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Research Article

Multilinear Riesz Potential on Morrey-Herz Spaces with Non-Doubling Measures

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The authors consider the multilinear Riesz potential operator defined by $\mathcal{I}_{\alpha,m}(\overrightarrow{f})(x) = \int_{(\mathbb{R}^d)^m}(f_1(y_1)f_2(y_2)\cdots f_m(y_m)/|(x-y_1,\ldots,x-y_m)|^{mn-\alpha})d\mu(y_1)\cdots d\mu(y_m)$, where $\overrightarrow{f}$ denotes the m-tuple $(f_1, f_2, \ldots, f_m)$, m, n the nonnegative integers with $n \geq 2$, $m \geq 1$, $0 < \alpha < mn$, and μ is a nonnegative n-dimensional Borel measure. In this paper, the boundedness for the operator $\mathcal{I}_{\alpha,m}$ on the product of homogeneous Morrey-Herz spaces in nonhomogeneous setting is found.

1. Introduction

Let $B(x,r)$ denote a ball centered at $x \in \mathbb{R}^d$ with radius $r > 0$, and for any $\ell > 0$, $B(x,\ell r)$ will mean the ball with the same center as $B(x,r)$ and with radius ℓr. A Borel measure μ on $\mathbb{R}^d$ is called a doubling measure if it satisfies the so-called doubling condition; that is, there exists a constant $C > 0$ such that

$$\mu(B(x,2r)) \leq C\mu(B(x,r)) \tag{1.1}$$

for every ball $B(x,r) \subset \mathbb{R}^d$. The doubling condition is a key feature for a homogeneous (metric) measure space. Many classical theories in Fourier analysis have been generalized to the homogeneous setting without too much difficulties. In the last decade, however, some researchers found that many results are still true without the assumption of the doubling condition on μ (see, e.g., [1–4]). This fact has encouraged other researchers to study various theories in the nonhomogeneous setting. By a nonhomogeneous space we mean a (metric) measure space, here we will consider only $\mathbb{R}^d$, equipped with a nonnegative n-dimensional Borel measure μ, that is, a measure satisfying the growth condition

$$\mu(B(x,r)) \leq Cr^n \tag{1.2}$$

for any ball $B(x,r) \subset \mathbb{R}^d$ and n is a fixed real number such that $0 < n \leq d$. Unless otherwise stated, throughout this paper we will always work in the nonhomogeneous setting.

As one of the most important operators in harmonic analysis and its applications, the Riesz potential operator I_α defined by

$$I_\alpha f(x) := \int_{\mathbb{R}^d} \frac{f(x-y)}{|y|^{n-\alpha}} d\mu(y), \quad 0 < \alpha < n, \tag{1.3}$$

was studied by García-Cuerva and Martell [1] in 2001. García-Cuerva and Martell proved that I_α is bounded from $L^p(\mu)$ to $L^q(\mu)$ for all $p > 1$ and $1/q = 1/p - \alpha/n > 0$ and that I_α is bounded from $L^1(\mu)$ to $WL^{n/(n-\alpha)}(\mu)$. Here $L^p(\mu)$ and $WL^p(\mu)$ denote the Lebesgue spaces and weak Lebesgue spaces with measure μ, respectively.

Simultaneously many classical multilinear operators on Euclidean spaces with Lebesgue measure have been generalized for nondoubling measures, that is, the case nonhomogeneous setting see [3, 4]. For example, based on the work of Kenig and Stein [5], Lian and Wu [4] studied multilinear Riesz potential operator

$$\mathcal{I}_{\alpha,m}\left(\vec{f}\right)(x) = \int_{(\mathbb{R}^d)^m} \frac{f_1(y_1) f_2(y_2) \cdots f_m(y_m)}{|(x-y_1,\ldots,x-y_m)|^{mn-\alpha}} d\mu(y_1) \cdots d\mu(y_m) \tag{1.4}$$

in the nonhomogeneous case, where, and throughout this paper, we denote by $\vec{f}$ the m-tuple $(f_1, f_2, \ldots, f_m)$, m, n the nonnegative integers with $n \geq 2, m \geq 1$. They obtained the following.

Proposition 1.1 (see [4]). *Let $m \in \mathbb{N}$, $1/s = 1/r_1 + 1/r_2 + \cdots + 1/r_m - \alpha/n > 0$ with $0 < \alpha < mn$, $1 \leq r_i \leq \infty$, then*

(a) *if each $r_i > 1$, $\mathcal{I}_{\alpha,m}$ is bounded from $L^{r_1}(\mu) \times \cdots \times L^{r_m}(\mu)$ to $L^s(\mu)$,*

(b) *if $r_i = 1$ for some i, $\mathcal{I}_{\alpha,m}$ is bounded from $L^{r_1}(\mu) \times \cdots \times L^{r_m}(\mu)$ to $WL^s(\mu)$.*

Obviously, it is Lemma 7 in [5] if μ is the Lebesgue measure in the proposition above and it is a multilinear setting of the result of García-Cuerva and Martell [1].

In addition, in the article [6–8], we have obtained the boundedness of the operator $\mathcal{I}_{\alpha,m}$ on the product of Morrey type spaces, (weak) homogeneous Morrey-Herz spaces, and Herz type Hardy spaces in the classical case and extended the result of Kenig and Stein. As a continuation of previous work in [4, 6–8], in this paper, we will study the operator $\mathcal{I}_{\alpha,m}$ in the product of (weak) homogeneous Morrey-Herz spaces in the nonhomogeneous setting.

The definitions of the (weak) homogeneous Morrey-Herz spaces ($WM\dot{K}_{p,q}^{\sigma,\lambda}(\mu)$) $M\dot{K}_{p,q}^{\sigma,\lambda}(\mu)$ and (weak) homogeneous Herz spaces ($W\dot{K}_q^{\sigma,p}(\mu)$)$\dot{K}_q^{\sigma,p}(\mu)$ will be given in Section 2, here we only point out that $WM\dot{K}_{p,p}^{0,0}(\mu) = WL^p(\mu)$ and $M\dot{K}_{p,p}^{0,0}(\mu) = L^p(\mu)$ for $1 \le p < \infty$.

We will establish the following boundedness of the multilinear Riesz potential operator $\mathcal{I}_{\alpha,m}$ on the homogeneous Morrey-Herz spaces.

Theorem 1.2. *Let $0 < \alpha < mn$, $0 \le \lambda_i < n - \alpha/m$, $0 < p_i \le \infty$, $1 < q_i < \infty$, and $\lambda_i + \alpha/m - n/q_i < \sigma_i < n(1 - 1/q_i)$ for $i = 1, 2, \ldots, m$. Suppose that $\lambda = \lambda_1 + \cdots + \lambda_m$, $\sigma = \sigma_1 + \cdots + \sigma_m$, $1/p = 1/p_1 + \cdots + 1/p_m - \alpha/n > 0$, $1/q = 1/q_1 + \cdots + 1/q_m - \alpha/n > 0$, then*

$$\left\| \mathcal{I}_{\alpha,m}\left(\overrightarrow{f}\right) \right\|_{M\dot{K}_{p,q}^{\sigma,\lambda}(\mu)} \le C \prod_{i=1}^{m} \| f_i \|_{M\dot{K}_{p_i,q_i}^{\sigma_i,\lambda_i}(\mu)} \tag{1.5}$$

with a constant $C > 0$ independent of $\overrightarrow{f}$.

In the case $\sigma_i = n(1 - 1/q_i)$ for $i = 1, 2, \ldots, m$, we will use the weak homogeneous Morrey-Herz spaces $WM\dot{K}_{p,q}^{\sigma,\lambda}(\mu)$ and weak homogeneous Herz spaces $W\dot{K}_q^{\sigma,p}(\mu)$ to derive the following boundedness for the operator $\mathcal{I}_{\alpha,m}$.

Theorem 1.3. *Let $0 < \alpha < mn$, $0 \le \lambda_i < n - \alpha/m$, $0 < p_i \le 1$ and $1 \le q_i < \infty$ for $i = 1, 2, \ldots, m$. Suppose that $\lambda = \lambda_1 + \cdots + \lambda_m$, $1/p = 1/p_1 + \cdots + 1/p_m - \alpha/n > 0$, $1/q = 1/q_1 + \cdots + 1/q_m - \alpha/n > 0$, then*

$$\left\| \mathcal{I}_{\alpha,m}\left(\overrightarrow{f}\right) \right\|_{WM\dot{K}_{p,q}^{n(m-1/q)-\alpha,\lambda}(\mu)} \le C \prod_{i=1}^{m} \| f_i \|_{M\dot{K}_{p_i,q_i}^{n(1-1/q_i),\lambda_i}(\mu)} \tag{1.6}$$

with a constant $C > 0$ independent of $\overrightarrow{f}$.

Remark 1.4. The restriction $0 < p_i \le 1$ in Theorem 1.3 cannot be removed see [9] for an counter-example when $m = 1$ and $n = d$.

In addition, we remark that the (weak) homogeneous Morrey-Herz spaces generalize the (weak) homogeneous Herz spaces. Particularly, $M\dot{K}_{p,q}^{\sigma,0}(\mu) = \dot{K}_q^{\sigma,p}(\mu)$ and $WM\dot{K}_{p,q}^{\sigma,0}(\mu) = W\dot{K}_q^{\sigma,p}(\mu)$ for $0 < p, q < \infty$ and $\sigma \in \mathbb{R}$. Moreover, we have $\dot{K}_p^{\sigma,p}(\mu) = L_{|x|^\sigma}^p(\mu)$, the weighted L^p spaces $L_w^p(\mu) = \{f : fw \in L^p(\mu)\}$ for $1 \le p < \infty$ and $\sigma \in \mathbb{R}$, for details, see Section 2.

Hence, it is easy to obtain the following corollaries from the theorems above.

Corollary 1.5. *Let $0 < \alpha < mn$, $1/p = 1/p_1 + \cdots + 1/p_m - \alpha/n > 0$, $1/q = 1/q_1 + \cdots + 1/q_m - \alpha/n > 0$.*

(i) *If $0 < p_i \le \infty$, $1 < q_i < \infty$, $\sigma = \sigma_1 + \cdots + \sigma_m$ with $\alpha/m - n/q_i < \sigma_i < n(1 - 1/q_i)$ for*

$i = 1, 2, \ldots, m$. Then

$$\left\| \mathcal{I}_{\alpha,m}\left(\vec{f}\right) \right\|_{\dot{K}_q^{\sigma,p}(\mu)} \leq C \prod_{i=1}^{m} \|f_i\|_{\dot{K}_{q_i}^{\sigma_i,p_i}(\mu)} \tag{1.7}$$

with a constant $C > 0$ independent of $\vec{f}$.

(ii) If $0 < p_i \leq 1$ and $1 \leq q_i < \infty$ for $i = 1, 2, \ldots, m$, then

$$\left\| \mathcal{I}_{\alpha,m}(\vec{f}) \right\|_{W\dot{K}_q^{n(m-1/q)-\alpha,p}(\mu)} \leq C \prod_{i=1}^{m} \|f_i\|_{\dot{K}_{q_i}^{n(1-1/q_i),p_i}(\mu)} \tag{1.8}$$

with a constant $C > 0$ independent of $\vec{f}$.

Corollary 1.6. *Let $0 < \alpha < mn$, $1/p = 1/p_1 + \cdots + 1/p_m - \alpha/n > 0$ with $1 < p_i < \infty$, $\sigma = \sigma_1 + \cdots + \sigma_m$ with $\alpha/m - n/p_i < \sigma_i < n(1 - 1/p_i)$ for $i = 1, 2, \ldots, m$, then*

$$\left\| \mathcal{I}_{\alpha,m}(\vec{f}) \right\|_{L^p_{|x|^\sigma}(\mu)} \leq C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}_{|x|^{\sigma_i}}(\mu)} \tag{1.9}$$

with a constant $C > 0$ independent of $\vec{f}$.

Throughout this paper, the letter C always remains to denote a positive constant that may vary at each occurrence but is independent of all essential variables.

2. The Definitions of Some Function Spaces

We start with some notations and definitions. Here and in what follows, denote by $B_k = B(0, 2^k) = \{x \in \mathbb{R}^d : |x| \leq 2^k\}$, $E_k = B_k \setminus B_{k-1}$, and $\chi_k = \chi_{E_k}$ for $k \in \mathbb{Z}$ the characteristic function of the set E_k.

Definition 2.1. Let $\sigma \in \mathbb{R}$, $0 < p \leq \infty$, and $0 < q < \infty$. The homogeneous Herz spaces $\dot{K}_q^{\sigma,p}(\mu)$ are defined to be the following space of functions:

$$\dot{K}_q^{\sigma,p}(\mu) = \left\{ f \in L^q_{\mathrm{loc}}\left(\mathbb{R}^d \setminus \{0\}\right) : \|f\|_{\dot{K}_q^{\sigma,p}(\mu)} < \infty \right\}, \tag{2.1}$$

where

$$\|f\|_{\dot{K}_q^{\sigma,p}(\mu)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\sigma p} \|f\chi_k\|^p_{L^q(\mu)} \right\}^{1/p}, \tag{2.2}$$

and the usual modification should be made when $p = \infty$.

Definition 2.2. Let $\sigma \in \mathbb{R}$, $0 < p \leq \infty$ and $0 < q < \infty$. The weak homogeneous Herz spaces $W\dot{K}_q^{\sigma,p}(\mu)$ are defined by

$$W\dot{K}_q^{\sigma,p}(\mu) = \left\{ f \text{ is a measurable function on } \mathbb{R}^d \text{ and } \|f\|_{W\dot{K}_q^{\sigma,p}(\mu)} < \infty \right\}, \tag{2.3}$$

where

$$\|f\|_{W\dot{K}_q^{\sigma,p}(\mu)} = \sup_{\gamma>0} \gamma \left\{ \sum_{k=-\infty}^{\infty} 2^{k\sigma p} \mu(\{x \in E_k : |f(x)| > \gamma\})^{p/q} \right\}^{1/p}, \tag{2.4}$$

and the usual modification should be made when $p = \infty$.

Definition 2.3. Let $\sigma \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q < \infty$, and $0 \leq \lambda < \infty$. The homogeneous Morrey-Herz spaces $M\dot{K}_{p,q}^{\sigma,\lambda}(\mu)$ are defined by

$$M\dot{K}_{p,q}^{\sigma,\lambda}(\mu) = \left\{ f \in L_{\mathrm{loc}}^q(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}_{p,q}^{\sigma,\lambda}(\mu)} < \infty \right\}, \tag{2.5}$$

where

$$\|f\|_{M\dot{K}_{p,q}^{\sigma,\lambda}(\mu)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\sigma p} \|f\chi_k\|_{L^q(\mu)}^p \right\}^{1/p}, \tag{2.6}$$

and the usual modifications should be made when $p = \infty$.

Definition 2.4. Let $\sigma \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q < \infty$ and $0 \leq \lambda < \infty$. The weak homogeneous Morrey-Herz spaces $WM\dot{K}_{p,q}^{\sigma,\lambda}(\mu)$ are defined by

$$WM\dot{K}_{p,q}^{\sigma,\lambda}(\mu) = \left\{ f \text{ is a measurable function on } \mathbb{R}^n \text{ and } \|f\|_{WM\dot{K}_{p,q}^{\sigma,\lambda}(\mu)} < \infty \right\}, \tag{2.7}$$

where

$$\|f\|_{WM\dot{K}_{p,q}^{\sigma,\lambda}(\mu)} = \sup_{\gamma>0} \gamma \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\sigma p} \mu(\{x \in E_k : |f(x)| > \gamma\})^{p/q} \right\}^{1/p}, \tag{2.8}$$

and the usual modifications should be made when $p = \infty$.

3. Proof of Theorems 1.2 and 1.3

Without loss of generality, in order to simplify the proof, we only consider the situation when $m = 2$. Actually, the similar procedure works for all $m \in \mathbb{N}$.

Indeed, we decompose f_i as

$$f_i(x) = \sum_{l_i=-\infty}^{\infty} f_i(x)\chi_{l_i}(x) =: \sum_{l_i=-\infty}^{\infty} f_{l_i}(x), \quad i = 1,2,\ l_i \in \mathbb{Z}. \tag{3.1}$$

To shorten the formulas below, we set

$$\begin{aligned}
\Lambda_1 &= \{(l_1,l_2) : l_1,l_2 \le k-2\},\\
\Lambda_2 &= \{(l_1,l_2) : l_1 \le k-2, k-1 \le l_2 \le k+1\},\\
\Lambda_3 &= \{(l_1,l_2) : l_1 \le k-2, l_2 \ge k+2\},\\
\Lambda_4 &= \{(l_1,l_2) : k-1 \le l_1 \le k+1, l_2 \le k-2\},\\
\Lambda_5 &= \{(l_1,l_2) : k-1 \le l_1, l_2 \le k+1\},\\
\Lambda_6 &= \{(l_1,l_2) : k-1 \le l_1 \le k+1, l_2 \ge k+2\},\\
\Lambda_7 &= \{(l_1,l_2) : l_1 \ge k+2, l_2 \le k-2\},\\
\Lambda_8 &= \{(l_1,l_2) : l_1 \ge k+2, k-1 \le l_2 \le k+1\},\\
\Lambda_9 &= \{(l_1,l_2) : l_1,l_2 \ge k+2\}.
\end{aligned} \tag{3.2}$$

It is easy to see that the case for $(l_1,l_2) \in \Lambda_2$ is analogous to the case for $(l_1,l_2) \in \Lambda_4$, the case for $(l_1,l_2) \in \Lambda_3$ is similar to the case for $(l_1,l_2) \in \Lambda_7$, and the case for $(l_1,l_2) \in \Lambda_6$ is analogous to the case for $(l_1,l_2) \in \Lambda_8$, respectively. Thus, by the symmetry of f_1 and f_2 in the operator $\mathcal{I}_{\alpha,2}$, we will only discuss the cases for (l_1,l_2) belong to Λ_1, Λ_2, Λ_3, Λ_5, Λ_6 and Λ_9, respectively.

By a direct computation, we have the following fact that, for $x \in E_k$,

$$\left|\mathcal{I}_{\alpha,2}(f_{l_1}, f_{l_2})(x)\right| \le \begin{cases} C2^{-k(2n-\alpha)}\|f_{l_1}\|_{L^1(\mu)}\|f_{l_2}\|_{L^1(\mu)} & \text{if } (l_1,l_2) \in \Lambda_1, \Lambda_2,\\ C2^{(k+l_2)(\alpha/2-n)}\|f_{l_1}\|_{L^1(\mu)}\|f_{l_2}\|_{L^1(\mu)} & \text{if } (l_1,l_2) \in \Lambda_3, \Lambda_6,\\ C2^{(l_1+l_2)(\alpha/2-n)}\|f_{l_1}\|_{L^1(\mu)}\|f_{l_2}\|_{L^1(\mu)} & \text{if } (l_1,l_2) \in \Lambda_9. \end{cases} \tag{3.3}$$

We will use estimates (3.3) in the proof of theorems below. In addition, we always let $1/g = 1/q_1 + 1/q_2$, $1/l = 1/p_1 + 1/p_2$, $1/h_i = 1/q_i - \alpha/2n$ for $i = 1,2$, and use the notations

$$G_i(s) = 2^{s[n(1/q_i-1)+\sigma_i]}, \quad H_i(s) = 2^{s[n/q_i-\alpha/2+\sigma_i]}, \quad i = 1,2. \tag{3.4}$$

It is easy to see that $\sum_{s=2}^{\infty} G_i(s) < \infty$ when $n(1/q_i - 1) + \sigma_i < 0$, and that $\sum_{s=2}^{\infty} H_2(-s) + \sum_{s=2}^{\infty} 2^{s\lambda_2} H_2(-s) < \infty$ when $\alpha/2 - n/q_2 \le \lambda_2 + \alpha/2 - n/q_2 < \sigma_2$.

We will also use repeatedly the inequality $(\sum_k |a_\kappa|)^\gamma \le \sum_k |a_\kappa|^\gamma$ for $0 < \gamma \le 1$.

Now we are ready to the proof of Theorem 1.2. Suppose that $f_1 \times f_2 \in M\dot{K}_{p_1,q_1}^{\sigma_1,\lambda_1}(\mu) \times M\dot{K}_{p_2,q_2}^{\sigma_2,\lambda_2}(\mu)$, by the decomposition of f_i above, we get

$$\begin{aligned} \left\| \mathcal{I}_{\alpha,2}\left(\overrightarrow{f}\right) \right\|_{M\dot{K}_{p,q}^{\sigma,\lambda}(\mu)} &= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\sigma p} \left\| \mathcal{I}_{\alpha,2}\left(\overrightarrow{f}\right)\chi_k \right\|_{L^q(\mu)}^p \right\}^{1/p} \\ &\le \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\sigma p} \left\| \sum_{l_1,l_2=-\infty}^{\infty} \left|\mathcal{I}_{\alpha,2}(f_{l_1}, f_{l_2})\right|\chi_k \right\|_{L^q(\mu)}^p \right\}^{1/p} \\ &\le C\sum_{\ell=1}^{9} \mathcal{V}_\ell, \end{aligned} \tag{3.5}$$

where

$$\mathcal{V}_\ell = \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\sigma p} \left\| \sum_{(l_1,l_2)\in\Lambda_\ell} \left|\mathcal{I}_{\alpha,2}(f_{l_1}, f_{l_2})\right|\chi_k \right\|_{L^q(\mu)}^p \right\}^{1/p}. \tag{3.6}$$

To estimate the term $\mathcal{V}_1$, we first note that inequality (3.3) and the growth condition (1.2) of μ imply that

$$\begin{aligned} \left\| \sum_{(l_1,l_2)\in\Lambda_1} \left|\mathcal{I}_{\alpha,2}(f_{l_1}, f_{l_2})\right|\chi_k \right\|_{L^q(\mu)} &\le C\|\chi_k\|_{L^q(\mu)} \prod_{i=1}^{2} \left(\sum_{l_i=-\infty}^{k-2} 2^{-k(n-\alpha/2)} \|f_{l_i}\|_{L^1(\mu)} \right) \\ &= C\mu\left(B\left(0,2^k\right)\right)^{1/q} \prod_{i=1}^{2} \left(\sum_{l_i=-\infty}^{k-2} 2^{-k(n-\alpha/2)} \|f_{l_i}\|_{L^1(\mu)} \right) \\ &\le C\prod_{i=1}^{2} \left(\sum_{l_i=-\infty}^{k-2} 2^{-k(n-\alpha/2)+kn/h_i} \|f_{l_i}\|_{L^1(\mu)} \right). \end{aligned} \tag{3.7}$$

Hence, by the fact $p < l$, the Cauchy inequality and the growth condition (1.2) of μ, we can show that

$$\begin{aligned}
\mho_1 &\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\sigma p}\prod_{i=1}^{2}\left(\sum_{l_i=-\infty}^{k-2} 2^{-k(n-\alpha/2)+kn/h_i}\|f_{l_i}\|_{L^1(\mu)}\right)^p\right\}^{1/p}\\
&\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\sigma p}\prod_{i=1}^{2}\left(\sum_{l_i=-\infty}^{k-2} 2^{kn(1/q_i-1)}\|f_{l_i}\|_{L^1(\mu)}\right)^p\right\}^{1/p}\\
&\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} \prod_{i=1}^{2}\left(2^{k\sigma_i}\sum_{l_i=-\infty}^{k-2} 2^{kn(1/q_i-1)}\|f_{l_i}\|_{L^1(\mu)}\right)^p\right\}^{1/p}\\
&\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} \prod_{i=1}^{2}\left(2^{k\sigma_i}\sum_{l_i=-\infty}^{k-2} 2^{kn(1/q_i-1)}\|f_{l_i}\|_{L^1(\mu)}\right)^l\right\}^{1/l}\\
&\le C \sup_{k_0\in\mathbb{Z}} \prod_{i=1}^{2} 2^{-k_0\lambda_i}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{l_i=-\infty}^{k-2} 2^{kn(1/q_i-1)+k\sigma_i}\|f_{l_i}\|_{L^1(\mu)}\right)^{p_i}\right\}^{1/p_i}\\
&\le C \sup_{k_0\in\mathbb{Z}} \prod_{i=1}^{2} 2^{-k_0\lambda_i}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{l_i=-\infty}^{k-2} 2^{kn(1/q_i-1)+k\sigma_i}\mu\left(B\left(0,2^{l_i}\right)\right)^{1-1/q_i}\|f_{l_i}\|_{L^{q_i}(\mu)}\right)^{p_i}\right\}^{1/p_i}\\
&\le C \sup_{k_0\in\mathbb{Z}} \prod_{i=1}^{2} 2^{-k_0\lambda_i}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{l_i=-\infty}^{k-2} 2^{kn(1/q_i-1)+k\sigma_i}2^{l_i n(1-1/q_i)}\|f_{l_i}\|_{L^{q_i}(\mu)}\right)^{p_i}\right\}^{1/p_i}\\
&=: C \sup_{k_0\in\mathbb{Z}} \mho_{11}(k_0)\times\mho_{12}(k_0),
\end{aligned} \tag{3.8}$$

where

$$\mho_{1i}(k_0) = 2^{-k_0\lambda_i}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{l_i=-\infty}^{k-2} G_i(k-l_i)2^{l_i\sigma_i}\|f_{l_i}\|_{L^{q_i}(\mu)}\right)^{p_i}\right\}^{1/p_i}. \tag{3.9}$$

In case $0 < p_i \leq 1$, using the fact $n(1/q_i - 1) + \sigma_i < 0$, we have

$$\begin{aligned}
\mho_{1i}(k_0) &\leq 2^{-k_0\lambda_i}\left\{\sum_{k=-\infty}^{k_0}\sum_{l_i=-\infty}^{k-2} G_i\left[p_i(k-l_i)\right]2^{l_i\sigma_i p_i}\left\|f_{l_i}\right\|_{L^{q_i}(\mu)}^{p_i}\right\}^{1/p_i} \\
&\leq 2^{-k_0\lambda_i}\left\{\sum_{l_i=-\infty}^{k_0-2}\sum_{k=l_i+2}^{k_0} G_i\left[p_i(k-l_i)\right]2^{l_i\sigma_i p_i}\left\|f_{l_i}\right\|_{L^{q_i}(\mu)}^{p_i}\right\}^{1/p_i} \\
&\leq 2^{-k_0\lambda_i}\left\{\sum_{l_i=-\infty}^{k_0-2} 2^{l_i\sigma_i p_i}\left\|f_{l_i}\right\|_{L^{q_i}(\mu)}^{p_i}\left(\sum_{k=l_i+2}^{k_0} G_i\left[p_i(k-l_i)\right]\right)\right\}^{1/p_i} \\
&\leq 2^{-k_0\lambda_i}\left\{\sum_{l_i=-\infty}^{k_0-2} 2^{l_i\sigma_i p_i}\left\|f_{l_i}\right\|_{L^{q_i}(\mu)}^{p_i}\left(\sum_{s=2}^{\infty} G_i(sp_i)\right)\right\}^{1/p_i} \\
&\leq C2^{-k_0\lambda_i}\left\{\sum_{l_i=-\infty}^{k_0-2} 2^{l_i\sigma_i p_i}\left\|f_i\chi_{l_i}\right\|_{L^{q_i}(\mu)}^{p_i}\right\}^{1/p_i} \\
&\leq C\left\|f_i\right\|_{M\dot{K}_{p_i,q_i}^{\sigma_i,\lambda_i}(\mu)}.
\end{aligned} \tag{3.10}$$

In case $1 < p_i < \infty$, using the Hölder inequality, we get

$$\begin{aligned}
\mho_{1i}(k_0) &= 2^{-k_0\lambda_i}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{l_i=-\infty}^{k-2} G_i(k-l_i)2^{l_i\sigma_i}\left\|f_{l_i}\right\|_{L^{q_i}(\mu)}\right)^{p_i}\right\}^{1/p_i} \\
&\leq 2^{-k_0\lambda_i}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{l_i=-\infty}^{k-2} G_i(k-l_i)2^{l_i\sigma_i p_i}\left\|f_{l_i}\right\|_{L^{q_i}(\mu)}^{p_i}\right)\left(\sum_{l_i=-\infty}^{k-2} G_i(k-l_i)\right)^{p_i-1}\right\}^{1/p_i} \\
&\leq 2^{-k_0\lambda_i}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{l_i=-\infty}^{k-2} G_i(k-l_i)2^{l_i\sigma_i p_i}\left\|f_{l_i}\right\|_{L^{q_i}(\mu)}^{p_i}\right)\left(\sum_{s=2}^{\infty} G_i(s)\right)^{p_i-1}\right\}^{1/p_i} \\
&\leq C2^{-k_0\lambda_i}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{l_i=-\infty}^{k-2} G_i(k-l_i)2^{l_i\sigma_i p_i}\left\|f_{l_i}\right\|_{L^{q_i}(\mu)}^{p_i}\right)\right\}^{1/p_i} \\
&\leq C2^{-k_0\lambda_i}\left\{\sum_{l_i=-\infty}^{k_0-2}\sum_{k=l_i+2}^{k_0} G_i(k-l_i)2^{l_i\sigma_i p_i}\left\|f_{l_i}\right\|_{L^{q_i}(\mu)}^{p_i}\right\}^{1/p_i} \\
&\leq C2^{-k_0\lambda_i}\left\{\sum_{l_i=-\infty}^{k_0-2} 2^{l_i\sigma_i p_i}\left\|f_{l_i}\right\|_{L^{q_i}(\mu)}^{p_i}\left(\sum_{s=2}^{\infty} G_i(s)\right)\right\}^{1/p_i} \\
&\leq C\left\|f_i\right\|_{M\dot{K}_{p_i,q_i}^{\sigma_i,\lambda_i}(\mu)}.
\end{aligned} \tag{3.11}$$

In case $p_i = \infty$, recalling the definition of Morrey-Herz spaces we get

$$\begin{aligned}
\mho_{1i}(k_0) &= 2^{-k_0\lambda_i} \sup_{k\le k_0} \left\{ \sup_{l_i\le k-2} 2^{l_i\sigma_i} \|f_{l_i}\|_{L^{q_i}(\mu)} \left(\sum_{l_i=-\infty}^{k-2} G_i(k-l_i) \right) \right\} \\
&\le C 2^{-k_0\lambda_i} \sup_{k\le k_0} \left\{ \sup_{l_i\le k-2} 2^{l_i\sigma_i} \|f_{l_i}\|_{L^{q_i}(\mu)} \left(\sum_{s=2}^{\infty} G_i(s) \right) \right\} \\
&\le C 2^{-k_0\lambda_i} \sup_{k\le k_0} \left\{ \sup_{l_i\le k-2} 2^{l_i\sigma_i} \|f_{l_i}\|_{L^{q_i}(\mu)} \right\} \\
&\le C 2^{-k_0\lambda_i} \sup_{k\le k_0} 2^{(k-2)\lambda_i} \|f_i\|_{M\dot{K}^{\sigma_i,\lambda_i}_{p_i,q_i}(\mu)} \\
&\le C \|f_i\|_{M\dot{K}^{\sigma_i,\lambda_i}_{p_i,q_i}(\mu)}.
\end{aligned} \tag{3.12}$$

Therefore, for any $0 < p_i \le \infty$, we have obtained that

$$\mho_1 \le C \sup_{k_0\in\mathbb{Z}} \mho_{11}(k_0) \times \mho_{12}(k_0) \le C \|f_1\|_{M\dot{K}^{\sigma_1,\lambda_1}_{p_1,q_1}(\mu)} \|f_2\|_{M\dot{K}^{\sigma_2,\lambda_2}_{p_2,q_2}(\mu)}. \tag{3.13}$$

For $\mho_2$, by the growth condition (1.2) of μ, inequality (3.3) and the Cauchy inequality, we use the analogous arguments as that of $\mho_1$ to deduce that

$$\begin{aligned}
\mho_2 &\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1} \left\{ \sum_{k=-\infty}^{k_0} \left(\sum_{l_1=-\infty}^{k-2} G_1(k-l_1) 2^{l_1\sigma_1} \|f_{l_1}\|_{L^{q_1}(\mu)} \right)^{p_1} \right\}^{1/p_1} \\
&\quad \times 2^{-k_0\lambda_2} \left\{ \sum_{k=-\infty}^{k_0} \left(\sum_{l_2=k-1}^{k+1} G_2(k-l_2) 2^{l_2\sigma_2} \|f_{l_2}\|_{L^{q_2}(\mu)} \right)^{p_2} \right\}^{1/p_2} \\
&=: C \sup_{k_0\in\mathbb{Z}} \mho_{21}(k_0) \times \mho_{22}(k_0).
\end{aligned} \tag{3.14}$$

We observe that $\mho_{21}(k_0)$ is equal to $\mho_{11}(k_0)$, and so we have

$$\mho_{21}(k_0) \le C \|f_1\|_{M\dot{K}^{\sigma_1,\lambda_1}_{p_1,q_1}(\mu)} \tag{3.15}$$

with a constant C independent of k_0.

For $\mho_{22}(k_0)$, noting that $1 < q_2$, one can see easily that

$$\begin{aligned}
\mho_{22}(k_0) &= 2^{-k_0\lambda_2}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{l_2=k-1}^{k+1}2^{(k-l_2)n(1/q_2-1)+k\sigma_2}\left\|f_{l_2}\right\|_{L^{q_2}(\mu)}\right)^{p_2}\right\}^{1/p_2} \\
&= 2^{-k_0\lambda_2}\left\{\sum_{k=-\infty}^{k_0}2^{k\sigma_2 p_2}\left(\sum_{l_2=k-1}^{k+1}2^{(k-l_2)n(1/q_2-1)}\left\|f_{l_2}\right\|_{L^{q_2}(\mu)}\right)^{p_2}\right\}^{1/p_2} \\
&\le C2^{-k_0\lambda_2}\left\{\sum_{k=-\infty}^{k_0}2^{k\sigma_2 p_2}\left\|f_2\chi_{\{2^{k-2}<|\cdot|\le 2^{k+1}\}}(\cdot)\right\|_{L^{q_2}(\mu)}^{p_2}\right\}^{1/p_2} \\
&\le C2^{-k_0\lambda_2}\left\{\sum_{k=-\infty}^{k_0+1}2^{k\sigma_2 p_2}\left\|f_2\chi_k\right\|_{L^{q_2}(\mu)}^{p_2}\right\}^{1/p_2} \\
&\le C\left\|f_2\right\|_{M\dot{K}_{p_2,q_2}^{\sigma_2,\lambda_2}(\mu)}
\end{aligned} \tag{3.16}$$

with a constant C independent of k_0.

Combining inequalities (3.15) and (3.16), we obtain

$$\mho_2 \le C\sup_{k_0\in\mathbb{Z}}\mho_{21}(k_0)\times\mho_{22}(k_0) \le C\left\|f_1\right\|_{M\dot{K}_{p_1,q_1}^{\sigma_1,\lambda_1}(\mu)}\left\|f_2\right\|_{M\dot{K}_{p_2,q_2}^{\sigma_2,\lambda_2}(\mu)}. \tag{3.17}$$

For $\mho_3$, by using the growth condition (1.2) of μ, estimate (3.3), and the Cauchy inequality, we obtain

$$\begin{aligned}
\mho_3 \le C\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}&\left\{\sum_{k=-\infty}^{k_0}2^{k\sigma p}\left(\sum_{l_1=-\infty}^{k-2}2^{-k(n-\alpha/2)+kn/h_1}\left\|f_{l_1}\right\|_{L^1(\mu)}\right)^{p}\right. \\
&\quad\left.\times\left(\sum_{l_2=k+2}^{\infty}2^{-l_2(n-\alpha/2)+kn/h_2}\left\|f_{l_2}\right\|_{L^1(\mu)}\right)^{p}\right\}^{1/p} \\
\le C\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}&\left\{\sum_{k=-\infty}^{k_0}2^{k\sigma l}\left(\sum_{l_1=-\infty}^{k-2}2^{-k(n-\alpha/2)+kn/h_1}\left\|f_{l_1}\right\|_{L^1(\mu)}\right)^{l}\right. \\
&\quad\left.\times\left(\sum_{l_2=k+2}^{\infty}2^{-l_2(n-\alpha/2)+kn/h_2}\left\|f_{l_2}\right\|_{L^1(\mu)}\right)^{l}\right\}^{1/l}
\end{aligned}$$

$$
\begin{aligned}
&\le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left\{\sum_{k=-\infty}^{k_0} 2^{k\sigma_1 p_1}\left(\sum_{l_1=-\infty}^{k-2} 2^{kn(1/q_1-1)}\left\|f_{l_1}\right\|_{L^1(\mu)}\right)^{p_1}\right\}^{1/p_1}\\
&\quad\times 2^{-k_0\lambda_2}\left\{\sum_{k=-\infty}^{k_0} 2^{k\sigma_2 p_2}\left(\sum_{l_2=k+2}^{\infty} 2^{-l_2(n-\alpha/2)+kn/h_2}\left\|f_{l_2}\right\|_{L^1(\mu)}\right)^{p_2}\right\}^{1/p_2}\\
&\le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{l_1=-\infty}^{k-2} G_1(k-l_1)2^{l_1\sigma_1}\left\|f_{l_1}\right\|_{L^{q_1}(\mu)}\right)^{p_1}\right\}^{1/p_1}\\
&\quad\times 2^{-k_0\lambda_2}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{l_2=k+2}^{\infty} H_2(k-l_2)2^{l_2\sigma_2}\left\|f_{l_2}\right\|_{L^{q_2}(\mu)}\right)^{p_2}\right\}^{1/p_2}\\
&=: C\sup_{k_0\in\mathbb{Z}} \mathcal{V}_{31}(k_0)\times\mathcal{V}_{32}(k_0).
\end{aligned}
\tag{3.18}
$$

Noting that $\mathcal{V}_{31}(k_0)=\mathcal{V}_{21}(k_0)=\mathcal{V}_{11}(k_0)$, by (3.15), we get

$$
\mathcal{V}_{31}(k_0)\le C\left\|f_1\right\|_{M\dot{K}^{\sigma_1,\lambda_1}_{p_1,q_1}(\mu)}
\tag{3.19}
$$

with a constant C independent of k_0.

As for $\mathcal{V}_{32}(k_0)$, we can also write

$$
\begin{aligned}
\mathcal{V}_{32}(k_0)&\le 2^{-k_0\lambda_2}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{l_2=k+2}^{k_0} H_2(k-l_2)2^{l_2\sigma_2}\left\|f_{l_2}\right\|_{L^{q_2}(\mu)}\right)^{p_2}\right\}^{1/p_2}\\
&\quad+2^{-k_0\lambda_2}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{l_2=k_0+1}^{\infty} H_2(k-l_2)2^{l_2\sigma_2}\left\|f_{l_2}\right\|_{L^{q_2}(\mu)}\right)^{p_2}\right\}^{1/p_2}\\
&=: \mathcal{V}_{32}^1(k_0)+\mathcal{V}_{32}^2(k_0).
\end{aligned}
\tag{3.20}
$$

Now, we estimate $\mathcal{V}_{32}^1(k_0)$ and $\mathcal{V}_{32}^2(k_0)$, respectively. For $\mathcal{V}_{32}^1(k_0)$, using similar methods as that for $\mathcal{V}_{1i}(k_0)$, we consider the following three cases.

If $0 < p_2 \leq 1$, the fact $\alpha/2 - n/q_2 \leq \lambda_2 + \alpha/2 - n/q_2 < \sigma_2$ implies that

$$\begin{aligned}
\mathcal{U}_{32}^{1}(k_0) &\leq 2^{-k_0\lambda_2}\left\{\sum_{k=-\infty}^{k_0}\sum_{l_2=k+2}^{k_0} H_2[p_2(k-l_2)]2^{l_2\sigma_2 p_2}\|f_{l_2}\|_{L^{q_2}(\mu)}^{p_2}\right\}^{1/p_2} \\
&\leq 2^{-k_0\lambda_2}\left\{\sum_{l_2=-\infty}^{k_0}\sum_{k=-\infty}^{l_2-2} H_2[p_2(k-l_2)]2^{l_2\sigma_2 p_2}\|f_{l_2}\|_{L^{q_2}(\mu)}^{p_2}\right\}^{1/p_2} \\
&\leq 2^{-k_0\lambda_2}\left\{\sum_{l_2=-\infty}^{k_0} 2^{l_2\sigma_2 p_2}\|f_{l_2}\|_{L^{q_2}(\mu)}^{p_2}\left(\sum_{s=2}^{\infty} H_2(-sp_2)\right)\right\}^{1/p_2} \\
&\leq C2^{-k_0\lambda_2}\left\{\sum_{l_2=-\infty}^{k_0} 2^{l_2\sigma_2 p_2}\|f_{l_2}\|_{L^{q_2}(\mu)}^{p_2}\right\}^{1/p_2} \\
&\leq C\|f_2\|_{M\dot{K}_{p_2,q_2}^{\sigma_2,\lambda_2}(\mu)}.
\end{aligned} \tag{3.21}$$

If $1 < p_2 < \infty$, the Hölder inequality and the fact $\alpha/2 - n/q_2 < \sigma_2$ yield that

$$\begin{aligned}
\mathcal{U}_{32}^{1}(k_0) &\leq 2^{-k_0\lambda_2}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{l_2=k+2}^{k_0} H_2(k-l_2)2^{l_2\sigma_2}\|f_{l_2}\|_{L^{q_2}(\mu)}\right)^{p_2}\right\}^{1/p_2} \\
&\leq C2^{-k_0\lambda_2}\left\{\sum_{l_2=-\infty}^{k_0}\sum_{k=-\infty}^{l_2-2} H_2(k-l_2)2^{l_2\sigma_2 p_2}\|f_{l_2}\|_{L^{q_2}(\mu)}^{p_2}\right\}^{1/p_2} \\
&\leq C2^{-k_0\lambda_2}\left\{\sum_{l_2=-\infty}^{k_0} 2^{l_2\sigma_2 p_2}\|f_{l_2}\|_{L^{q_2}(\mu)}^{p_2}\left(\sum_{s=2}^{\infty} H_2(-s)\right)\right\}^{1/p_2} \\
&\leq C2^{-k_0\lambda_2}\left\{\sum_{l_2=-\infty}^{k_0} 2^{l_2\sigma_2 p_2}\|f_{l_2}\|_{L^{q_2}(\mu)}^{p_2}\right\}^{1/p_2} \\
&\leq C\|f_2\|_{M\dot{K}_{p_2,q_2}^{\sigma_2,\lambda_2}(\mu)}.
\end{aligned} \tag{3.22}$$

If $p_2 = \infty$, we get

$$\begin{aligned}
\mho_{32}^1(k_0) &\le 2^{-k_0\lambda_2}\sup_{k\le k_0}\left\{\sum_{l_2=k+2}^{k_0} H_2(k-l_2)2^{l_2\sigma_2}\|f_{l_2}\|_{L^{q_2}(\mu)}\right\} \\
&\le 2^{-k_0\lambda_2}\sup_{k\le k_0}\left\{\sup_{l_2\le k_0} 2^{l_2\sigma_2}\|f_{l_2}\|_{L^{q_2}(\mu)}\left(\sum_{l_2=k+2}^{k_0} H_2(k-l_2)\right)\right\} \\
&\le 2^{-k_0\lambda_2}\sup_{k\le k_0}\left\{2^{k_0\lambda_2}\|f_2\|_{M\dot{K}_{p_2,q_2}^{\sigma_2,\lambda_2}(\mu)}\left(\sum_{z=2}^{\infty} H_2(-z)\right)\right\} \\
&\le C\|f_2\|_{M\dot{K}_{p_2,q_2}^{\sigma_2,\lambda_2}(\mu)}.
\end{aligned} \tag{3.23}$$

Thus, we get

$$\mho_{32}^1(k_0) \le C\|f_2\|_{M\dot{K}_{p_2,q_2}^{\sigma_2,\lambda_2}(\mu)} \tag{3.24}$$

with a constant C independent of k_0.

For $\mho_{32}^2(k_0)$, by the fact $\alpha/2 - n/q_2 \le \lambda_2 + \alpha/2 - n/q_2 < \sigma_2$, we have

$$\begin{aligned}
\mho_{32}^2(k_0) &\le 2^{-k_0\lambda_2}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{l_2=k_0+1}^{\infty} H_2(k-l_2)2^{l_2\sigma_2}\|f_{l_2}\|_{L^{q_2}(\mu)}\right)^{p_2}\right\}^{1/p_2} \\
&\le 2^{-k_0\lambda_2}\left\{\sum_{k=-\infty}^{k_0}\left[\sum_{l_2=k_0+1}^{\infty} H_2(k-l_2)\left(\sum_{j=-\infty}^{l_2} 2^{j\sigma_2p_2}\|f_2\chi_j\|_{L^{q_2}(\mu)}^{p_2}\right)^{1/p_2}\right]^{p_2}\right\}^{1/p_2} \\
&\le C2^{-k_0\lambda_2}\left\{\sum_{k=-\infty}^{k_0}\left[\sum_{l_2=k_0+1}^{\infty} H_2(k-l_2)2^{l_2\lambda_2}\|f_2\|_{M\dot{K}_{p_2,q_2}^{\sigma_2,\lambda_2}(\mu)}\right]^{p_2}\right\}^{1/p_2} \\
&\le C\|f_2\|_{M\dot{K}_{p_2,q_2}^{\sigma_2,\lambda_2}(\mu)}2^{-k_0\lambda_2}\left\{\sum_{k=-\infty}^{k_0} H_2(kp_2)\right\}^{1/p_2}\left(\sum_{l_2=k_0+1}^{\infty} 2^{l_2\lambda_2}H_2(-l_2)\right) \\
&\le C\|f_2\|_{M\dot{K}_{p_2,q_2}^{\sigma_2,\lambda_2}(\mu)}.
\end{aligned} \tag{3.25}$$

Therefore, the inequality above and inequalities (3.19), (3.20), and (3.24) yield

$$\mho_3 \le C\|f_1\|_{M\dot{K}_{p_1,q_1}^{\sigma_1,\lambda_1}(\mu)}\|f_2\|_{M\dot{K}_{p_2,q_2}^{\sigma_2,\lambda_2}(\mu)}. \tag{3.26}$$

To estimate the term $\mho_5$, using Proposition 1.1, the L^q-boundedness for $\mathcal{I}_{\alpha,2}$, we obtain

$$\|\mathcal{I}_{\alpha,2}(f_{l_1}f_{l_2})\chi_k\|_{L^q(\mu)} \le C\|f_{l_1}\|_{L^{q_1}(\mu)}\|f_{l_2}\|_{L^{q_2}(\mu)}. \tag{3.27}$$

Thus, a similar argument shows that

$$
\begin{aligned}
\mathcal{U}_5 &\le \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\sigma p}\left(\sum_{(l_1,l_2)\in\Lambda_5}\left\|\mathcal{I}_{\alpha,2}(f_{l_1},f_{l_2})\chi_k\right\|_{L^q(\mu)}\right)^p\right\}^{1/p}\\
&\le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\sigma l}\left(\sum_{l_1=k-1}^{k+1}\left\|f_{l_1}\right\|_{L^{q_1}(\mu)}\right)^l\left(\sum_{l_2=k-1}^{k+1}\left\|f_{l_2}\right\|_{L^{q_2}(\mu)}\right)^l\right\}^{1/l}\\
&\le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\prod_{i=1}^{2}\left\{\sum_{k=-\infty}^{k_0} 2^{k\sigma_1 p_i}\left(\sum_{l_i=k-1}^{k+1}\left\|f_{l_i}\right\|_{L^{q_i}(\mu)}\right)^{p_i}\right\}^{1/p_i}\\
&\le C\left\|f_1\right\|_{M\dot{K}^{\sigma_1,\lambda_1}_{p_1,q_1}(\mu)}\left\|f_2\right\|_{M\dot{K}^{\sigma_2,\lambda_2}_{p_2,q_2}(\mu)}.
\end{aligned}
\tag{3.28}
$$

For $\mathcal{U}_6$, by condition (1.2) and inequality (3.3), we have

$$
\begin{aligned}
\mathcal{U}_6 &\le \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{l_1=k-1}^{k+1} G_1(k-l_1)2^{l_1\sigma_1}\left\|f_{l_1}\right\|_{L^{q_1}(\mu)}\right)^{p_1}\right\}^{1/p_1}\\
&\quad\times 2^{-k_0\lambda_2}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{l_2=k+2}^{\infty} H_2(k-l_2)2^{l_2\sigma_2}\left\|f_{l_2}\right\|_{L^{q_2}(\mu)}\right)^{p_2}\right\}^{1/p_2}\\
&=: \sup_{k_0\in\mathbb{Z}}\mathcal{U}_{61}(k_0)\times\mathcal{U}_{62}(k_0) \le C\left\|f_1\right\|_{M\dot{K}^{\sigma_1,\lambda_1}_{p_1,q_1}(\mu)}\left\|f_2\right\|_{M\dot{K}^{\sigma_2,\lambda_2}_{p_2,q_2}(\mu)}.
\end{aligned}
\tag{3.29}
$$

Finally to estimate the term $\mathcal{U}_9$, by the Hölder inequality, condition (1.2), and inequality (3.3), one sees that

$$
\begin{aligned}
\mathcal{U}_9 &\le \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{l_1=k+2}^{\infty} H_1(k-l_1)2^{l_1\sigma_1}\left\|f_{l_1}\right\|_{L^{q_1}(\mu)}\right)^{p_1}\right\}^{1/p_1}\\
&\quad\times 2^{-k_0\lambda_2}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{l_2=k+2}^{\infty} H_2(k-l_2)2^{l_2\sigma_2}\left\|f_{l_2}\right\|_{L^{q_2}(\mu)}\right)^{p_2}\right\}^{1/p_2}\\
&=: \sup_{k_0\in\mathbb{Z}}\mathcal{U}_{91}(k_0)\times\mathcal{U}_{92}(k_0) \le C\left\|f_1\right\|_{M\dot{K}^{\sigma_1,\lambda_1}_{p_1,q_1}(\mu)}\left\|f_2\right\|_{M\dot{K}^{\sigma_2,\lambda_2}_{p_2,q_2}(\mu)}.
\end{aligned}
\tag{3.30}
$$

Combining all the estimates for $\mathcal{U}_i$ for $i = 1, 2, \ldots, 9$, we get

$$
\left\|\mathcal{I}_{\alpha,2}(f_{l_1}f_{l_2})\right\|_{M\dot{K}^{\sigma,\lambda}_{p,q}(\mu)} \le C\left\|f_1\right\|_{M\dot{K}^{\sigma_1,\lambda_1}_{p_1,q_1}(\mu)}\left\|f_2\right\|_{M\dot{K}^{\sigma_2,\lambda_2}_{p_2,q_2}(\mu)}.
\tag{3.31}
$$

This is the desired estimate of Theorem 1.2.

The proof of Theorem 1.2 is completed.

Next we turn to the proof of Theorem 1.3. Let f_1, f_2 be functions in $M\dot{K}_{p_1,q_1}^{n(1-1/q_1),\lambda_1}(\mu)$ and $M\dot{K}_{p_2,q_2}^{n(1-1/q_2),\lambda_2}(\mu)$, respectively. Obviously, to prove the theorem, we only need to find a constant $C>0$ independent of $\vec{f}$ such that

$$\begin{aligned}&\gamma \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{kn(2-1/g)p}\mu\left(\left\{x\in E_k : \left|\mathcal{I}_{\alpha,2}\left(\vec{f}\right)(x)\right|>9\gamma\right\}\right)^{p/q}\right\}^{1/p}\\&\quad\le C\|f_1\|_{M\dot{K}_{p_1,q_1}^{n(1-1/p_1),\lambda_1}(\mu)}\|f_2\|_{M\dot{K}_{p_2,q_2}^{n(1-1/p_2),\lambda_2}(\mu)}\end{aligned}\tag{3.32}$$

for all $\gamma>0$.

By the decomposition of f_i above, we get

$$\begin{aligned}&\gamma \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{kn(2-1/g)p}\mu\left(\left\{x\in E_k : \left|\mathcal{I}_{\alpha,2}\left(\vec{f}\right)(x)\right|>9\gamma\right\}\right)^{p/q}\right\}^{1/p}\\&\quad\le C\gamma \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{kn(2-1/g)p}\mu\left(\left\{x\in E_k : \sum_{l_i=-\infty}^{\infty}\left|\mathcal{I}_{\alpha,2}(f_{l_1},f_{l_2})(x)\right|>9\gamma\right\}\right)^{p/q}\right\}^{1/p}\\&\quad\le C\sum_{\ell=1}^{9}\left(\gamma\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\mathcal{G}_\ell(k_0)\right)=:C\sum_{\ell=1}^{9}\mathcal{H}_\ell,\end{aligned}\tag{3.33}$$

where

$$\mathcal{G}_\ell(k_0)=\left\{\sum_{k=-\infty}^{k_0} 2^{kn(2-1/g)p}\mu\left(\left\{x\in E_k : \sum_{(l_1,l_2)\in\Lambda_\ell}\left|\mathcal{I}_{\alpha,2}(f_{l_1},f_{l_2})(x)\right|>\gamma\right\}\right)^{p/q}\right\}^{1/p}. \tag{3.34}$$

Similar to the proof of Theorem 1.2, we only need to estimate $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$, $\mathcal{H}_5$, $\mathcal{H}_6$, and $\mathcal{H}_9$, respectively.

For $\mathcal{H}_1$, using the Chebychev inequality, the Hölder inequality, (1.2), and (3.3), we obtain

$$\begin{aligned}&\mu\left(\left\{x\in E_k : \sum_{(l_1,l_2)\in\Lambda_1}\left|\mathcal{I}_{\alpha,2}(f_{l_1},f_{l_2})(x)\right|>\gamma\right\}\right)^{1/q}\\&\quad\le C\gamma^{-1}\prod_{i=1}^{2}\left(\sum_{l_i=-\infty}^{k-2}2^{n(l_i-k)(1-1/q_i)}\|f_{l_i}\|_{L^{q_i}(\mu)}\right).\end{aligned}\tag{3.35}$$

Therefore, the facts $l < p$, $0 < p_i \leq 1$ and the Cauchy inequality imply that

$$\begin{aligned}
\mathcal{H}_1 &\leq C\prod_{i=1}^{2}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_i}\left\{\sum_{k=-\infty}^{k_0}\sum_{l_i=-\infty}^{k-2} 2^{l_i n(1-1/q_i)p_i}\left\|f_{l_i}\right\|_{L^{q_i}(\mu)}^{p_i}\right\}^{1/p_i} \\
&\leq C\prod_{i=1}^{2}\left\|f_i\right\|_{M\dot{K}_{p_i,q_i}^{n(1-1/q_i),\lambda_i}(\mu)}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_i}\left\{\sum_{k=-\infty}^{k_0} 2^{(k-2)\lambda_i p_i}\right\}^{1/p_i} \\
&\leq C\left\|f_1\right\|_{M\dot{K}_{p_1,q_1}^{n(1-1/q_1),\lambda_1}(\mu)}\left\|f_2\right\|_{M\dot{K}_{p_2,q_2}^{n(1-1/q_2),\lambda_2}(\mu)}.
\end{aligned} \tag{3.36}$$

Now, we consider the estimate of the term $\mathcal{H}_2$. Using a similar argument as that of $\mathcal{H}_1$, we can deduce that

$$\begin{aligned}
\mathcal{H}_2 \leq{}& C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left\{\sum_{k=-\infty}^{k_0} 2^{kn(1-1/q_1)p_1}\left(\sum_{l_1=-\infty}^{k-2} 2^{n(l_1-k)(1-1/q_1)}\left\|f_{l_1}\right\|_{L^{q_1}(\mu)}\right)^{p_1}\right\}^{1/p_1} \\
&\times 2^{-k_0\lambda_2}\left\{\sum_{k=-\infty}^{k_0} 2^{kn(1-1/q_2)p_2}\left(\sum_{l_2=k-1}^{k+1} 2^{n(l_2-k)(1-1/q_2)}\left\|f_{l_2}\right\|_{L^{q_2}(\mu)}\right)^{p_2}\right\}^{1/p_2} \\
\leq{}& C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left\{\sum_{k=-\infty}^{k_0}\sum_{l_1=-\infty}^{k-2} 2^{l_1 n(1-1/q_1)p_1}\left\|f_{l_1}\right\|_{L^{q_1}(\mu)}^{p_1}\right\}^{1/p_1} \\
&\times 2^{-k_0\lambda_2}\left\{\sum_{k=-\infty}^{k_0+1} 2^{kn(1-1/q_2)p_2}\left\|f_2\chi_k\right\|_{L^{q_2}(\mu)}^{p_2}\right\}^{1/p_2} \\
\leq{}& C\left\|f_1\right\|_{M\dot{K}_{p_1,q_1}^{n(1-1/q_1),\lambda_1}(\mu)}\left\|f_2\right\|_{M\dot{K}_{p_2,q_2}^{n(1-1/q_2),\lambda_2}(\mu)},
\end{aligned} \tag{3.37}$$

as desired.

To estimate $\mathcal{H}_3$, we point out the fact

$$\begin{aligned}
&\mu\left(\left\{x\in E_k : \sum_{l_1=-\infty}^{k-2}\sum_{l_2=k+2}^{\infty}\left|\mathcal{I}_{\alpha,2}\left(f_{l_1},f_{l_2}\right)(x)\right|>\gamma\right\}\right)^{1/q} \\
&\quad\leq C\gamma^{-1}\left(\sum_{l_1=-\infty}^{k-2} 2^{n(l_1-k)(1-1/q_1)}\left\|f_{l_1}\right\|_{L^{q_1}(\mu)}\right)\left(\sum_{l_2=k+2}^{\infty} 2^{(l_2-k)(\alpha/2-n/q_2)}\left\|f_{l_2}\right\|_{L^{q_2}(\mu)}\right).
\end{aligned} \tag{3.38}$$

So we can show that

$$\begin{aligned}
\mathscr{H}_3 &\leq C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\left\{\sum_{k=-\infty}^{k_0}\sum_{l_1=-\infty}^{k-2} 2^{l_1 n(1-1/q_1)p_1}\left\|f_{l_1}\right\|_{L^{q_1}(\mu)}^{p_1}\right\}^{1/p_1}\\
&\quad\times 2^{-k_0\lambda_2}\left\{\sum_{k=-\infty}^{k_0}\sum_{l_2=k+2}^{\infty} 2^{(l_2-k)(\alpha/2-n)p_2+l_2 n(1-1/q_2)p_2}\left\|f_{l_2}\right\|_{L^{q_2}(\mu)}^{p_2}\right\}^{1/p_2}\\
&=: \mathscr{H}_{31}(k_0)\times\mathscr{H}_{32}(k_0).
\end{aligned}\tag{3.39}$$

From the estimates of $\mathscr{H}_1$, we know $\mathscr{H}_{31}(k_0)\leq C\|f_1\|_{M\dot{K}_{p_1,q_1}^{n(1-1/q_1),\lambda_1}(\mu)}$, so we only need to show that $\mathscr{H}_{32}(k_0)\leq C\|f_2\|_{M\dot{K}_{p_2,q_2}^{n(1-1/q_2),\lambda_2}(\mu)}$.

For $\mathscr{H}_{32}(k_0)$, we write

$$\begin{aligned}
\mathscr{H}_{32}(k_0) &\leq 2^{-k_0\lambda_2}\left\{\sum_{k=-\infty}^{k_0}\sum_{l_2=k+2}^{k_0} 2^{(l_2-k)(\alpha/2-n)p_2+l_2 n(1-1/q_2)p_2}\left\|f_{l_2}\right\|_{L^{q_2}(\mu)}^{p_2}\right\}^{1/p_2}\\
&\quad+ 2^{-k_0\lambda_2}\left\{\sum_{k=-\infty}^{k_0}\sum_{l_2=k_0+1}^{\infty} 2^{(l_2-k)(\alpha/2-n)p_2+l_2 n(1-1/q_2)p_2}\left\|f_{l_2}\right\|_{L^{q_2}(\mu)}^{p_2}\right\}^{1/p_2}\\
&=: \mathscr{H}_{32}^1(k_0)+\mathscr{H}_{32}^2(k_0).
\end{aligned}\tag{3.40}$$

First, the fact $0<\alpha<2n$ yields that

$$\begin{aligned}
\mathscr{H}_{32}^1(k_0) &\leq 2^{-k_0\lambda_2}\left\{\sum_{l_2=-\infty}^{k_0}\sum_{k=-\infty}^{l_2-2} 2^{(l_2-k)(\alpha/2-n)p_2+l_2 n(1-1/q_2)p_2}\left\|f_{l_2}\right\|_{L^{q_2}(\mu)}^{p_2}\right\}^{1/p_2}\\
&\leq 2^{-k_0\lambda_2}\left\{\sum_{l_2=-\infty}^{k_0} 2^{l_2 n(1-1/q_2)p_2}\left\|f_{l_2}\right\|_{L^{q_2}(\mu)}^{p_2}\left(\sum_{s=2}^{\infty}2^{s(\alpha/2-n)p_2}\right)\right\}^{1/p_2}\\
&\leq C\|f_2\|_{M\dot{K}_{p_2,q_2}^{n(1-1/q_2),\lambda_2}(\mu)}.
\end{aligned}\tag{3.41}$$

Second, the fact $\lambda_2 < n - \alpha/2$ implies

$$
\begin{aligned}
\mathscr{K}_{32}^2(k_0) &\le 2^{-k_0\lambda_2}\left\{\sum_{k=-\infty}^{k_0}\sum_{l_2=k+2}^{\infty}2^{(l_2-k)(\alpha/2-n)p_2}2^{l_2 n(1-1/q_2)p_2}\|f_{l_2}\|_{L^{q_2}(\mu)}^{p_2}\right\}^{1/p_2} \\
&\le 2^{-k_0\lambda_2}\left\{\sum_{k=-\infty}^{k_0}\sum_{l_2=k+2}^{\infty}2^{(l_2-k)(\alpha/2-n)p_2}\left(\sum_{j=-\infty}^{l_2}2^{jn(1-1/q_2)p_2}\|f_2\chi_j\|_{L^{q_2}(\mu)}^{p_2}\right)\right\}^{1/p_2} \\
&\le C\|f_2\|_{M\dot{K}_{p_2,q_2}^{n(1-1/q_2),\lambda_2}(\mu)}2^{-k_0\lambda_2}\left\{\sum_{k=-\infty}^{k_0}\sum_{l_2=k+2}^{\infty}2^{(l_2-k)(\alpha/2-n)p_2}2^{l_2\lambda_2}\right\}^{1/p_2} \\
&\le C\|f_2\|_{M\dot{K}_{p_2,q_2}^{n(1-1/q_2),\lambda_2}(\mu)},
\end{aligned}
\tag{3.42}
$$

as desired.

To estimate the term $\mathscr{K}_5$, by Proposition 1.1, the weak L^q-boundedness for $\mathcal{I}_{\alpha,2}$, we obtain

$$
\|\mathcal{I}_{\alpha,2}(f_{l_1}f_{l_2})\chi_k\|_{WL^q(\mu)} \le C\|f_{l_1}\|_{L^{q_1}(\mu)}\|f_{l_2}\|_{L^{q_2}(\mu)}. \tag{3.43}
$$

Similar to the estimates of $\mathcal{V}_5$, we have

$$
\begin{aligned}
\mathscr{K}_5 &\le C\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}2^{kn(2-1/g)p}\left(\sum_{(l_1,l_2)\in\Lambda_5}\|f_{l_1}\|_{L^{q_1}(\mu)}\|f_{l_2}\|_{L^{q_2}(\mu)}\right)^p\right\}^{1/p} \\
&\le C\sup_{k_0\in\mathbb{Z}}\prod_{i=1}^{2}2^{-k_0\lambda_i}\left\{\sum_{k=-\infty}^{k_0}2^{kn(1-1/q_i)p_i}\left(\sum_{l_i=k-1}^{k+1}\|f_{l_i}\|_{L^{q_i}(\mu)}\right)^{p_i}\right\}^{1/p_i} \\
&\le C\|f_1\|_{M\dot{K}_{p_1,q_1}^{n(1-1/q_1),\lambda_1}(\mu)}\|f_2\|_{M\dot{K}_{p_2,q_2}^{n(1-1/q_2),\lambda_2}(\mu)}.
\end{aligned}
\tag{3.44}
$$

For $\mathcal{H}_6$, we obtain

$$\begin{aligned}
\mathcal{H}_6 \le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1} \left\{ \sum_{k=-\infty}^{k_0} 2^{kn(1-1/q_1)p_1} \left(\sum_{l_i=k-1}^{k+1} 2^{n(l_1-k)(1-1/q_1)} \|f_{l_1}\|_{L^{q_1}(\mu)} \right)^{p_1} \right\}^{1/p_1} \\
\times 2^{-k_0\lambda_2} \left\{ \sum_{k=-\infty}^{k_0} 2^{kn(1-1/q_2)p_2} \left(\sum_{l_2=k+2}^{\infty} 2^{(l_2-k)(\alpha/2-n/q_2)} \|f_{l_2}\|_{L^{q_2}(\mu)} \right)^{p_2} \right\}^{1/p_2} \\
\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1} \left\{ \sum_{k=-\infty}^{k_0+1} 2^{kn(1-1/q_1)p_1} \|f_1\chi_k\|_{L^{q_1}(\mu)}^{p_1} \right\}^{1/p_1} \\
\times 2^{-k_0\lambda_2} \left\{ \sum_{k=-\infty}^{k_0} \sum_{l_2=k+2}^{\infty} 2^{(l_2-k)(\alpha/2-n)p_2+l_2n(1-1/q_2)p_2} \|f_{l_2}\|_{L^{q_2}(\mu)}^{p_2} \right\}^{1/p_2} \\
\le C \|f_1\|_{M\dot{K}_{p_1,q_1}^{n(1-1/q_1),\lambda_1}(\mu)} \|f_2\|_{M\dot{K}_{p_2,q_2}^{n(1-1/q_2),\lambda_2}(\mu)}.
\end{aligned} \tag{3.45}$$

Finally, we have to estimate the term $\mathcal{H}_9$. It can be deduced that

$$\begin{aligned}
\mathcal{H}_9 \le \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1} \left\{ \sum_{k=-\infty}^{k_0} 2^{kn(1-1/q_1)p_1} \left(\sum_{l_1=k+2}^{\infty} 2^{(l_1-k)(\alpha/2-n/q_1)} \|f_{l_1}\|_{L^{q_1}(\mu)} \right)^{p_1} \right\}^{1/p_1} \\
\times 2^{-k_0\lambda_2} \left\{ \sum_{k=-\infty}^{k_0} 2^{kn(1-1/q_2)p_2} \left(\sum_{l_2=k+2}^{\infty} 2^{(l_2-k)(\alpha/2-n/q_2)} \|f_{l_2}\|_{L^{q_2}(\mu)} \right)^{p_2} \right\}^{1/p_2} \\
=: \sup_{k_0\in\mathbb{Z}} \mathcal{H}_{91}(k_0) \times \mathcal{H}_{92}(k_0) \le C \|f_1\|_{M\dot{K}_{p_1,q_1}^{\sigma_1,\lambda_1}(\mu)} \|f_2\|_{M\dot{K}_{p_2,q_2}^{\sigma_2,\lambda_2}(\mu)}.
\end{aligned} \tag{3.46}$$

Here the estimate of $\mathcal{H}_{91}(k_0)$ and $\mathcal{H}_{92}(k_0)$ is similar to that of $\mathcal{H}_{32}(k_0)$.

Finally, a combination for the estimates of $\mathcal{H}_i$ $(i = 1, 2, \ldots, 9)$ finishes the proof of Theorem 1.3.

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