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Research Article

Common Fixed Points of Weakly Contractive and Strongly Expansive Mappings in Topological Spaces

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Using the notion of weakly F-contractive mappings, we prove several new common fixed point theorems for commuting as well as noncommuting mappings on a topological space X. By analogy, we obtain a common fixed point theorem of mappings which are strongly F-expansive on X.

1. Introduction

It is well known that if X is a compact metric space and $f : X \to X$ is a weakly contractive mapping (see Section 2 for the definition), then f has a fixed point in X (see [1, p. 17]). In late sixties, Furi and Vignoli [2] extended this result to α-condensing mappings acting on a bounded complete metric space (see [3] for the definition). A generalized version of Furi-Vignoli's theorem using the notion of weakly F-contractive mappings acting on a topological space was proved in [4] (see also [5]).

On the other hand, in [6] while examining KKM maps, the authors introduced a new concept of lower (upper) semicontinuous function (see Definition 2.1, Section 2) which is more general than the classical one. In [7], the authors used this definition of lower semicontinuity to redefine weakly F-contractive mappings and strongly F-expansive mappings (see Definition 2.6, Section 2) to formulate and prove several results for fixed points.

In this article, we have used the notions of weakly F-contractive mappings ($f : X \to X$ where X is a topological space) to prove a version of the above-mentioned fixed point theorem [7, Theorem 1] for common fixed points (see Theorem 3.1). We also prove a common fixed point theorem under the assumption that certain iteration of the mappings in question is weakly F-contractive. As a corollary to this fact, we get an extension (to common fixed points) of [7, Theorem 3] for Banach spaces with a quasimodulus endowed with a suitable transitive binary relation. The most interesting result of this section is Theorem 3.8 wherein the strongly F-expansive condition on f (with some other conditions) implies that f and g have a unique common fixed point.

In Section 4, we define a new class of noncommuting self-maps and prove some common fixed point results for this new class of mappings.

2. Preliminaries

Definition 2.1 (see [6]). Let X be a topological space. A function $f : X \to \mathbb{R}$ is said to be *lower semi-continuous from above* (*lsca*) at x_0 if for any net $(x_\lambda)_{\lambda\in\Lambda}$ convergent to x_0 with

$$f(x_{\lambda_1}) \le f(x_{\lambda_2}) \quad \text{for } \lambda_2 \le \lambda_1, \tag{2.1}$$

we have

$$f(x_0) \le \lim_{\lambda\in\Lambda} f(x_\lambda). \tag{2.2}$$

A function $f : X \to \mathbb{R}$ is said to be lsca if it is lsca at every $x \in X$.

Example 2.2. (i) Let $X = \mathbb{R}$. Define $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} x+1, & \text{when } x > 0, \\ \dfrac{1}{2}, & \text{when } x = 0, \\ -x+1, & \text{when } x < 0. \end{cases} \tag{2.3}$$

Let $(z_n)_{n\ge1}$ be a sequence of nonnegative terms such that $(z_n)_{n\ge1}$ converges to 0. Then

$$f(z_{n+1}) \le f(z_n) \quad \text{for } \lambda_2 = n \le n+1 = \lambda_1, \quad f(0) = \frac{1}{2} < 1 = \lim_{n\to\infty} f(z_n). \tag{2.4}$$

Similarly, if $(z'{}_n)_{n\ge1}$ is a sequence in X of negative terms such that $(z'_n)_{n\ge1}$ converges to 0, then

$$f(z'_{n+1}) \le f(z'_n) \quad \text{for } \lambda_2 = n \le n+1 = \lambda_1, \quad f(0) = \frac{1}{2} < 1 = \lim_{n\to\infty} f(z'_n). \tag{2.5}$$

Thus, f is lsca at 0.

(ii) Every lower semi-continuous function is lsca but not conversely. One can check that the function $f : X \to \mathbb{R}$ with $X = \mathbb{R}$ defined below is lsca at 0 but is not lower semi-continuous at 0:

$$f(x) = \begin{cases} x+1, & \text{when } x \geq 0, \\ x, & \text{when } x < 0. \end{cases} \tag{2.6}$$

The following lemmas state some properties of lsca mappings. The first one is an analogue of Weierstrass boundedness theorem and the second one is about the composition of a continuous function and a function lsca.

Lemma 2.3 (see [6]). *Let X be a compact topological space and $f : X \to \mathbb{R}$ a function lsca. Then there exists $x_0 \in X$ such that $f(x_0) = \inf\{f(x) : x \in X\}$.*

Lemma 2.4 (see [7]). *Let X be a topological space and $f : X \to Y$ a continuous function. If $g : X \to \mathbb{R}$ is a function lsca, then the composition function $h = g \circ f : X \to \mathbb{R}$ is also lsca.*

Proof. Fix $x_0 \in X \times X$ and consider a net $(x_\lambda)_{\lambda\in\Lambda}$ in X convergent to x_0 such that

$$h(x_{\lambda_1}) \leq h(x_{\lambda_2}) \quad \text{for } \lambda_2 \leq \lambda_1. \tag{2.7}$$

Set $z_\lambda = f(x_\lambda)$ and $z = f(x_0)$. Then since f is continuous, $\lim_\lambda f(x_\lambda) = f(x_0) \in X$, and g lsca implies that

$$g(z) = g\big(f(x_0)\big) \leq \lim_\lambda g\big(f(x_\lambda)\big) = \lim_\lambda g(z_\lambda) \tag{2.8}$$

with $g(z_{\lambda_1}) \leq g(z_{\lambda_2})$ for $\lambda_2 \leq \lambda_1$. Thus $h(x_0) \leq \lim_\lambda h(x_\lambda))$ and h is lsca. □

Remark 2.5 (see [6]). Let X be topological space. Let $f : X \to X$ be a continuous function and $F : X \times X \to \mathbb{R}$ lsca. Then $g : X \to \mathbb{R}$ defined by $g(x) = F(x, f(x))$ is also lsca. For this, let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in X convergent to $x \in X$. Since f is continuous, $\lim_\lambda f(x_\lambda) = f(x)$. Suppose that

$$g(x_{\lambda_1}) \leq g(x_{\lambda_2}) \quad \text{for } \lambda_2 \leq \lambda_1. \tag{2.9}$$

Then since F is lsca, we have

$$g(x) = F\big(x, f(x)\big) \leq \lim_\lambda F\big(x_\lambda, f(x_\lambda)\big) = \lim_\lambda g(x_\lambda). \tag{2.10}$$

Definition 2.6 (see [7]). Let X be a topological space and $F : X \times X \to \mathbb{R}$ be lsca. The mapping $f : X \to X$ is said to be

(i) weakly F-contractive if $F(f(x), f(y)) < F(x, y)$ for all $x, y, \in X$ such that $x \neq y$,

(ii) strongly F-expansive if $F(f(x), f(y)) > F(x, y)$ for all $x, y \in X$ such that $x \neq y$.

If X is a metric space with metric d and $F = d$, then we call f, respectively, weakly contractive and strongly expansive.

Let $f,\ g : X \to X$. The set of fixed points of f (resp., g) is denoted by $F(f)$ (resp., $F(g)$). A point $x \in M$ is a coincidence point (common fixed point) of f and g if $fx = gx$ ($x = fx = gx$). The set of coincidence points of f and g is denoted by $C(f, g)$. Maps $f,\ g : X \to X$ are called (1) commuting if $fgx = gfx$ for all $x \in X$, (2) weakly compatible [8] if they commute at their coincidence points, that is, if $fgx = gfx$ whenever $fx = gx$, and (3) occasionally weakly compatible [9] if $fgx = gfx$ for some $x \in C(f, g)$.

3. Common Fixed Point Theorems for Commuting Maps

In this section we extend some results in [7] to the setting of two mappings having a unique common fixed point.

Theorem 3.1. *Let X be a topological space, $x_0 \in X$, and $f, g : X \to X$ self-mappings such that for every countable set $U \subseteq X$,*

$$U = f(U) \cup \{g(x_0)\} \Longrightarrow U \text{ is relatively compact} \tag{3.1}$$

and f, g commute on X. If

(i) *f is continuous and weakly F-contractive or*

(ii) *g is continuous and weakly F-contractive with $g(U) \subseteq U$,*

then f and g have a unique common fixed point.

Proof. Let $x_1 = g(x_0)$ and define the sequence $(x_n)_{n \geq 1}$ by setting $x_{n+1} = f(x_n)$ for $n \geq 1$. Let $A = \{x_n : n \geq 1\}$. Then

$$A = f(A) \cup \{g(x_0)\}, \tag{3.2}$$

so by hypothesis $\overline{A}$ is compact. Define $\varphi : \overline{A} \longrightarrow \mathbb{R}$, by

$$\varphi(x) = \begin{cases} F(x, f(x)) & \text{if } f \text{ is continuous,} \\ F(x, g(x)) & \text{if } g \text{ is continuous.} \end{cases} \tag{3.3}$$

Now if f or g is continuous and since F is lsca, then by Remark 2.5, φ is lsca. So by Lemma 2.3, φ has a minimum at, say, $a \in \overline{A}$.

(i) *Suppose that f is continuous and weakly F-contractive*. Then $\varphi(x) = F(x, f(x))$ as f is continuous. Now observe that if $a \in \overline{A}$, f is continuous, and $f(A) \subseteq A$, then $f(a) \in \overline{A}$. We show that $f(a) = a$. Suppose that $f(a) \neq a$; then

$$\varphi(f(a)) = F(f(a), f(f(a))) < F(a, f(a)) = \varphi(a), \tag{3.4}$$

a contradiction to the minimality of φ at a. Having $f(a) = a$, one can see that $g(a) = a$. Indeed, if $g(a) \neq a$ then we have

$$F(a, g(a)) = F(f(a), gf(a)) = F(f(a), fg(a)) < F(a, g(a)) \tag{3.5}$$

a contradiction.

(ii) *Suppose that* g *is continuous and weakly F-contractive* with $g(U) \subseteq U$. Then $\varphi(x) = F(x, g(x))$ as g is continuous. Put $U = A$; then $a \in \overline{A}$, g is continuous, and $g(A) \subseteq A$ implies that $g(a) \in \overline{A}$. We claim that $g(a) = a$, for otherwise we will have

$$\varphi(g(a)) = F(g(a), g(g(a))) < F(a, g(a)) = \varphi(a) \tag{3.6}$$

which is a contradiction. Hence the claim follows.

Now suppose that $f(a) \neq a$ then we have

$$F(a, f(a)) = F(g(a), fg(a)) = F(g(a), gf(a)) < F(a, f(a)), \tag{3.7}$$

a contradiction, hence $f(a) = a$.

In both cases, uniqueness follows from the contractive conditions: suppose there exists $b \in \overline{A}$ such that $f(b) = b = g(b)$. Then we have

$$\begin{aligned} F(a,b) &= F(f(a), f(b)) < F(a,b), \\ F(a,b) &= F(g(a), g(b)) < F(a,b) \end{aligned} \tag{3.8}$$

which is false. Thus f and g have a unique common fixed point.

If $g = id_X$, then Theorem 3.1(i) reduces to [7, Theorem 1]. □

Corollary 3.2 (see [7, Theorem 1]). *Let X be a topological space, $x_0 \in X$, and $f : X \to X$ continuous and weakly F-contractive. If the implication $U \subseteq X$,*

$$U = f(U) \cup \{x_0\} \Longrightarrow \quad U \text{ is relatively compact}, \tag{3.9}$$

holds for every countable set $U \subseteq X$, then f has a unique fixed point.

Example 3.3. Let $(c_0, \|\cdot\|_\infty)$ be the Banach space of all null real sequences. Define

$$X = \{x = (x_n)_{n\geq 1} \in c_0 : x_n \in [0,1], \text{ for } n \geq 1\}. \tag{3.10}$$

Let $k \in \mathbb{N}$ and $(p_n)_{n\geq 1} \subseteq [0,1)$ a sequence such that

$$(p_n)_{n\leq k} \subseteq \{0\}, \qquad (p_n)_{n>k} \subseteq (0,1) \tag{3.11}$$

with $p_n \to 1$ as $n \to \infty$. Define the mappings $f, g : X \to X$ by

$$f(x) = \left(f_{\mathrm{n}}(x_n)\right)_{n\geq 1}, \qquad g(x) = \left(g_n(x_n)\right)_{n\geq 1}, \tag{3.12}$$

where $x \in X, x_n \in [0,1]$ and $f_n, g_n : [0,1] \to [0,1]$ are such that for $1 \leq n \leq k$,

$$\left|f_n(x_n) - f_n(y_n)\right| = \frac{\left|x_n - y_n\right|}{2}, \tag{3.13}$$

$$\left|g_n(x_n) - g_n(y_n)\right| = \frac{\left|x_n - y_n\right|}{3}, \tag{3.14}$$

and for $n > k$

$$f_n(x_n) = \frac{p_n x_n}{2}, \qquad g_n(x_n) = \frac{p_n x_n}{3}. \tag{3.15}$$

We verify the hypothesis of Theorem 3.1.

(i) Observe that f and g are, clearly, continuous by their definition.

(ii) For $x, y \in$ X, we have

$$\begin{aligned} \|f(x) - f(y)\| &= \sup_{n\geq 1}\left|f_n(x_n) - f_n(y_n)\right|, \\ \|g(x) - g(y)\| &= \sup_{n\geq 1}\left|g_n(\mathrm{x}_n) - g_n(y_n)\right|. \end{aligned} \tag{3.16}$$

Since the sequences $(f_n(x_n))_{n\geq 1}$ and $(g_n(x_n))_{n\geq 1}$ are null sequences, there exists $N \in \mathbb{N}$ such that

$$\begin{aligned} \sup_{n\geq 1}\left|f_n(x_n) - f_n(y_n)\right| &= \left|f_N(x_N) - f_N(y_N)\right|, \\ \sup_{n\geq 1}\left|g_n(x_n) - g_n(y_n)\right| &= \left|g_N(x_N) - g_N(y_N)\right|. \end{aligned} \tag{3.17}$$

Hence

$$\begin{aligned} \left\|f_n(x_n) - f_n(y_n)\right\| &= \left|f_N(x_N) - f_N(y_N)\right| < \left|x_N - y_N\right| = \sup_{n\geq 1}\left|x_n - y_n\right| = \left\|x_n - y_n\right\|, \\ \left\|g_n(x_n) - g_n(y_n)\right\| &= \left|g_N(x_N) - g_N(y_N)\right| < \left|x_N - y_N\right| = \sup_{n\geq 1}\left|x_n - y_n\right| = \left\|x_n - y_n\right\|. \end{aligned} \tag{3.18}$$

This implies that f and g are weakly contractive. Thus f and g are continuous and weakly contractive. Next suppose that for any countable set $U \subseteq X$, we have

$$U = f(U) \cup \left\{g(0_{c_0})\right\}, \tag{3.19}$$

then by the definition of f, we can consider $U \subseteq [0,1]$. Hence closure of U being closed subset of a compact set is compact. Also

$$fg(x) = \left(\frac{(p_n)^2}{2} x_n \right)_{n \geq N} = gf(x) \quad \text{for every } x \in \overline{U}. \tag{3.20}$$

So by Theorem 3.1, f and g have a unique common fixed point.

Corollary 3.4. *Let (X, d) be a metric space, $x_0 \in X$, and $f, g : X \to X$ self-mappings such that for every countable set $U \subseteq X$,*

$$U = f(U) \cup \{g(x_0)\} \Longrightarrow U \text{ is relatively compact}, \tag{3.21}$$

and f, g commute on X. If

(i) *f is continuous and weakly contractive or*

(ii) *g is continuous and weakly contractive with $g(U) \subseteq U$,*

then f and g have a unique common fixed point.

Proof. It is immediate from Theorem 3.1 with $F = d$. □

Corollary 3.5. *Let X be a compact metric space, $x_0 \in X$, and $f, g : X \to X$ self-mappings such that for every countable set $U \subseteq X$,*

$$U = f(U) \cup \{g(x_0)\} \Longrightarrow U \text{ is closed} \tag{3.22}$$

and f, g commute on X. If

(i) *f is continuous and weakly contractive or*

(ii) *g is continuous and weakly F-contractive with $g(U) \subseteq U$,*

then f and g have a unique common fixed point.

Proof. It is immediate from Theorem 3.1. □

Theorem 3.6. *Let X be a topological space, $x_0 \in X$, and $f, g : X \to X$ self-mappings such that for every countable set $U \subseteq X$,*

(1) $U = f(U) \cup \{g(x_0)\} \Longrightarrow U$ is relatively compact;

(2) $U = f^k(U) \cup \{g(x_0)\} \Longrightarrow U$ is relatively compact for some $k \in \mathbb{N}$;

(3) $U = f^k(U) \cup \{g^k(x_0)\} \Longrightarrow U$ is relatively compact for some $k \in \mathbb{N}$.

And f, g commute on X. Further, if

$$\begin{aligned} &(i) \quad f \text{ is continuous and } f^k \text{ weakly } F\text{-contractive or} \\ &(ii) \quad g \text{ is continuous and } g^k \text{ weakly } F\text{-contractive with } g(U) \subseteq U, \end{aligned} \tag{3.23}$$

then f and g have a unique common fixed point.

Proof. Part (3): we proceed as in Theorem 3.1. Let $x_1 = g^k(x_0)$ for some $k \in \mathbb{N}$ and define the sequence $(x_n)_{n\geq 1}$ by setting $x_{n+1} = f^k(x_n)$ for $n \geq 1$. Let $A = \{x_n : n \geq 1\}$. Then

$$A = f^k(A) \cup \left\{g^k(x_0)\right\}, \tag{3.24}$$

so by hypothesis (3), $\overline{A}$ is compact. Define $\varphi : \overline{A} \to \mathbb{R}$ by

$$\varphi(x) = \begin{cases} F(x, f^k(x)) & \text{if } f \text{ is continuous,} \\ F(x, g^k(x)) & \text{if } g \text{ is continuous.} \end{cases} \tag{3.25}$$

Now since F is lsca and if f or g is continuous, then by Remark 2.5 φ would be lsca and hence by Lemma 2.3, φ would have a minimum, say, at $a \in \overline{A}$.

(i) *Suppose that f is continuous and f^kweakly F-contractive*. Then $\varphi(x) = F(x, f^k(x))$ as f is continuous. Now observe that $a \in \overline{A}$, f is continuous, and $f(A) \subsetneq A$ implies that f^k is continuous and $f^k(A) \subsetneq A$ and so $f^k(a) \in \overline{A}$ for some $k \in \mathbb{N}$. We show that $f^k(a) = a$. Suppose that $f^k(a) \neq a$ for any $k \in \mathbb{N}$, then

$$\varphi\left(f^k(a)\right) = F\left(f^k(a), f^k\left(f^k(a)\right)\right) < F\left(a, f^k(a)\right) = \varphi(a), \tag{3.26}$$

a contradiction to the minimality of φ at a. Therefore, $f^k(a) = a$, for some $k \in \mathbb{N}$. One can check that $g(a) = a$. Suppose that $g^k(a) \neq a$, then we have

$$\begin{aligned} \mathrm{F}\left(a, g^k(a)\right) &= F\left(f^k(a), g^k\left(f^k(a)\right)\right) \\ &= F\left(f^k(a), f^k\left(g^k(a)\right)\right) < F\left(a, g^k(a)\right) \end{aligned} \tag{3.27}$$

a contradiction. Thus a is a common fixed point of f^k and g^k and hence of f and g.

(ii) *Suppose that g is continuous and g^k weakly F-contractive* with $g(U) \subsetneq \mathrm{U}$. Then $\varphi(x) = F(x, g^k(x))$ as g is continuous. Put $U = A$. Then $a \in \overline{A}$, g continuous and $g(A) \subsetneq A$ imply that $g^k(a) \in \overline{A}$. We claim that $g^k(a) = a$, for otherwise we will have

$$\varphi\left(g^k(a)\right) = F\left(g^k(a), g^k\left(g^k(a)\right)\right) < F\left(a, g^k(a)\right) = \varphi(a) \tag{3.28}$$

which is a contradiction. Hence the claim follows.

Now suppose that $f^k(a) \neq a$ then we have

$$\begin{aligned} F\left(a, f^k(a)\right) &= F\left(g^k(a), f^k\left(g^k(a)\right)\right) \\ &= F\left(g^k(a), g^k\left(f^k(a)\right)\right) < F\left(a, f^k(a)\right) \end{aligned} \tag{3.29}$$

a contradiction, hence $f^k(a) = a$. Thus a is a common fixed point of f^k and g^k and hence of f and g.

Now we establish the uniqueness of a. Suppose there exists $b \in \overline{A}$ such that $f^k(b) = b = g^k(b)$ for some $k \in \mathbb{N}$. Now if f is continuous and f^k is weakly F-contractive, then we have

$$F(a,b) = F\left(f^k(a), f^k(b)\right) < F(a,b) \tag{3.30}$$

and if g is continuous and g^k is weakly F-contractive, then we have

$$F(a,b) = F\left(g^k(a), g^k(b)\right) < F(a,b) \tag{3.31}$$

which is false. Thus f^k and g^k have a unique common fixed point which obviously is a unique common fixed point of f and g.

Part (2). The conclusion follows if we set $h = g^k$ in part (3).

Part (1). The conclusion follows if we set $S = f^k$ and $T = g^k$ in part (3).

A nice consequence of Theorem 3.6 is the following theorem where X is taken as a Banach space equipped with a transitive binary relation. □

Theorem 3.7. *Let* $X = (X, \|\cdot\|)$ *be a Banach space with a transitive binary relation* $\preccurlyeq$ *such that* $\|x\| \leq \|y\|$ *for* $x, y \in X$ *with* $x \preccurlyeq y$. *Suppose, further, that the mappings* $A, m : X \to X$ *are such that the following conditions are satisfied:*

(i) $0 \preccurlyeq m(x)$ *and* $\|m(x)\| = \|x\|$ *for all* $x \in X$;

(ii) $0 \preccurlyeq x \preccurlyeq y$, *then* $Ax \preccurlyeq Ay$;

(iii) *A is bounded linear operator and* $\|A^k x\| < \|x\|$ *for some* $k \in \mathbb{N}$ *and for all* $x \in X$ *such that* $x \neq 0$ *with* $0 \preccurlyeq x$.

If either

$$\begin{aligned} &(a)\ m(f(x) - f(y)) \preccurlyeq Am(g(x) - g(y)) \text{ and } g \text{ is contractive,} \\ &(b)\ m(g(x) - g(y)) \preccurlyeq Am(f(x) - f(y)) \text{ and } f \text{ is contractive,} \end{aligned} \tag{3.32}$$

for all $x, y \in X$ *with* f, g *commuting on* X *and if one of the conditions, (1)–(3), of Theorem 3.6 holds, then* f *and* g *have a unique common fixed point.*

Proof. (a) Suppose that $m(f(x)-f(y)) \preccurlyeq Am(g(x)-g(y))$ for all $x, y \in X$ with f, g commuting on X and g is contractive. Then we have

$$\begin{aligned} 0 &\preccurlyeq m(f(x) - f(y)) \\ &\preccurlyeq Am(g(x) - g(y)). \end{aligned} \tag{3.33}$$

Next

$$\begin{aligned} 0 &\preccurlyeq m\left(f^2(x) - f^2(y)\right) \\ &\preccurlyeq Am(gf(x) - gf(y)) \\ &= Am(fg(x) - fg(y)) \\ &\preccurlyeq A^2 m(g(x) - g(y)). \end{aligned} \tag{3.34}$$

Therefore, after k-steps, $k \in \mathbb{N}$, we get

$$\begin{aligned} 0 &\preccurlyeq m\left(f^k(x) - f^k(y)\right) \\ &\preccurlyeq A^k m(g(x) - g(y)). \end{aligned} \tag{3.35}$$

Hence,

$$\begin{aligned} \left\| f^k(x) - f^k(y) \right\| &= \left\| m\left(f^k(x) - f^k(y)\right) \right\| \\ &\le \left\| A^k m(g(x) - g(y)) \right\| \\ &< \| m(g(x) - g(y)) \| \\ &= \| g(x) - g(y) \| \\ &\le \| x - y \|. \end{aligned} \tag{3.36}$$

So f^k is weakly contractive. Since f is continuous (as A is bounded and g contractive) by Theorem 3.6, f and g have a unique common fixed point.

(b) Suppose that $m(g(x)-g(y)) \preccurlyeq Am(f(x)-f(y))$ and f is contractive for all $x, y \in X$ with f, g commuting on X and f being contractive. The proof now follows if we mutually interchange f, g in (a) above. □

Theorem 3.8. *Let X be a topological space, $Y \subset Z \subset X$ with Y closed and $x_0 \in Y$. Let $f, g : Y \to Z$ be mappings such that for every countable set $U \subseteq Y$,*

$$f(U) = U \cup \{g(x_0)\} \Longrightarrow U \text{ is relatively compact} \tag{3.37}$$

and f, g commute on X. If f is a homeomorphism and strongly F-expansive, then f and g have a unique common fixed point.

Proof. Suppose that f is a homeomorphism and strongly F-expansive. Let $z, w \in Z$ with $z \neq w$. Then there exists $x, y \in Y$ such that $z = f(x)$ and $w = f(y)$ or $f^{-1}(z) = x$ and $f^{-1}(w) = y$. Since f is strongly F-expansive, we have

$$F(z, w) = F(f(x), f(y)) > F(x, y) = F\left(f^{-1}(z), f^{-1}(w)\right), \tag{3.38}$$

or

$$F\left(f^{-1}(z), f^{-1}(w)\right) < F(z, w). \tag{3.39}$$

So f^{-1} is a weakly F-contractive mapping. Choose any countable subset V of Z and set $B = V \cap Y$. Suppose that

$$B = f^{-1}(B) \cup \{g(x_0)\}. \tag{3.40}$$

Then $f^{-1}(B) = U$ for some $U \subseteq Y$ and we get

$$f(U) = U \cup \{g(x_0)\}. \tag{3.41}$$

So by hypothesis $\overline{U}$ is compact and since f is a homeomorphism, $(f(\overline{U}) =)\overline{B}$ is compact. Since $fg(x) = gf(x)$ for every $x \in \overline{U}$ and $f^{-1}(B) = U$, we have

$$f^{-1}g(x) = f^{-1}g\left(ff^{-1}(x)\right) = f^{-1}(gf)\left(f^{-1}(x)\right) = f^{-1}(fg)\left(f^{-1}(x)\right) = gf^{-1}(x) \tag{3.42}$$

for every $x \in \overline{B}$. Thus

$$B = f^{-1}(B) \cup \{g(x_0)\} \Longrightarrow B \text{ is relatively compact} \tag{3.43}$$

and $f^{-1}g(x) = gf^{-1}(x)$ for every $x \in \overline{B}$. Since f^{-1} is continuous and weakly F-contractive, by Theorem 3.1, the mappings f^{-1} and g have a unique common fixed point, say, $a \in \overline{B}$. Since $f^{-1}(a) = a$ implies that $a = f(a)$, so a is a unique common fixed point of f and g. □

The following example illustrates Theorem 3.8.

Example 3.9. Let $X = \mathbb{R}^2$ with the River metric $d : X \times X \to \mathbb{R}_+$ defined by

$$d(x, y) = \begin{cases} \delta(x, y) & \text{if } x, y \text{ are collinear,} \\ \delta(x, 0) + \delta(0, y), & \text{otherwise,} \end{cases} \tag{3.44}$$

where $x = (x_1, y_1)$, $y = (x_2, y_2)$, and δ denotes the Euclidean metric on X. Then X is a topological space with a topology induced by the metric d. Consider the sets Y, Z defined by

$$\begin{aligned} Y &= \left\{(u, v) \in \mathbb{R}^2 : u = v \in [0, 1]\right\}, \\ Z &= \left\{(u, v) \in \mathbb{R}^2 : u = v \in \left[0, \frac{3}{2}\right]\right\}. \end{aligned} \tag{3.45}$$

Let the mappings $f, g : Y \to Z$ be defined by $f(u,v) = ((3/2)u, (3/2)v)$ and $g(u,v) = ((2/3)u, (2/3)v)$ for $(u,v) \in Y$. Then f is clearly a homeomorphism and for an arbitrary countable subset A of Y and $x_0 = (0,0) \in Y$,

$$f(A) = A \cup \{g(x_0)\}. \tag{3.46}$$

If and only if $A = \{(0,0)\}$. Indeed, if $(u,v) \in A$ such that $(u,v) \neq (0,0)$, then

$$f(A) = \frac{3}{2}A \neq A \cup \{(0,0)\} = A \cup \{g(x_0)\}. \tag{3.47}$$

Further, $fg(u,v) = gf(u,v)$ for every $(u,v) \in Y$. Set $F(u,v) = \rho(u,v)$ where $\rho : X \times X \to \mathbb{R}_+$ is the Radial metric defined by

$$\rho(x,y) = \begin{cases} |y_1 - y_2| & \text{if } x_1 = x_2, \\ |y_1| + |y_2| + |x_1 - x_2| & \text{if } x_1 \neq x_2, \end{cases} \tag{3.48}$$

and $x = (x_1, y_1)$; $y = (x_2, y_2)$. Now for $x, y \in Y$, since

$$F(f(x), f(y)) = \rho(f(x), f(y)) = \frac{3}{2}\rho(x,y) > \rho(x,y) = F(x,y), \tag{3.49}$$

f is strongly F-expansive. Also $F = \rho : (X,d) \times (X,d) \to \mathbb{R}_+$ is lower semi-continuous and hence *lsca*. Thus all the conditions of Theorem 3.8 are satisfied and f and g have a unique common fixed point.

4. Occasionally Banach Operator Pair and Weak F-Contractions

In this section, we define a new class of noncommuting self-maps and prove some common fixed point results for this new class of maps.

The pair (T, I) is called a Banach operator pair [10] if the set $F(I)$ is T-invariant, namely, $T(F(I)) \subseteq F(I)$. Obviously, commuting pair (T, I) is a Banach operator pair but converse is not true, in general; see [10–13]. If (T, I) is a Banach operator pair, then (I, T) need not be a Banach operator pair.

Definition 4.1. The pair (T, I) is called *occasionally Banach operator pair* if

$$d(u, Tu) \leq \operatorname{diam} F(I) \text{ for some } u \in F(I). \tag{4.1}$$

Clearly, Banach operator pair (BOP) (T, I) is occasionally Banach operator pair (OBOP) but not conversely, in general.

Example 4.2. Let $X = \mathbb{R} = M$ with usual norm. Define $I, T : M \to M$ by $Ix = x^2$ and $Tx = 2 - x^2$, for $x \neq -1$ and $I(-1) = T(-1) = 1/2$. $F(I) = \{0, 1\}$ and $C(I, T) = \{-1, 1\}$. Obviously (T, I) is OBOP but not BOP as $T0 = 2 \notin F(I)$. Further, (T, I) is not weakly compatible and hence not commuting.

Example 4.3. Let $X = R$ with usual norm and $M = [0,1]$. Define $T, I : M \to M$ by

$$Tx = \begin{cases} \frac{1}{2}, & \text{if } x \in \left[0, \frac{1}{4}\right], \\ 1 - 2x, & \text{if } x \in \left[\frac{1}{4}, \frac{1}{2}\right], \\ 0, & \text{if } x \in \left[\frac{1}{2}, 1\right], \end{cases} \tag{4.2}$$

$$Ix = \begin{cases} 2x, & \text{if } x \in \left[0, \frac{1}{2}\right], \\ 1, & \text{if } x \in \left[\frac{1}{2}, 1\right]. \end{cases} \tag{4.3}$$

Here $F(I) = \{0,1\}$ and $T(0) = 1/2 \notin F(I)$ implies that (T, I) is not Banach operator pair. Similarly, (I, T) is not Banach operator pair. Further,

$$|0 - T(0)| = \left|0 - \frac{1}{2}\right| = \frac{1}{2} \leq 1 = \operatorname{diam}(F(I)) \tag{4.4}$$

imply that (T, I) is OBOP. Further, note that $C(T, I) = \{1/4\}$ and $TI(1/4) \neq IT(1/4)$. Hence $\{T, I\}$ is not occasionally weakly compatible pair.

Definition 4.4. Let X be a nonempty set and $d : X \times X \to [0, \infty)$ be a mapping such that

$$d(x, y) = 0 \text{ if and only if } x = y. \tag{4.5}$$

For a space (X, d) satisfying (4.5) and $A \subseteq X$, the diameter of A is defined by

$$\operatorname{diam}(A) = \sup\{d(x, y) : x, y \in A\}. \tag{4.6}$$

Here we extend this concept to the space (X, d) satisfying condition (4.5).

Definition 4.5. Let (X, d) be a space satisfying (4.5). The pair (T, I) is called *occasionally Banach operator pair* on X iff there is a point u in X such that $u \in F(I)$ and

$$d(u, Tu) \leq \operatorname{diam}(F(I)), \qquad d(Tu, u) \leq \operatorname{diam}(F(I)). \tag{4.7}$$

Theorem 4.6. *Let X be a topological space, $x_0 \in X$, and $f, g : X \to X$ self-mappings such that for every countable set $U \subseteq X$,*

$$U = f(U) \cup \{x_0\} \Longrightarrow U \text{ is relatively compact}. \tag{4.8}$$

If f is continuous and weakly F-contractive, F satisfies condition (4.5), and the pair (g, f) is occasionally Banach operator pair, then f and g have a unique common fixed point.

Proof. By Corollary 3.2, $F(f)$ is a singleton. Let $u \in F(f)$. Then, by our hypothesis,

$$d(u, gu) \leq \operatorname{diam} \quad F(f) = 0. \tag{4.9}$$

Therefore, $u = gu = fu$. That is, u is unique common fixed point of f and g. □

Corollary 4.7. *Let (X, d) be a metric space, $x_0 \in X$, and $f, g : X \to X$ self-mappings such that for every countable set $U \subseteq X$,*

$$U = f(U) \cup \{x_0\} \Longrightarrow U \text{ is relatively compact}. \tag{4.10}$$

If f is continuous and weakly contractive and the pair (g, f) is occasionally Banach operator pair, then f and g have a unique common fixed point.

Proof. It is immediate from Theorem 4.6 with $F = d$. □

Corollary 4.8. *Let X be a compact metric space, $x_0 \in X$, and $f, g : X \to X$ self-mappings such that for every countable set $U \subseteq X$,*

$$U = f(U) \cup \{x_0\} \Longrightarrow U \text{ is closed}. \tag{4.11}$$

If f is continuous and weakly contractive and the pair (g, f) is occasionally Banach operator pair, then f and g have a unique common fixed point.

Proof. It is immediate from Theorem 4.6. □

Theorem 4.6 holds for a Banach operator pair without condition (4.5) as follows.

Theorem 4.9. *Let X be a topological space, $x_0 \in X$, and $f, g : X \to X$ self-mappings such that for every countable set $U \subseteq X$,*

$$U = f(U) \cup \{x_0\} \Longrightarrow U \text{ is relatively compact}. \tag{4.12}$$

If f is continuous and weakly F-contractive and the pair (g, f) is a Banach operator pair, then f and g have a unique common fixed point.

Proof. By Corollary 3.2, $F(f)$ is a singleton. Let $u \in F(f)$. As (g, f) is a Banach operator pair, by definition $g(F(f)) \subset F(f)$. Thus $gu \in F(f)$ and hence $u = gu = fu$. That is, u is unique common fixed point of f and g. □

Corollary 4.10. *Let (X, d) be a metric space, $x_0 \in X$, and $f, g : X \to X$ self-mappings such that for every countable set $U \subseteq X$,*

$$U = f(U) \cup \{x_0\} \Longrightarrow U \text{ is relatively compact}. \tag{4.13}$$

If f is continuous and weakly contractive and the pair (g, f) is a Banach operator pair, then f and g have a unique common fixed point.

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