F(0) = 1$, and $\operatorname{Re} f(z) > 0$ $(z \in \mathcal{U})$. Thus $F$ is the Carathéodory function. Since the extremal function for the Carathéodory function $F$ is given by

$$F(z) = \frac{1+z}{1-z}, \tag{A.4}$$

we can write

$$\frac{1 + (1/\gamma)\left(D^{n+1} f^{(q)}(z)/D^n f^{(q)}(z) - p + q + n\right) - i(1 - p + q + n)\operatorname{Im}(1/\gamma)}{1 + (1 - p + q + n)\operatorname{Re}(1/\gamma)} = \frac{1+z}{1-z}. \tag{A.5}$$

This shows us that

$$\frac{D^{n+1} f^{(q)}(z)}{D^n f^{(q)}(z)} - p + q + n + \gamma - i\gamma(1 - p + q + n)\operatorname{Im}\left(\frac{1}{\gamma}\right) = \gamma\left(1 + (1 - p + q + n)\operatorname{Re}\left(\frac{1}{\gamma}\right)\right)\frac{1+z}{1-z}. \tag{A.6}$$

Noting that

$$D^{n+1} f^{(q)}(z) = \frac{1}{p-q} z\left(D^n f^{(q)}(z)\right)', \tag{A.7}$$

we see that

$$\begin{aligned} &\frac{1}{p-q}\frac{\left(D^n f^{(q)}(z)\right)'}{D^n f^{(q)}(z)} - \frac{1}{z}\left(p - q - n - \gamma + i\gamma(1 - p + q + n)\operatorname{Im}\left(\frac{1}{\gamma}\right)\right) \\ &\quad = \gamma\left(1 + (1 - p + q + n)\operatorname{Re}\left(\frac{1}{\gamma}\right)\right)\left(\frac{2}{1-z} + \frac{1}{z}\right), \end{aligned} \tag{A.8}$$

that is,

$$\frac{1}{p-q}\frac{\left(D^n f^{(q)}(z)\right)'}{D^n f^{(q)}(z)} - \frac{1}{z} = 2\gamma\left(1 + (1 - p + q + n)\operatorname{Re}\left(\frac{1}{\gamma}\right)\right)\frac{1}{1-z}. \tag{A.9}$$

It follows from the above that

$$\int_0^z \left(\frac{1}{p-q}\frac{\left(D^n f^{(q)}(t)\right)'}{D^n f^{(q)}(t)} - \frac{1}{t}\right)dt = 2\gamma\left(1 + (1 - p + q + n)\operatorname{Re}\left(\frac{1}{\gamma}\right)\right)\int_0^z \frac{1}{1-t}dt. \tag{A.10}$$

Calculating the above integrations, we have that

$$\frac{1}{p-q}\log D^n f^{(q)}(z) - \log z = -2\gamma\left(1 + (1 - p + q + n)\operatorname{Re}\left(\frac{1}{\gamma}\right)\right)\log(1-z). \tag{A.11}$$

Therefore, we obtain that

$$\left(D^n f^{(q)}(z)\right)^{1/(p-q)} = \frac{z}{(1-z)^{2\gamma(1+(1-p+q+n)\operatorname{Re}(1/\gamma))}}, \tag{A.12}$$

that is,

$$D^n f^{(q)}(z) = \left(\frac{z}{(1-z)^{2\gamma(1+(1-p+q+n)\operatorname{Re}(1/\gamma))}}\right)^{p-q}. \tag{A.13}$$

Consequently, the function $f$ defined by the above is the extremal function for the class $G^n_{1,p}(q,\gamma)$. But our class $E^n_{m,p}(q,\gamma)$ is defined with analytic functions $f$ with negative coefficients. Thus we do not know how we can consider the extremal function for this class.

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