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Research Article

Existence of Solutions for Weighted $p(r)$-Laplacian Impulsive Integro-Differential System Periodic-Like Boundary Value Problems

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This paper investigates the existence of solutions for weighted $p(r)$-Laplacian impulsive integro-differential system periodic-like boundary value problems via Leray-Schauder's degree. The sufficient conditions for the existence of solutions are given.

1. Introduction

In this paper, we consider the existence of solutions for the weighted $p(r)$-Laplacian integro-differential system

$$-\left(w(r)\left|u'\right|^{p(r)-2}u'\right)' + f\left(r, u, (w(r))^{1/\left(p(r)-1\right)}u', S(u)\right) = 0, \quad r \in (0,T),\ r \neq r_i, \tag{1.1}$$

where $u : [0,1] \to \mathbb{R}^N$, with the following impulsive boundary value conditions:

$$\lim_{r\to r_i^+} u(r) - \lim_{r\to r_i^-} u(r) = A_i\left(\lim_{r\to r_i^-} u(r), \lim_{r\to r_i^-} (w(r))^{1/\left(p(r)-1\right)} u'(r)\right), \quad i = 1,\ldots,k, \tag{1.2}$$

$$\lim_{r\to r_i^+}(w(r))^{1/(p(r)-1)}u'(r)-\lim_{r\to r_i^-}(w(r))^{1/(p(r)-1)}u'(r)$$
$$=B_i\left(\lim_{r\to r_i^-}u(r),\lim_{r\to r_i^-}(w(r))^{1/(p(r)-1)}u'(r)\right),\quad i=1,\dots,k, \tag{1.3}$$

$$u(0)=u(T),\qquad \lim_{r\to 0^+}w(r)\left|u'\right|^{p(r)-2}u'(r)=\lim_{r\to T^-}w(r)\left|u'\right|^{p(r)-2}u'(r), \tag{1.4}$$

where $p\in C([0,T],\mathbb{R})$ and $p(r)>1$, $-(w(r)|u'|^{p(r)-2}u')'$ is called the weighted $p(r)$-Laplacian; $0<r_1<r_2<\cdots<r_k<T$; $A_i,B_i\in C(\mathbb{R}^N\times\mathbb{R}^N,\mathbb{R}^N)$; $S(u)(t)=\int_0^T\phi(s,t)u(s)ds$, where $0\le\phi(\cdot,\cdot)\in C([0,T]\times[0,T],\mathbb{R})$.

Throughout the paper, $o(1)$ means function which uniformly convergent to 0 (as $n\to+\infty$); for any $v\in\mathbb{R}^N$, v^j will denote the jth component of v; the inner product in $\mathbb{R}^N$ will be denoted by $\langle\cdot,\cdot\rangle$, $|\cdot|$ will denote the absolute value and the Euclidean norm on $\mathbb{R}^N$. Denote $J=[0,T]$, $J'=[0,T]\setminus\{r_0,r_1,\dots,r_k,r_{k+1}\}$, $J_0=[r_0,r_1]$, $J_i=(r_i,r_{i+1}]$, $i=1,\dots,k$, where $r_0=0$, $r_{k+1}=T$. Denote J_i^o the interior of J_i, $i=0,1,\dots,k$. Let $\mathrm{PC}(J,\mathbb{R}^N)=\{x: J\to\mathbb{R}^N\mid x\in C(J_i,\mathbb{R}^N),\ i=0,1,\dots,k$, and $x(r_i^+)$ exists for $i=1,\dots,k\}$; $w\in\mathrm{PC}(J,\mathbb{R})$ satisfies $0<w(r),\forall r\in J'$, and $(w(r))^{-1/(p(r)-1)}\in L^1(0,T)$; $\mathrm{PC}^1(J,\mathbb{R}^N)=\{x\in\mathrm{PC}(J,\mathbb{R}^N)\mid x'\in C(J_i^o,\mathbb{R}^N)$, $\lim_{r\to r_i^+}(w(r))^{1/(p(r)-1)}x'(r)$ and $\lim_{r\to r_{i+1}^-}(w(r))^{1/(p(r)-1)}x'(r)$ exists for $i=0,1,\dots,k\}$. The equivalent $\lim_{r\to 0^+}(w(r))^{1/(p(r)-1)}u'(r)=\lim_{r\to T^-}(w(r))^{1/(p(r)-1)}u'(r)$ means $\lim_{r\to 0^+}(w(r))^{1/(p(r)-1)}u'(r)$ and $\lim_{r\to T^-}(w(r))^{1/(p(r)-1)}u'(r)$ both exist and equal. For any $x=(x^1,\dots,x^N)\in\mathrm{PC}(J,\mathbb{R}^N)$, denote $|x^i|_0=\sup\{|x^i(r)|\mid r\in J'\}$. Obviously, $\mathrm{PC}(J,\mathbb{R}^N)$ is a Banach space with the norm $\|x\|_0=(\sum_{i=1}^k|x^i|_0^2)^{1/2}$, $\mathrm{PC}^1(J,\mathbb{R}^N)$ is a Banach space with the norm $\|x\|_1=\|x\|_0+\|(w(r))^{1/(p(r)-1)}x'\|_0$. In the following, $\mathrm{PC}(J,\mathbb{R}^N)$ and $\mathrm{PC}^1(J,\mathbb{R}^N)$ will be simply denoted by PC and PC^1, respectively. Let $L^1=L^1(J,\mathbb{R}^N)$ with the norm $\|x\|_{L^1}=(\sum_{i=1}^N|x^i|_{L^1}^2)^{1/2}$, $\forall x\in L^1$, where $|x^i|_{L^1}=\int_0^T|x^i(r)|dr$. We denote

$$w(0)\left|u'\right|^{p(0)-2}u'(0)=\lim_{r\to 0^+}w(r)\left|u'\right|^{p(r)-2}u'(r),$$
$$w(T)\left|u'\right|^{p(T)-2}u'(T)=\lim_{r\to T^-}w(r)\left|u'\right|^{p(r)-2}u'(r). \tag{1.5}$$

The study of differential equations and variational problems with nonstandard $p(r)$-growth conditions is a new and interesting topic. It arises from nonlinear elasticity theory, electrorheological fluids, image processing, and so forth. [1–3]. Many results have been obtained on this kind of problems, for example [1–20]. If $p(r)\equiv p$ (a constant), (1.1) is the well-known p-Laplacian problem. But if $p(r)$ is a general function, the $-\Delta_{p(r)}$ is more complicated than $-\Delta_p$, since it represents a nonhomogeneity and possesses more nonlinearity, many methods and results for p-Laplacian problems are invalid for $p(x)$-Laplacian problems; for example, if $\Omega\subset\mathbb{R}^N$ is a bounded domain, the Rayleigh quotient

$$\lambda_{p(x)}=\inf_{u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}}\frac{\int_\Omega(1/p(x))|\nabla u|^{p(x)}dx}{\int_\Omega(1/p(x))|u|^{p(x)}dx} \tag{1.6}$$

is zero in general, and only under some special conditions $\lambda_{p(x)} > 0$ (see [9]), but the fact that $\lambda_p > 0$ is very important in the study of p-Laplacian problems.

Impulsive differential equations have been studied extensively in the recent years. Such equations arise in many applications such as spacecraft control, impact mechanics, chemical engineering and inspection process in operations research (see [21–23] and the references therein). On the Laplacian impulsive differential equations boundary value problems, there are many papers (see [24–27]). The methods includ subsuper-solution method, fixed point theorem, monotone iterative method, and coincidence degree, and so forth. Because of the nonlinearity, results on the existence of solutions for p-Laplacian impulsive differential equation boundary value problems are rare (see [23, 28–30]).

In [29], Tian and Ge have studied nonlinear IBVP

$$
\begin{gathered}
-\left(\rho(t)\Phi_p\left(x'(t)\right)\right)' + s(t)\Phi_p(x(t)) = f(t,x(t)), \quad t \neq t_i, \text{ a.e. } t \in [a,b], \\
\lim_{t\to t_i^+}\rho(t)\Phi_p\left(x'(t)\right) - \lim_{t\to t_i^-}\rho(t)\Phi_p\left(x'(t)\right) = I_i(x(t_i)), \quad i = 1,\ldots,l, \\
\alpha x'(a) - \beta x(a) = \sigma_1, \qquad \gamma x'(b) + \sigma x(b) = \sigma_2,
\end{gathered}
\tag{1.7}
$$

where $\Phi_p(x) = |x|^{p-2}x$, $p > 1$, $\rho, s \in L^\infty[a,b]$ with $\operatorname{essinf}_{[a,b]}\rho > 0$, and $\operatorname{essinf}_{[a,b]}s > 0$, $0 < \rho(a)$, $p(b) < \infty$, $\sigma_1 \le 0$, $\sigma_2 \ge 0$, $\alpha,\beta,\gamma,\sigma > 0$, $a = t_0 < t_1 < \cdots < t_l < t_{l+1} = b$, $I_i \in C([0,+\infty),[0,\infty))$, $i = 1,\ldots,l$, $f \in C([a,b] \times 0,+\infty),[0,\infty))$, $f(\cdot,0)$ is nontrivial. By using variational methods, the existence of at least two positive solutions was obtained.

On the existence of solutions for $p(r)$-Laplacian impulsive differential equation boundary value problems, we refer to [31, 32]. If $w(0) = w(T) \neq 0$ and $p(0) = p(T)$, then (1.4) is the periodic boundary value condition, so we call condition (1.4) the periodic-like boundary value condition. In [31], the present author deals with the existence of solutions of (1.1) with (1.2), (1.4) and the following impulsive boundary value condition:

$$
\begin{aligned}
&\lim_{r\to r_i^+} w(r)\left|u'\right|^{p(r)-2}u'(r) - \lim_{r\to r_i^-} w(r)\left|u'\right|^{p(r)-2}u'(r) \\
&\quad = D_i\left(\lim_{r\to r_i^-} u(r), \lim_{r\to r_i^-} (w(r))^{1/(p(r)-1)}u'(r)\right), \quad i = 1,\ldots,k,
\end{aligned}
\tag{1.8}
$$

the method in [31] is the coincidence degree.

In this paper, when $p(r)$ is a general function, we investigate the existence of solutions for the weighted $p(r)$-Laplacian impulsive integro-differential system periodic-like boundary value problems via Leray-Schauder's degree, problems with the impulsive condition (1.8) has been discussed also. The homotopy transformation of this paper is different from [31], our main results partly generalized the results of [23, 28–31].

Let $N \geq 1$, the function $f : J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be Caratheodory, by this we mean:

(i) for almost every $t \in J$ the function $f(t, \cdot, \cdot, \cdot)$ is continuous;

(ii) for each $(x, y, z) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ the function $f(\cdot, x, y, z)$ is measurable on J;

(iii) for each $R > 0$ there is a $\alpha_R \in L^1(J, \mathbb{R})$ such that, for almost every $t \in J$ and every $(x, y, z) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ with $|x| \leq R$, $|y| \leq R$, $|z| \leq R$, one has

$$\left|f(t, x, y, z)\right| \leq \alpha_R(t). \tag{1.9}$$

We say a function $u : J \to \mathbb{R}^N$ is a solution of (1.1) if $u \in \mathrm{PC}^1$ with $w(r)|u'|^{p(r)-2}u'(r)$ absolutely continuous on J_i^o, $i = 0, 1, \ldots, k$, which satisfies (1.1) a.e. on J.

This paper is divided into four sections; in the second section, we present some preliminary. In the third section, we give the existence of solutions for system (1.1), (1.2), (1.3) (or (1.8)) and (1.4). Finally, in the fourth section, we give several examples.

2. Preliminary

For any $(r, y) \in (J \times \mathbb{R}^N)$, denote $\varphi(r, y) = |y|^{p(r)-2}y$. Obviously, φ has the following properties.

Lemma 2.1. *φ is a continuous function and satisfies the following.*

(i) *For any $r \in [0, T]$, $\varphi(r, \cdot)$ is strictly monotone, that is,*

$$\langle \varphi(r, y_1) - \varphi(r, y_2), y_1 - y_2 \rangle > 0, \quad \text{for any } y_1, y_2 \in \mathbb{R}^N,\ y_1 \neq y_2, \tag{2.1}$$

(ii) *There exists a function $\alpha : [0, +\infty) \to [0, +\infty)$, $\alpha(s) \to +\infty$ as $s \to +\infty$, such that*

$$\langle \varphi(r, y), y \rangle \geq \alpha(|y|)|y|, \quad \forall y \in \mathbb{R}^N. \tag{2.2}$$

It is well known that $\varphi(r, \cdot)$ is a homeomorphism from $\mathbb{R}^N$ to $\mathbb{R}^N$ for any fixed $r \in J$. Denote

$$\varphi^{-1}(r, y) = |y|^{(2-p(r))/(p(r)-1)}y, \quad \text{for } y \in \mathbb{R}^N \setminus \{0\},\ \varphi^{-1}(r, 0) = 0. \tag{2.3}$$

It is clear that $\varphi^{-1}(r, \cdot)$ is continuous and send bounded sets into bounded sets. Let us now consider the following simple impulsive problem:

$$\begin{gathered}
\left(w(r)\varphi(r, u'(r))\right)' = f(r), \quad r \in (0, T),\ r \neq r_i, \\
\lim_{r \to r_i^+} u(r) - \lim_{r \to r_i^-} u(r_i) = a_i, \quad i = 1, \ldots, k, \\
\lim_{r \to r_i^+} w(r)|u'|^{p(r)-2}u'(r) - \lim_{r \to r_i^-} w(r)|u'|^{p(r)-2}u'(r) = b_i, \quad i = 1, \ldots, k, \\
u(0) = u(T), \qquad \lim_{r \to 0^+} w(r)|u'|^{p(r)-2}u'(r) = \lim_{r \to T^-} w(r)|u'|^{p(r)-2}u'(r),
\end{gathered} \tag{2.4}$$

where $a_i, b_i \in \mathbb{R}^N$; $f \in L^1$ and satisfies

$$\int_0^T f(r)dr + \sum_{i=1}^{k} b_i = 0. \tag{2.5}$$

If u is a solution of (2.4), by integrating (2.4) from 0 to r, we find that

$$w(r)\varphi(r, u'(r)) = w(0)\varphi(0, u'(0)) + \sum_{r_i<r} b_i + \int_0^r f(t)dt. \tag{2.6}$$

Denote $\rho = w(0)\varphi(0, u'(0))$. Define the operator $F : L^1 \to \mathrm{PC}$ as

$$F(f)(r) = \int_0^r f(t)dt, \quad \forall r \in J,\ \forall f \in L^1. \tag{2.7}$$

By solving for u' in (2.6) and integrating, we find

$$u(r) = u(0) + \sum_{r_i<r} a_i + F\left\{\varphi^{-1}\left[r, (w(r))^{-1}\left(\rho(f) + \sum_{r_i<r} b_i + F(f)(r)\right)\right]\right\}(r), \quad \forall r \in J. \tag{2.8}$$

The boundary value conditions imply that

$$\sum_{i=1}^{k} a_i + \int_0^T \varphi^{-1}\left\{r, (w(r))^{-1}\left[\rho + \sum_{r_i<r} b_i + F(f)(r)\right]\right\}dr = 0. \tag{2.9}$$

Denote $a = (a_1, \ldots, a_k) \in \mathbb{R}^{kN}$, $b = (b_1, \ldots, b_k) \in \mathbb{R}^{kN}$. It is easy to see that ρ is dependent on a, b and f. Denote $W = \mathbb{R}^{2kN} \times \mathrm{PC}$ with the norm

$$\|\varpi\|_W = \sum_{i=1}^{k} |a_i| + \sum_{i=1}^{k} |b_i| + \|h\|_0, \quad \forall \varpi = (a, b, h) \in W, \tag{2.10}$$

then W is a Banach space.

For fixed $\varpi \in W$, we denote

$$\Lambda_{\varpi}(\rho) = \sum_{i=1}^{k} a_i + \int_0^T \varphi^{-1}\left\{r, (w(r))^{-1}\left[\rho + \sum_{r_i<r} b_i + h(r)\right]\right\}dr, \quad \text{where } \varpi = (a, b, h). \tag{2.11}$$

Lemma 2.2. *The function $\Lambda_{\varpi}(\cdot)$ has the following properties.*

(i) *For any fixed $\varpi \in W$, the equation*

$$\Lambda_{\varpi}(\rho) = 0 \tag{2.12}$$

has a unique solution $\tilde{\rho}(\varpi) \in \mathbb{R}^N$.

(ii) *The function $\tilde{\rho} : W \to \mathbb{R}^N$, defined in (i), is continuous and sends bounded sets to bounded sets. Moreover $|\tilde{\rho}(\varpi)| \leq 3N[(((E+1)/E)\sum_{i=1}^{k}|a_i|)^{p^{\#}-1} + \sum_{i=1}^{k}|b_i| + \|h\|_0]$, where $\varpi = (a,b,h) \in W$, $E = \int_0^T (w(r))^{-1/(p(r)-1)}dr$, the notation $M^{p^{\#}-1}$ means $M^{p^{\#}-1} = \begin{cases} M^{p^{+}-1}, & M>1 \\ M^{p^{-}-1}, & M\leq 1 \end{cases}$.*

Proof. (i) From Lemma 2.1, it is immediate that

$$\langle \Lambda_{\varpi}(y_1) - \Lambda_{\varpi}(y_2), y_1 - y_2 \rangle > 0, \quad \text{for } y_1 \neq y_2, \tag{2.13}$$

and hence, if (2.12) has a solution, then it is unique.

Let $R_0 = 3N[(((E+1)/E)\sum_{i=1}^{k}|a_i|)^{p^{\#}-1} + \sum_{i=1}^{k}|b_i| + \|h\|_0]$. Since $(w(r))^{-1/(p(r)-1)} \in L^1(0,T)$ and $h \in \mathrm{PC}$, if $|\rho| > R_0$, it is easy to see that there exists an $j_0 \in \{1,\ldots,N\}$ such that, the absolute value of the j_0th component ρ^{j_0} of ρ is bigger than $3[(((E+1)/E)\sum_{i=1}^{k}|a_i|)^{p^{\#}-1} + \sum_{i=1}^{k}|b_i| + \|h\|_0]$. Thus the j_0th component of $\rho + \sum_{r_i<r} b_i + h(r)$ keeps the same sign of ρ^{j_0} on J, namely,

$$\left(\rho^{j_0} + \sum_{r_i<r} b_i^{j_0} + h^{j_0}(r)\right)\rho^{j_0} > 2\left[\left(\frac{E+1}{E}\sum_{i=1}^{k}|a_i|\right)^{p^{\#}-1} + \sum_{i=1}^{k}|b_i| + \|h\|_0\right]^2, \quad \text{for any } r \in J, \tag{2.14}$$

then it is easy to see that the j_0th component of $\Lambda_{\varpi}(\rho)$ keep the same sign of ρ^{j_0}, thus

$$\sum_{i=1}^{k} a_i + \int_0^T \varphi^{-1}\left\{r, (w(r))^{-1}\left[\rho + \sum_{r_i<r} b_i + h(r)\right]\right\}dr \neq 0. \tag{2.15}$$

Let us consider the equation

$$\lambda\Lambda_{\varpi}(\rho) + (1-\lambda)\rho = 0, \quad \lambda \in [0,1]. \tag{2.16}$$

According to the former discussion, all the solutions of (2.16) belong to $b(R_0+1) = \{x \in \mathbb{R}^N \mid |x| < R_0 + 1\}$. So, we have

$$d_B[\Lambda_{\varpi}(\rho), b(R_0+1), 0] = d_B[I, b(R_0+1), 0] \neq 0. \tag{2.17}$$

It means the existence of solutions of $\Lambda_{\varpi}(\rho) = 0$.

In this way, we define a function $\widetilde{\rho}(\varpi) : W \to \mathbb{R}^N$, which satisfies

$$\sum_{i=1}^{k} a_i + \int_0^T \varphi^{-1}\left\{r, (w(r))^{-1}\left[\widetilde{\rho}(\varpi) + \sum_{r_i<r} b_i + h(r)\right]\right\} dr = 0. \tag{2.18}$$

(ii) By the proof of (i), we also obtain $\widetilde{\rho}$ the sends bounded set to bounded set, and

$$\left|\widetilde{\rho}(\varpi)\right| \leq 3N\left[\left(\frac{E+1}{E}\sum_{i=1}^{k}|a_i|\right)^{p^{\#}-1} + \left\|\sum_{r_i<r} b_i\right\|_0 + \|h\|_0\right], \quad \forall \varpi = (a,b,h) \in W. \tag{2.19}$$

Finally to show that the continuity of $\widetilde{\rho}$. Let $\{\varpi_n = (a_n, b_n, h_n)\}$ is a convergent sequence in W and $\varpi_n \to \varpi = (a,b,h)$, as $n \to +\infty$. Since $\{\widetilde{\rho}(\varpi_n)\}$ is bounded, it contains a convergent subsequence $\{\widetilde{\rho}(\varpi_{n_j})\}$. Set $\widetilde{\rho}(\varpi_{n_j}) \to \rho_0$ as $j \to +\infty$. Obviously,

$$\sum_{i=1}^{k} a_{i,n_j} + \int_0^T \varphi^{-1}\left\{r, (w(r))^{-1}\left[\widetilde{\rho}\left(\varpi_{n_j}\right) + \sum_{r_i<r} b_{i,n_j} + h_{n_j}(r)\right]\right\} dr = 0, \tag{2.20}$$

where $a_{n_j} = (a_{1,n_j}, \ldots, a_{k,n_j})$, $b_{n_j} = (b_{1,n_j}, \ldots, b_{k,n_j})$. Letting $j \to +\infty$, we have

$$\sum_{i=1}^{k} a_i + \int_0^T \varphi^{-1}\left\{r, (w(r))^{-1}\left[\rho_0 + \sum_{r_i<r} b_i + h(r)\right]\right\} dr = 0, \tag{2.21}$$

from (i) we get $\rho_0 = \widetilde{\rho}(\varpi)$, it means that $\widetilde{\rho}$ is continuous.

This completes the proof. □

Now, we define $\rho : \mathbb{R}^N \times \mathbb{R}^N \times L^1 \to \mathbb{R}^N$ be defined by

$$\rho(a,b,u) = \widetilde{\rho}(a,b,F(u)). \tag{2.22}$$

It is clear that ρ is a continuous function which send bounded sets of $\mathbb{R}^N \times \mathbb{R}^N \times L^1$ into bounded sets of $\mathbb{R}^N$, and hence it is a compact continuous mapping.

Now, we continue with our argument previous to Lemma 2.2.

Denote

$$\begin{aligned} A_i &= A_i\left(\lim_{r\to r_i^-} u(r), \lim_{r\to r_i^-} (w(r))^{1/(p(r)-1)} u'(r)\right), \\ D_i &= D_i\left(\lim_{r\to r_i^-} u(r), \lim_{r\to r_i^-} (w(r))^{1/(p(r)-1)} u'(r)\right). \end{aligned} \tag{2.23}$$

From the definition of B_i and D_i, we have

$$D_i = \varphi\left(r_i, \lim_{r\to r_i^-}(w(r))^{1/(p(r)-1)}u'(r) + B_i\left(\lim_{r\to r_i^-}u(r), \lim_{r\to r_i^-}(w(r))^{1/(p(r)-1)}u'(r)\right)\right) - \varphi\left(r_i, \lim_{r\to r_i^-}(w(r))^{1/(p(r)-1)}u'(r)\right). \tag{2.24}$$

Let us define

$$P : \mathrm{PC}^1 \longrightarrow \mathrm{PC}^1, \qquad u \longmapsto u(0); \qquad Q : L^1 \longrightarrow L^1, \qquad h \longmapsto \frac{1}{T}\int_0^T h(r)dr;$$
$$\Theta_b : L^1 \longrightarrow L^1, \qquad h \longmapsto (I-Q)h - \frac{1}{T}\sum_{i=1}^{k} b_i; \tag{2.25}$$

and $K_{(a,b)} : L^1 \to \mathrm{PC}^1$ as

$$K_{(a,b)}(h)(r) = F\left\{\varphi^{-1}\left[r, (w(r))^{-1}\left(\rho(a,b,h) + \sum_{r_i<r} b_i + F(h)\right)\right]\right\}(r), \quad \forall r \in J. \tag{2.26}$$

Lemma 2.3. *The operator $(K_{(a,b)} \circ \Theta_b)(\cdot)$ is continuous and send equi-integrable sets of L^1 into relatively compact sets of PC^1.*

Proof. It is easy to check that $(K_{(a,b)} \circ \Theta_b)(h)(\cdot) \in PC^1$. Since $(w(r))^{-1/(p(r)-1)} \in L^1$ and

$$(K_{(a,b)} \circ \Theta_b)(h)'(t) = \varphi^{-1}\left\{t, (w(t))^{-1}\left[\rho(\Theta_b(h)) + \sum_{r_i<r} b_i + F(\Theta_b(h))\right]\right\}, \quad \forall t \in J, \tag{2.27}$$

it is easy to check that $(K_{(a,b)} \circ \Theta_b)(\cdot)$ is a continuous operator from L^1 to PC^1.

Let now U be an equi-integrable set in L^1, then there exist $\beta \in L^1$, such that, for any $u \in L^1$

$$|u(t)| \le \beta(t) \quad \text{a.e. in } J. \tag{2.28}$$

We want to show that $\overline{(K_{(a,b)} \circ \Theta_b)(U)} \subset \mathrm{PC}^1$ is a compact set.

Let $\{u_n\}$ be a sequence in $(K_{(a,b)} \circ \Theta_b)(U)$, then there exists a sequence $\{h_n\} \in U$ such that $u_n = (K_{(a,b)} \circ \Theta_b)(h_n)$. For any $t_1, t_2 \in J$, we have that

$$|F(\Theta_b(h_n))(t_1) - F(\Theta_b(h_n))(t_2)| \le \left|\int_{t_1}^{t_2} \beta(t)dt\right| + |t_1 - t_2|\frac{1}{T}\left(\int_0^T \beta(t)dt + \left|\sum_{i=1}^{k} b_i\right|\right). \tag{2.29}$$

Hence the sequence $\{F(\Theta_b(h_n))\}$ is uniformly bounded and equicontinuous. By Ascoli-Arzela theorem, there exists a subsequence of $\{F(\Theta_b(h_n))\}$ (which we rename the same) is convergent in PC. According to the bounded continuity of the operator ρ, we can choose a subsequence of $\{\rho(\Theta_b(h_n)) + F(\Theta_b(h_n))\}$ (which we still denote by $\{\rho(\Theta_b(h_n)) + F(\Theta_b(h_n))\}$) is convergent in PC, then

$$w(t)\varphi\big(t,(K\circ\Theta_b)(h_n)'(t)\big) = \rho(\Theta_b(h_n)) + \sum_{r_i<t} b_i + F(\Theta_b(h_n)) \tag{2.30}$$

is convergent according to the norm of PC. Since

$$\big(K_{(a,b)}\circ\Theta_b\big)(h_n)(t) = F\left\{\varphi^{-1}\left[t,(w(r))^{-1}\Big(\rho(\Theta_b(h_n)) + \sum_{r_i<t} b_i + F(\Theta_b(h_n))\Big)\right]\right\}(t), \quad \forall t\in J, \tag{2.31}$$

according to the continuous of φ^{-1} and the integrability of $(w(t))^{-1/(p(t)-1)}$ in L^1, we can see that $(K_{(a,b)}\circ\Theta_b)(h_n)$ is convergent in PC. Thus $\{u_n\}$ convergent in PC^1. This completes the proof. □

We denote $N_f(u) : \mathrm{PC}^1\times J \to L^1$ the Nemytski operator associated to f defined by

$$N_f(u)(r) = f\Big(r,u(r),(w(r))^{1/(p(r)-1)}u'(r),S(u)\Big), \quad \text{a.e. on } J. \tag{2.32}$$

Denote $A=(A_1,\ldots,A_k)$, $D=(D_1,\ldots,D_k)$ and

$$\Theta_f(u) = (I-Q)N_f(u) - \frac{1}{T}\sum_{i=1}^{k} D_i, \tag{2.33}$$

$$\begin{gathered}\rho(u) = \rho\big(A,D,\Theta_f\big)(u),\\ K(u)(r) = F\left\{\varphi^{-1}\left[r,(w(r))^{-1}\Big(\rho(u) + \sum_{r_i<r} D_i + F(\Theta_f(u))\Big)\right]\right\}(r), \quad \forall r\in J.\end{gathered} \tag{2.34}$$

Lemma 2.4. *u is a solution of* (1.1), (1.2), (1.8) *and* (1.4), *if and only if u is a solution of the following abstract equation:*

$$u = Pu + \sum_{r_i<r} A_i + \frac{1}{T}\sum_{i=1}^{k} D_i + QN_f(u) + K(u). \tag{2.35}$$

Proof. If u is a solution of (1.1), (1.2), (1.8) and (1.4), it is clear that u is a solution of (2.35).

Conversely, if u is a solution of (2.35), then (1.2) is satisfied and

$$\frac{1}{T}\sum_{i=1}^{k} D_i + QN_f(u) = 0. \tag{2.36}$$

Thus $\Theta_f(u) = N_f(u)$.

By the condition of the mapping ρ, we have

$$\sum_{i=1}^{k} A_i + F\left\{\varphi^{-1}\left[r, w^{-1}(r)\left(\rho(u) + \sum_{r_i<r} D_i + F\left(\Theta_f(u)\right)\right)\right]\right\}(T) = 0, \tag{2.37}$$

then $u(0) = u(T)$. From (2.35) and (2.36), we also have

$$w(r)\varphi(r, u') = \rho(u) + \sum_{r_i<r} D_i + F\left(\Theta_f(u)\right)(r), \quad r \in (0,T),\ r \neq r_i, \tag{2.38}$$

$$\left(w(r)\varphi(r, u')\right)' = N_f(u)(r), \quad r \in (0,T),\ r \neq r_i. \tag{2.39}$$

From (2.38), we get that (1.8) is satisfied. Since $\sum_{i=1}^{k} D_i + F(\Theta_f(u))(T) = 0$, we have

$$w(0)\varphi(0, u'(0)) = w(T)\varphi(T, u'(T)). \tag{2.40}$$

Hence u is a solutions of (1.1), (1.2), (1.8) and (1.4). This completes the proof. □

3. Main Results and Proofs

In this section, we will apply Leray-Schauder's degree to deal with the existence of solutions for (1.1) with impulsive periodic-like boundary value conditions (1.2), (1.8) (or (1.3)) and (1.4).

Theorem 3.1. *Assume that Ω is an open bounded set in PC^1 such that the following conditions hold.*

(1^0) *For each $\lambda \in (0,1)$ the problem*

$$\begin{gathered}
\left(w(r)|u'|^{p(r)-2}u'\right)' = \lambda f\left(r, u, (w(r))^{1/(p(r)-1)}u', S(u)\right), \quad r \in (0,T),\ r \neq r_i, \\
\lim_{r\to r_i^+} u(r) - \lim_{r\to r_i^-} u(r) = \lambda A_i\left(\lim_{r\to r_i^-} u(r), \lim_{r\to r_i^-} (w(r))^{1/(p(r)-1)}u'(r)\right), \quad i = 1,\ldots,k, \\
\lim_{r\to r_i^+} w(r)|u'|^{p(r)-2}u'(r) - \lim_{r\to r_i^-} w(r)|u'|^{p(r)-2}u'(r) \\
= \lambda D_i\left(\lim_{r\to r_i^-} u(r), \lim_{r\to r_i^-} (w(r))^{1/(p(r)-1)}u'(r)\right), \quad i = 1,\ldots,k, u(0) \\
u(0) = u(T), \qquad \lim_{r\to 0^+} w(r)|u'|^{p(r)-2}u'(r) = \lim_{r\to T^-} w(r)|u'|^{p(r)-2}u'(r),
\end{gathered} \tag{3.1}$$

has no solution on $\partial\Omega$.

(2^0) *The equation*

$$\omega(l) := \frac{1}{T}\int_0^T f(r,l,0,S(l))dr + \frac{1}{T}\sum_{i=1}^{k} D_i(l,0) = 0, \tag{3.2}$$

has no solution on $\partial\Omega \cap \mathbb{R}^N$.

(3^0) *The Brouwer degree* $d_B[\omega, \Omega \cap \mathbb{R}^N, 0] \neq 0$.

Then (1.1) with (1.2), (1.8) and (1.4) has a solution on $\overline{\Omega}$.

Proof. Let us consider the following impulsive equation:

$$\begin{gathered}
\left(w(r)|u'|^{p(r)-2}u'\right)' = \lambda N_f(u) + (1-\lambda)\left[QN_f(u) + \frac{1}{T}\sum_{i=1}^{k} D_i\right], \quad r \in (0,T),\ r \neq r_i, \\
\lim_{r\to r_i^+} u(r) - \lim_{r\to r_i^-} u(r) = \lambda A_i, \quad i = 1,\dots,k, \\
\lim_{r\to r_i^+} w(r)|u'|^{p(r)-2}u'(r) - \lim_{r\to r_i^-} w(r)|u'|^{p(r)-2}u'(r) = \lambda D_i, \quad i = 1,\dots,k, \\
u(0) = u(T), \qquad \lim_{r\to 0^+} w(r)|u'|^{p(r)-2}u'(r) = \lim_{r\to T^-} w(r)|u'|^{p(r)-2}u'(r),
\end{gathered} \tag{3.3}$$

where $A = (A_1,\dots,A_k)$ and $D = (D_1,\dots,D_k)$ are defined in (2.23).

For any $\lambda \in (0,1]$, if u is a solution to (3.1) or u is a solution to (3.3), we have necessarily

$$QN_f(u) + \frac{1}{T}\sum_{i=1}^{k} D_i = 0. \tag{3.4}$$

It means that (3.1) and (3.3) has the same solutions for $\lambda \in (0,1]$.

We denote $N(\cdot,\cdot) : \mathrm{PC}^1 \times [0,1] \to L^1$ defined by

$$N(u,\lambda) = \lambda N_f(u) + (1-\lambda)\left[QN_f(u) + \frac{1}{T}\sum_{i=1}^{k} D_i\right], \tag{3.5}$$

where $N_f(u)$ is defined by (2.32). Denote

$$\begin{gathered}
\Theta_\lambda : L^1 \longrightarrow L^1, \qquad u \longmapsto (I-Q)N(u,\lambda) - \frac{\lambda}{T}\sum_{i=1}^{k} D_i; \\
\rho_\lambda(u) = \rho(\lambda A, \lambda B, \Theta_\lambda),
\end{gathered} \tag{3.6}$$

$$K_\lambda(u)(t) = F\left\{\varphi^{-1}\left[r, \left(w(r))^{-1}(\rho_\lambda(u) + \lambda\sum_{r_i<r} D_i + F(\Theta_\lambda(u))\right)\right]\right\}(t), \quad \forall t \in 0,T]. \tag{3.7}$$

Let

$$\begin{aligned}\Phi_f(u,\lambda) &= Pu + \lambda\sum_{r_i<r} A_i + \lambda\frac{1}{T}\sum_{i=1}^{k} D_i + QN(u,\lambda) + K_\lambda(u)\\ &= Pu + \lambda\sum_{r_i<r} A_i + \frac{1}{T}\sum_{i=1}^{k} D_i + QN_f(u) + K_\lambda(u),\end{aligned} \tag{3.8}$$

the fixed point of $\Phi_f(u,1)$ is a solution for (1.1) with (1.2), (1.8) and (1.4). Also problem (3.3) can be written in the equivalent form

$$u = \Phi_f(u,\lambda). \tag{3.9}$$

Since f is Caratheodory, it is easy to see that $N(\cdot,\cdot)$ is continuous and sends bounded sets into equi-integrable sets. According to Lemma 2.3 we can conclude that Φ_f is continuous and compact for any $\lambda \in [0,1]$. We assume that for $\lambda = 1$, (3.9) does not have a solution on $\partial\Omega$, otherwise we complete the proof. Now from hypothesis (1^0) it follows that (3.9) has no solutions for $(u,\lambda) \in \partial\Omega \times (0,1]$. For $\lambda = 0$, (3.3) is equivalent to the following usual problem:

$$\begin{aligned}&-\left(w(r)\left|u'\right|^{p(r)-2}u'\right)' = QN_f(u) + \frac{1}{T}\sum_{i=1}^{k} D_i\left(\lim_{r\to r_i^-} u(r), \lim_{r\to r_i^-}(w(r))^{1/(p(r)-1)}u'(r)\right), \quad r \in (0,T),\\ &u(0) = u(T), \qquad \lim_{r\to 0^+} w(r)\left|u'\right|^{p(r)-2}u'(r) = \lim_{r\to T^-} w(r)\left|u'\right|^{p(r)-2}u'(r).\end{aligned} \tag{3.10}$$

If u is a solution to this problem, we must have

$$\int_0^T f\left(r, u(r), (w(r))^{1/(p(r)-1)}u'(r), S(u)\right)dr + \sum_{i=1}^{k} D_i\left(\lim_{r\to r_i^-} u(r), \lim_{r\to r_i^-}(w(r))^{1/(p(r)-1)}u'(r)\right) = 0. \tag{3.11}$$

When $\lambda = 0$, the problem is a usual differential equation. Hence

$$w(r)\left|u'\right|^{p(r)-2}u' \equiv c, \tag{3.12}$$

where $c \in \mathbb{R}^N$ is a constant. Since $u(0) = u(T)$, there exist $t_0^i \in (0,T)$, such that $(u^i)'(t_0^i) = 0$, hence $(u^i)' \equiv 0$, it holds $u \equiv l$, a constant. Thus, by (3.11) we have

$$\int_0^T f(r,l,0,S(l))dr + \sum_{i=1}^{k} D_i(l,0) = 0, \tag{3.13}$$

which together with hypothesis (2^0), implies that $u = l \notin \partial\Omega$. Thus, we have proved that (3.9) has no solution (u,λ) on $\partial\Omega \times [0,1]$, then we get that the Leray-Schauder's degree $d_{LS}[I - \Phi_f(\cdot,\lambda),\Omega,0]$ is well defined for $\lambda \in [0,1]$, and from the homotopy invariant property of that degree, we have

$$d_{LS}\left[I-\Phi_f(\cdot,1),\Omega,0\right]=d_{LS}\left[I-\Phi_f(\cdot,0),\Omega,0\right]. \tag{3.14}$$

Now it is clear that the problem

$$u=\Phi_f(u,1) \tag{3.15}$$

is equivalent to problem (1.1) with (1.2), (1.8) and (1.4), and (3.14) tell us that problem (3.15) will have a solution if we can show that

$$d_{LS}\left[I-\Phi_f(\cdot,0),\Omega,0\right]\neq 0. \tag{3.16}$$

Since $K_0(\cdot)\equiv 0$, we have

$$\Phi_f(u,0)=Pu+\frac{1}{T}\sum_{i=1}^{k}D_i+QN_f(u)+K_0(u)=Pu+\frac{1}{T}\sum_{i=1}^{k}D_i+QN_f(u), \tag{3.17}$$

and then

$$u-\Phi_f(u,0)=u-Pu-\frac{1}{T}\sum_{i=1}^{k}D_i-QN_f(u)=-\frac{1}{T}\sum_{i=1}^{k}D_i-QN_f(u),\quad \text{on } \overline{\Omega}. \tag{3.18}$$

By the properties of the Leray-Schauder degree, we have

$$d_{LS}\left[I-\Phi_f(\cdot,0),\Omega,0\right]=(-1)^N d_B\left[\omega,\Omega\cap\mathbb{R}^N,0\right], \tag{3.19}$$

where the function ω is defined in (3.2) and d_B denotes the Brouwer degree. Since by hypothesis (3^0), this last degree is different from zero. This completes the proof. □

Our next theorem is a consequence of Theorem 3.1. As an application of Theorem 3.1, let us consider the system

$$\begin{aligned}\left(w(r)|u'|^{p(r)-2}u'\right)'&=g\left(r,u,(w(r))^{1/(p(r)-1)}u',S(u)\right)\\&\quad+e\left(r,u(r),(w(r))^{1/(p(r)-1)}u'(r),S(u)\right),\quad r\in J',\end{aligned} \tag{3.20}$$

with (1.2), (1.8) and (1.4), where $e: J\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N$ is Caratheodory, $g=(g^1,\ldots,g^N): J\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N$ is continuous, and for any fixed $y_0\in\mathbb{R}^N$, $g^i(r,y_0,0,S(y_0))\neq 0$ if $y_0^i\neq 0$, $\forall r\in J$, $i=1,\ldots,N$. Denote

$$z^-=\min_{r\in J}z(r),\qquad z^+=\max_{r\in J}z(r),\quad \text{for } z\in C(J,\mathbb{R}). \tag{3.21}$$

Theorem 3.2. *Assume that the following conditions hold:*

(1^0) $g(r,kx,ky,kz) = k^{q(r)-1}g(r,x,y,z)$ *for all* $k>0$ *and all* $(r,x,y,z)\in J\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N$, *where* $q(r)\in C(J,\mathbb{R})$, *and* $1<q^-\le q^+<p^-$;

(2^0) $\lim_{|u|+|v|\to+\infty}(e(r,u,v,S(u))/(|u|+|v|)^{q(r)-1})=0$, *for* $r\in J$ *uniformly;*

(3^0) $\sum_{i=1}^k|A_i(u,v)|\le C(1+|u|+|v|)^{\theta}, \forall(u,v)\in\mathbb{R}^N\times\mathbb{R}^N$, *where* $0<\theta<(p^--1)/(p^+-1)$;

(4^0) $\sum_{i=1}^k|D_i(u,v)|\le C(1+|u|+|v|)^{\beta-1}, \forall(u,v)\in\mathbb{R}^N\times\mathbb{R}^N$, *where* $1\le\beta<q^+$;

(5^0) *For large enough* $R_0>0$, *the equation*

$$\omega_g(l) := \frac{1}{T}\int_0^T g(r,l,0,S(l))dr + \frac{1}{T}\sum_{i=1}^k D_i(l,0)=0, \tag{3.22}$$

has no solution on $\partial B(R_0)\cap\mathbb{R}^N$, *where* $B(R_0)=\{u\in PC^1 \mid \|u\|_1<R_0\}$;

(6^0) $d_B[\omega_g,b(R_0),0]\neq 0$ *for large enough* $R_0>0$, *where* $b(R_0)=\{x\in\mathbb{R}^N \mid |x|<R_0\}$.

Then problem (3.20) with (1.2), (1.8) and (1.4) has at least one solution.

Proof. For any $u\in \mathrm{PC}^1$, $\lambda\in[0,1]$, we denote

$$N_{f_\lambda}(u) = g\Big(r,u,(w(r))^{1/(p(r)-1)}u',S(u)\Big)+\lambda e\Big(r,u(r),(w(r))^{1/(p(r)-1)}u'(r),S(u)\Big). \tag{3.23}$$

At first, we consider the following problem:

$$\begin{gathered}\Big(w(r)|u'|^{p(r)-2}u'\Big)' = N_{f_\lambda}(u), \quad r\in(0,T),\ r\neq r_i,\\ \lim_{r\to r_i^+}u(r)-\lim_{r\to r_i^-}u(r)=A_i, \quad i=1,\ldots,k,\\ \lim_{r\to r_i^+}w(r)|u'|^{p(r)-2}u'(r)-\lim_{r\to r_i^-}w(r)|u'|^{p(r)-2}u'(r)=D_i, \quad i=1,\ldots,k,\\ u(0)=u(T), \qquad \lim_{r\to 0^+}w(r)|u'|^{p(r)-2}u'(r)=\lim_{r\to T^-}w(r)|u'|^{p(r)-2}u'(r),\end{gathered} \tag{3.24}$$

where A_i and D_i are defined in (2.23).

According to the proof of Theorem 3.1, we know that (3.24) has the same solutions of

$$u=\Phi_f(u,\lambda)=Pu+\sum_{r_i<r}A_i+\frac{1}{T}\sum_{i=1}^k D_i+QN_{f_\lambda}(u)+K\big(\Theta_{f_\lambda}(u)\big), \tag{3.25}$$

where Θ_{f_λ} is defined in (2.33).

We claim that all the solutions of (3.24) are bounded for each $\lambda \in [0,1]$. In fact, if it is false, we can find a sequence (u_n, λ_n) of solutions for (3.24) such that $\|u_n\|_1 > 1$ and $\|u_n\|_1 \to +\infty$ when $n \to +\infty$. Since (u_n, λ_n) are solutions of (3.24), we have

$$w(r)\varphi(r, u_n'(r)) = \rho(u_n) + \sum_{r_i<r} D_i + F(N_{f_{\lambda_n}}(u_n))(r), \tag{3.26}$$

$$u_n(r) = u_n(0) + \sum_{r_i<r} A_i + F\left\{\varphi^{-1}\left[r, (w(r))^{-1}\left(\rho(u_n) + \sum_{r_i<r} D_i + F(N_{f_{\lambda_n}}(u_n))(r)\right)\right]\right\}(r). \tag{3.27}$$

Since $u_n(0) = u_n(T)$, we have

$$\sum_{i=1}^{k} A_i + F\left\{\varphi^{-1}\left[r, (w(r))^{-1}\left(\rho(u_n) + \sum_{r_i<r} D_i + F(N_{f_{\lambda_n}}(u_n))(r)\right)\right]\right\}(T) = 0. \tag{3.28}$$

From Lemma 2.2, we have

$$|\rho(u_n)| \leq 3NC\left(1 + \left(\frac{E+1}{E}\right)^{(p^+-1)} \|u_n\|_1^{\theta(p^+-1)} + \|u_n\|_1^{q^+-1}\right). \tag{3.29}$$

From (3^0), (4^0), (3.26) and (3.29), we can see that

$$\left\|(w(r))^{1/(p(r)-1)} u_n'\right\|_0 \leq o(1)\|u_n\|_1. \tag{3.30}$$

From (3.30), we have

$$\lim_{n\to+\infty} \frac{\|u_n\|_0}{\|u_n\|_1} = 1. \tag{3.31}$$

Denote $\delta_n = (|u_n^1|_0/\|u_n\|_0, |u_n^2|_0/\|u_n\|_0, \ldots, |u_n^N|_0/\|u_n\|_0)$, then $\delta_n \in \mathbb{R}^N$ and $|\delta_n| = 1$ $(n = 1,2,\ldots)$. Since $\{\delta_n\}$ possesses a convergent subsequence (which still denoted by δ_n), there exists a vector $\delta_0 = (\delta_0^1, \delta_0^2, \ldots, \delta_0^N) \in \mathbb{R}^N$ such that

$$|\delta_0| = 1, \qquad \lim_{n\to+\infty} \delta_n = \delta_0. \tag{3.32}$$

With out loss of generality, we assume that $\delta_0^1 > 0$. Since $u_n \in \mathrm{PC}^1$, there exist $\eta_n^i \in J$ such that

$$\left|u_n^i(\eta_n^i)\right| \geq \left(1 - \frac{1}{n}\right)\left|u_n^i\right|_0, \quad i = 1,2,\ldots,N, \ n = 1,2,\ldots. \tag{3.33}$$

Obviously

$$0 \le \left|u_n^1(r) - u_n^1\left(\eta_n^1\right)\right| = \left|\int_{\eta_n^1}^{r} \left(u_n^1\right)'(t)dt + \sum_{\eta_n^1 < r_i < r} A_i^1\right| \le o(1)\|u_n\|_1 \int_0^T (w(t))^{-1/(p(t)-1)} dt + \sum_{i=1}^{k} \left|A_i^1\right|. \tag{3.34}$$

Since $\|u_n\|_1 \to +\infty$ (as $n \to +\infty$) and $\delta_0^1 > 0$, according to (3.31), (3.33) and (3^0), we have

$$\lim_{n\to+\infty} \frac{1}{\left|u_n^1(\eta_n^1)\right|} \left\{ o(1)\|u_n\|_1 \int_0^T (w(t))^{-1/(p(t)-1)} dt + \sum_{i=1}^{k} \left|A_i^1\right| \right\} = 0. \tag{3.35}$$

From (3.31), (3.34) and (3.35) we have

$$\lim_{n\to+\infty} \frac{u_n^1(r)}{u_n^1(\eta_n^1)} = 1, \quad \text{for } r \in J \text{ uniformly}. \tag{3.36}$$

So we get

$$\lim_{n\to+\infty} \frac{u_n(r)}{\|u_n\|_1} = \delta_*, \qquad \lim_{n\to+\infty} \frac{(w(r))^{1/(p(r)-1)} u_n'(r)}{\|u_n\|_1} = 0, \quad \text{for } r \in J \text{ uniformly}, \tag{3.37}$$

where $\delta_* \in \mathbb{R}^N$ with $|\delta_*^i| = \delta_0^i$, $i = 1, \ldots, N$.

From (1.4), we have

$$\begin{aligned} \sum_{i=1}^{k} D_i + \int_0^T \Big\{ g\Big(r, u_n, (w(r))^{1/(p(r)-1)} u_n', S(u_n)\Big) + \\ + e\Big(r, u_n(r), (w(r))^{1/(p(r)-1)} u_n'(r)\Big), S(u_n) \Big\} dr = 0. \end{aligned} \tag{3.38}$$

Since $g^1(t, \delta_0, 0, S(\delta_0)) \neq 0$, according to (3.37), ($4^0$) and the continuity of g, we have

$$\sum_{i=1}^{k} D_i + \int_0^T \|u_n\|_1^{q(t)-1} \{ g[t, \delta_0 + o(1), o(1), S(\delta_0)] + o(1) \} dt \neq 0, \tag{3.39}$$

it is a contradiction to (3.38). This implies that there is a large enough $R_0 > 0$ such that all the solutions of (3.24) belong to $B(R_0)$, then we have

$$d_{LS}\left[I - \Phi_f(\cdot, 1), B(R_0), 0\right] = d_{LS}\left[I - \Phi_f(\cdot, 0), B(R_0), 0\right]. \tag{3.40}$$

If we prove that $d_{LS}[I - \Phi_f(\cdot, 0), B(R_0), 0] \neq 0$, then we obtain the existence of solutions (3.20) with (1.2), (1.8) and (1.4).

Now we consider the following equation:

$$\begin{gathered}
\left(w(r)\left|u'\right|^{p(r)-2}u'\right)' = \lambda N_g(u) + (1-\lambda)\left[QN_g(u) + \frac{1}{T}\sum_{i=1}^{k} D_i\right], \quad r \in J', \\
\lim_{r\to r_i^+} u(r) - \lim_{r\to r_i^-} u(r) = \lambda A_i, \quad i = 1,\dots,k, \\
\lim_{r\to r_i^+} w(r)\left|u'\right|^{p(r)-2}u'(r) - \lim_{r\to r_i^-} w(r)\left|u'\right|^{p(r)-2}u'(r) = D_i, \quad i = 1,\dots,k, \\
u(0) = u(T), \qquad \lim_{r\to 0^+} w(r)\left|u'\right|^{p(r)-2}u'(r) = \lim_{r\to T^-} w(r)\left|u'\right|^{p(r)-2}u'(r),
\end{gathered} \tag{3.41}$$

where $N_g(u) = g(r, u, (w(r))^{1/(p(r)-1)}u', S(u))$, A_i and D_i are defined in (2.23).

Similar to the above discussions, for any $\lambda \in (0,1]$, all the solutions of (3.41) are uniformly bounded.

If u is a solution of the following usual equation with (1.4):

$$\left(w(r)\left|u'\right|^{p(r)-2}u'\right)' = QN_g(u) + \frac{1}{T}\sum_{i=1}^{k} D_i, \quad r \in (0,T), \tag{3.42}$$

then we have

$$QN_g(u) + \frac{1}{T}\sum_{i=1}^{k} D_i = 0, \qquad w(r)\left|u'\right|^{p(r)-2}u' \equiv c. \tag{3.43}$$

Since $u(0) = u(T)$, we have $w(r)|u'|^{p(r)-2}u' \equiv 0$, it means that u is a solution of

$$\omega_g(l) = \frac{1}{T}\int_{T_1}^{T_2} g(r, l, 0, S(l))dr + \frac{1}{T}\sum_{i=1}^{k} D_i(l, 0) = 0, \tag{3.44}$$

according to hypothesis (5^0), (3.41) has no solution on $[0,1] \times \partial B(R_0)$, from Theorem 3.1 we obtain that (3.20) with (1.2), (1.8) and (1.4) has at least one solution. This completes the proof. □

Corollary 3.3. *If $e : J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is Caratheodory, conditions (2^0), (3^0) and (4^0) of Theorem 3.2 are satisfied, $g(r, u, v, S(u)) = \varphi(r)(|u|^{q(r)-2}u + |v|^{q(r)-2}v + |S(u)|^{q(r)-2}S(u))$, where $\varphi(\cdot)$, $q(\cdot) \in C(J, \mathbb{R})$ are positive functions, and satisfies $1 < q^- \le q^+ < p^-$; then* (3.20) *with* (1.2), (1.8) *and* (1.4) *has at least one solution.*

Proof. Denote

$$H(l, \lambda) = \frac{1}{T}\int_{0}^{T} g(r, l, 0, S(l))dr + \frac{1}{T}\sum_{i=1}^{k} \lambda D_i(l, 0). \tag{3.45}$$

From condition (4^0), we have

$$|D_i(l,0)| \leq C(1+|l|)^{\beta-1}, \quad 1 \leq \beta < q^+. \tag{3.46}$$

Since ϕ is nonnegative, from the above inequality, we can see that all the solutions of $H(l,\lambda) = 0$ are uniformly bounded for $\lambda \in [0,1]$. Thus the Leray-Schauder degree $d_B[H(l,\lambda), b(R_0), 0]$ is well defined for $\lambda \in [0,1]$, and we have

$$\begin{gathered} d_B\left[\omega_g, b(R_0), 0\right] = d_B[H(l,1), b(R_0), 0] = d_B[H(l,0), b(R_0), 0]. \\ H(l,0) = \frac{1}{T}\int_0^T g(r,l,0,S(l))dr = \frac{1}{T}\int_0^T \varphi(r)\left(|l|^{q(r)-2}l + S(l)\right)dr, \end{gathered} \tag{3.47}$$

then it is easy to see that $H(l,0) = 0$ has only one solution in $\mathbb{R}^N$ and

$$d_B\left[\omega_g, b(R_0), 0\right] = d_B[I, b(R_0), 0] \neq 0. \tag{3.48}$$

According to Theorem 3.2 we get that (3.20) with (1.2), (1.8) and (1.4) has at least a solution. This completes the proof. □

Let us consider

$$-\left(w(r)\left|u'\right|^{p(r)-2}u'\right)' + f\left(r,u,(w(r))^{1/(p(r)-1)}u', S(u), \delta\right) = 0, \quad r \in (0,1),\ r \neq r_i, \tag{3.49}$$

where δ is a parameter, and

$$\begin{aligned} &f\left(r,u,(w(r))^{1/(p(r)-1)}u', S(u), \delta\right) \\ &\quad = g\left(r,u,(w(r))^{1/(p(r)-1)}u', S(u)\right) + \delta h\left(r,u,(w(r))^{1/(p(r)-1)}u', S(u)\right), \end{aligned} \tag{3.50}$$

where $g, h : J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ are Caratheodory.

We have the following theorem.

Theorem 3.4. *One assumes that conditions of* (1^0) *and* (3^0)–(6^0) *of Theorem 3.2 are satisfied, then problem* (3.49) *with* (1.2), (1.8) *and* (1.4) *has at least a solution when the parameter* δ *is small enough.*

Proof. Denote

$$f_{\lambda\delta}\Big(r,u,(w(r))^{1/(p(r)-1)}u',S(u)\Big) = g\Big(r,u,(w(r))^{1/(p(r)-1)}u',S(u)\Big) + \lambda\delta h\Big(r,u,(w(r))^{1/(p(r)-1)}u',S(u)\Big). \tag{3.51}$$

Let us consider the existence of solutions of the following

$$-\Big(w(r)|u'|^{p(r)-2}u'\Big)' + f_{\lambda\delta}\Big(r,u,(w(r))^{1/(p(r)-1)}u',S(u)\Big) = 0, \quad r \in (0,1),\ r \neq r_i, \tag{3.52}$$

with (1.2), (1.8) and (1.4).

Denote

$$\begin{aligned} \rho^{\#}_{\lambda}(u,\delta) &= \rho\big(A,D,\Theta_{f_{\lambda\delta}}\big)(u), \\ K^{\#}_{\lambda}(u,\delta) &= F\left\{\varphi^{-1}\left[r,(w(r))^{-1}\left(\rho^{\#}_{\lambda}(u,\delta) + \sum_{r_i<r} D_i + F\big(\Theta_{f_{\lambda\delta}}(u)\big)(r)\right)\right]\right\}, \\ \Psi_{\delta}(u,\lambda) &= P(u) + \sum_{r_i<r} A_i + \frac{1}{T}\sum_{i=1}^{k} D_i + QN_{f_{\lambda\delta}}(u) + K^{\#}_{\lambda}(u,\delta), \end{aligned} \tag{3.53}$$

where $N_{f_{\lambda\delta}}(u)$ is defined in (2.32), $\Theta_{f_{\lambda\delta}}$ is defined in (2.33).

We know that (3.52) with (1.2), (1.8) and (1.4) has the same solution of

$$u = \Psi_{\delta}(u,\lambda). \tag{3.54}$$

Obviously, $f_0 = g$. Thus $\Psi_{\delta}(u,0) = \Phi_g(u,1)$. From the proof of Theorem 3.2, we can see that all the solutions of $u = \Psi_{\delta}(u,0)$ are uniformly bounded, then there exists a large enough $R_0 > 0$ such that all the solutions of $u = \Psi_{\delta}(u,0)$ belong to $B(R_0) = \{u \in PC^1 \mid \|u\|_1 < R_0\}$. Since $\Psi_{\delta}(\cdot,0)$ is compact continuous from PC^1 to PC^1, we have

$$\inf_{u\in\partial B(R_0)} \|u - \Psi_{\delta}(u,0)\|_1 > 0. \tag{3.55}$$

Since g,h are Caratheodory, we have

$$\begin{aligned} \big\|F\big(N_{f_{\lambda\delta}}(u)\big) - F\big(N_{f_0}(u)\big)\big\|_0 &\longrightarrow 0 \quad \text{for } (u,\lambda) \in \overline{B(R_0)} \times [0,1] \text{ uniformly, as } \delta \longrightarrow 0, \\ \Big|\rho^{\#}_{\lambda}(u,\delta) - \rho^{\#}_{0}(u,\delta)\Big| &\longrightarrow 0 \quad \text{for } (u,\lambda) \in \overline{B(R_0)} \times [0,1] \text{ uniformly, as } \delta \longrightarrow 0, \\ \Big\|K^{\#}_{\lambda}(u,\delta) - K^{\#}_{0}(u,\delta)\Big\|_1 &\longrightarrow 0 \quad \text{for } (u,\lambda) \in \overline{B(R_0)} \times [0,1] \text{ uniformly, as } \delta \longrightarrow 0. \end{aligned} \tag{3.56}$$

Thus

$$\|\Psi_{\delta}(u,\lambda) - \Psi_0(u,\lambda)\|_1 \longrightarrow 0 \text{ for } (u,\lambda) \in \overline{B(R_0)} \times [0,1] \text{ uniformly, as } \delta \longrightarrow 0. \tag{3.57}$$

Obviously, $\Psi_0(u,\lambda) = \Psi_\delta(u,0) = \Psi_0(u,0)$. Therefore

$$\|\Psi_\delta(u,\lambda) - \Psi_\delta(u,0)\|_1 \longrightarrow 0 \quad \text{for } (u,\lambda) \in \overline{B(R_0)} \times [0,1] \text{ uniformly, as } \delta \longrightarrow 0. \tag{3.58}$$

Thus, when δ is small enough, from (3.55), we can conclude

$$\begin{aligned} &\inf_{(u,\lambda)\in\partial B(R_0)\times[0,1]} \|u - \Psi_\delta(u,\lambda)\|_1 \\ &\quad \geq \inf_{u\in\partial B(R_0)} \|u - \Psi_\delta(u,0)\|_1 - \sup_{(u,\lambda)\in\overline{B(R_0)}\times[0,1]} \|\Psi_\delta(u,0) - \Psi_\delta(u,\lambda)\|_1 > 0. \end{aligned} \tag{3.59}$$

Thus $u = \Psi_\delta(u,\lambda)$ has no solution on $\partial B(R_0)$ for any $\lambda \in [0,1]$, when δ is small enough. It means that the Leray-Schauder degree $d_{LS}[I - \Psi_\delta(u,\lambda), B(R_0), 0]$ is well defined for any $\lambda \in [0,1]$, and

$$d_{LS}[I - \Psi_\delta(u,\lambda), B(R_0), 0] = d_{LS}[I - \Psi_\delta(u,0), B(R_0), 0]. \tag{3.60}$$

From the proof of Theorem 3.2, we can see that the right-hand side is nonzero, then (3.49) with (1.2), (1.8) and (1.4) has at least a solution, when the parameter δ is small enough. □

Now, let us consider the existence of solutions of (1.1) with (1.2), (1.3) and (1.4).

Theorem 3.5. *One assumes that conditions of* (1^0)–(3^0) *and* (5^0)-(6^0) *of Theorem 3.2 are satisfied, one also assumes that B satisfy*

$$|B_i(u,v)| \leq C(1 + |u| + |v|)^{\alpha_i}, \quad \forall (u,v) \in \mathbb{R}^N \times \mathbb{R}^N, \ i = 1,\dots,k, \tag{3.61}$$

where

$$\alpha_i < \frac{q^+ - 1}{p(r_i) - 1}, \qquad p(r_i) - 1 < q^+ - \alpha_i, \quad i = 1,\dots,k, \tag{3.62}$$

then problem (3.20) *with* (1.2), (1.3) *and* (1.4) *has at least a solution.*

Proof. We only need to prove that condition (4^0) of Theorem 3.2 is satisfied.

(a) When $|v| \leq M|B_i(u,v)|$, where M is a large enough positive constant, from the definition of D, we can see that

$$|D_i(u,v)| \leq C_1 |B_i(u,v)|^{p(r_i)-1} \leq C_2(1 + |u| + |v|)^{\alpha_i(p(r_i)-1)}. \tag{3.63}$$

Since $\alpha_i < (q^+ - 1)/(p(r_i) - 1)$, we have

$$\alpha_i(p(r_i) - 1) < q^+ - 1. \tag{3.64}$$

Thus condition (4^0) of Theorem 3.2 is satisfied.

(b) When $|v| \geq M|B_i(u,v)|$, we can see that

$$|D_i(u,v)| \leq C_3|v|^{p(r_i)-1}\frac{|B_i(u,v)|}{|v|} = C_4|v|^{p(r_i)-2}|B_i(u,v)|. \tag{3.65}$$

(i) If $p(r_i) - 1 \geq 1$, since $p(r_i) - 1 < q^+ - \alpha_i$, we have

$$\begin{gathered} p(r_i) - 2 + \alpha_i < q^+ - 1, \\ |D_i(u,v)| \leq C_5|v|^{p(r_i)-2}|B_i(u,v)| \leq C_6(1 + |u| + |v|)^{p(r_i)-2+\alpha_i}. \end{gathered} \tag{3.66}$$

Thus the condition (4^0) of Theorem 3.2 is satisfied.

(ii) If $p(r_i) - 1 < 1$, since $\alpha_i < (q^+ - 1)/(p(r_i) - 1)$, we have

$$\begin{gathered} \alpha_i(p(r_i) - 1) < q^+ - 1, \\ |D_i(u,v)| \leq C_7|v|^{p(r_i)-2}|B_i(u,v)| \leq C_8|B_i(u,v)|^{p(r_i)-1} \leq C_9(1 + |u| + |v|)^{\alpha_i(p(r_i)-1)}. \end{gathered} \tag{3.67}$$

Thus the condition (4^0) of Theorem 3.2 is satisfied.

Summarizing the discussion, we can see that condition (4^0) of Theorem 3.2 is satisfied. Thus problem (3.20) with (1.2), (1.3) and (1.4) has at least a solution. □

Corollary 3.6. *If $e : J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is Caratheodory,* (3.61), (3.62) *and conditions* (2^0) *and* (3^0) *of Theorem 3.2 are satisfied, $g(r,u,v,S(u)) = \varphi(r)(|u|^{q(r)-2}u + |v|^{q(r)-2}v + |S(u)|^{q(r)-2}S(u))$, where $\varphi(r)$, $q(r) \in C(J,\mathbb{R})$ are positive functions, and satisfies $1 < q^- \leq q^+ < p^-$; then* (3.20) *with* (1.2), (1.3) *and* (1.4) *has at least one solution.*

Proof. It is easy to see that from the proof of Corollary 3.3 and Theorem 3.5. We omit it here. □

4. Examples

Example 4.1. Consider the following problem:

$$\begin{gathered} -\left(w(t)\left|u'\right|^{p(t)-2}u'\right)' + |u|^{q(t)-2}u + w(t)\left|u'\right|^{q(t)-2}u' + S(u)(t) = 0, \quad t \in J', \\ \lim_{t\to t_i^+} u(t) - \lim_{t\to t_i^-} u(t) = \lim_{t\to t_i^-} |u(t)|^{-1/2}u(t) + \lim_{t\to t_i^-}\left|(w(t))^{1/(p(t)-1)}u'(t)\right|^{-1/2}(w(t))^{1/(p(t)-1)}u'(t), \\ \lim_{t\to t_i^+} w(t)\left|u'\right|^{p(t)-2}u'(t) - \lim_{t\to t_i^-} w(t)\left|u'\right|^{p(t)-2}u'(t) = \lim_{t\to t_i^-}|u(t)|^2u(t) + \lim_{t\to t_i^-}(w(t))^{3/(p(t)-1)}\left|u'(t)\right|^2u'(t), \\ u(0) = u(T), \qquad \lim_{r\to 0^+} w(r)\left|u'\right|^{p(r)-2}u'(r) = \lim_{r\to T^-} w(r)\left|u'\right|^{p(r)-2}u'(r), \end{gathered} \tag{S_1}$$

where $p(t) = 6 + \sin t$, $q(t) = 3 + \cos t$, $S(u)(t) = \int_0^T (\sin st + 1)u(s)ds$.

Obviously, $|u|^{q(t)-2}u + w(t)|u'|^{q(t)-2}u' + S(u)(t)$ is Caratheodory, $q(t) \le 4 < 5 \le p(t)$, and the conditions of Theorem 3.2 are satisfied, then (S_1) has a solution.

Example 4.2. Consider the following problem:

$$\begin{gathered}
-\left(w(t)\left|u'\right|^{p(t)-2}u'\right)' + f\left(r, u, (w(r))^{1/(p(r)-1)}u', S(u)\right) \\
+\delta h\left(r, u, (w(r))^{1/(p(r)-1)}u', S(u)\right) = 0, \qquad t \in J', \\
\lim_{t\to t_i^+} u(t) - \lim_{t\to t_i^-} u(t) = \lim_{t\to t_i^-} |u(t)|^{-1/3}u(t) + \lim_{t\to t_i^-}\left|(w(t))^{1/(p(t)-1)}u'(t)\right|^{-1/3}(w(t))^{1/(p(t)-1)}u'(t), \\
\lim_{t\to t_i^+} w(t)\left|u'\right|^{p(t)-2}u'(t) - \lim_{t\to t_i^-} w(t)\left|u'\right|^{p(t)-2}u'(t) = \lim_{t\to t_i^-}|u(t)|^2u(t) + \lim_{t\to t_i^-}(w(t))^{3/(p(t)-1)}\left|u'(t)\right|^2u'(t), \\
u(0) = u(T), \quad \lim_{r\to 0^+} w(r)\left|u'\right|^{p(r)-2}u'(r) = \lim_{r\to T^-} w(r)\left|u'\right|^{p(r)-2}u'(r),
\end{gathered} \tag{S_2}$$

where

$$\begin{gathered}
f\left(r, u, (w(r))^{1/(p(r)-1)}u', S(u)\right) = |u|^{q(t)-2}u + w(t)\left|u'\right|^{q(t)-2}u' + \delta u e^{|u| + \left|(w(t))^{1/(p(t)-1)}u'\right|} + S(u)(t), \\
p(t) = 7 + \cos 3t, \quad q(t) = 4 + \sin 2t, \quad S(u)(t) = \int_0^T (\cos st + 1)u(s)ds.
\end{gathered} \tag{4.1}$$

Obviously, $|u|^{q(t)-2}u + w(t)|u'|^{q(t)-2}u' + \delta u e^{|u| + \left|(w(t))^{1/(p(t)-1)}u'\right|} + S(u)(t)$ is Caratheodory, $q(t) \le 5 < 6 \le p(t)$, and the conditions of Theorem 3.4 are satisfied, then (S_2) has a solution when δ is small enough.

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