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Research Article
Generalization of Stolarsky Type Means

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We generalize means of Stolarsky type and show the monotonicity of these generalized means.

1. Introduction and Preliminaries

The following double inequality is well known in the literature as the Hermite-Hadamard (H.H) integral inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

provided that $f : [a,b] \to \mathbb{R}$ is a convex function [1, page 137], [2, page 1].

This result for convex functions plays an important role in nonlinear analysis. These classical inequalities have been improved and generalized in a number of ways and applied for special means including Stolarsky type, logarithmic, and p-logarithmic means. A generalization of H.H inequalities was obtained in [3–5], [2, page 5], and [1, page 143].

Theorem 1.1. *Let p, q be positive real numbers and a_1, a, b, b_1 be real numbers such that $a_1 \leq a < b \leq b_1$. Then the inequalities*

$$f\left(\frac{pa+qb}{p+q}\right) \leq \frac{1}{2y}\int_{A-y}^{A+y} f(x)dx \leq \frac{pf(a)+qf(b)}{p+q} \tag{1.2}$$

hold for $A = (pa+qb)/(p+q)$, $y > 0$, *and all continuous convex functions*

$$f : [a_1, b_1] \to \mathbb{R} \text{ if and only if } y \leq (b-a)/(p+q) \min\{p, q\}. \tag{1.3}$$

Remark 1.2. The inequalities given by (1.2) are strict if f is a continuous strictly convex on $[a_1, b_1]$.

If we keep the assumptions as stated in Theorem 1.1, we also have [1, page 146]

$$\frac{1}{2y}\int_{A-y}^{A+y} f(x)dx - f\left(\frac{pa+qb}{p+q}\right) \leq \frac{pf(a)+qf(b)}{p+q} - \frac{1}{2y}\int_{A-y}^{A+y} f(x)dx. \tag{1.4}$$

The above inequality is strict, when f is strictly convex continuous function.

Let us define $F^i : C[a,b] \to \mathbb{R}$ for $i = 1,2,3$ by differences of (1.2) and (1.4)

$$\begin{aligned} F^1(f; p, q; a, b, y) &= \frac{pf(a)+qf(b)}{p+q} - \frac{1}{2y}\int_m^M f(x)dx, \\ F^2(f; p, q; a, b, y) &= \frac{1}{2y}\int_m^M f(x)dx - f\left(\frac{pa+qb}{p+q}\right), \\ F^3(f; p, q; a, b, y) &= \frac{pf(a)+qf(b)}{p+q} + f\left(\frac{pa+qb}{p+q}\right) - \frac{1}{y}\int_m^M f(x)dx, \end{aligned} \tag{1.5}$$

where $m = A - y$, $M = A + y$.

Remark 1.3. It is clear from inequalities (1.2) and (1.4) that if the conditions of Theorem 1.1 are satisfied and $f \in K_2[a,b]$ (f is continuous convex on $[a,b]$), then

$$F^i(f; p, q; a, b, y) \geq 0, \quad \text{for } i = 1,2,3. \tag{1.6}$$

Consider the following means:

$$E_{r,t}(x,y) = \begin{cases} \left(\dfrac{r(y^t - x^t)}{t(y^r - x^r)}\right)^{1/(t-r)}, & tr(t-r) \neq 0, \\ \left(\dfrac{y^r - x^r}{r(\log y - \log x)}\right)^{1/r}, & r \neq 0,\ t = 0, \\ e^{-1/r}\left(\dfrac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r - y^r)}, & t = r \neq 0, \\ \sqrt{xy}, & t = r = 0, \end{cases} \tag{1.7}$$

where $x, y \in (0, \infty)$ such that $x \neq y$ and $r, t \in \mathbb{R}$. These means are known as Stolarsky means. Namely, Stolarsky introduced these means in 1975 (see [1, page 120]) and proved that for $r \leq u$ and $t \leq v$ one can get

$$E_{r,t}(x, y) \leq E_{u,v}(x, y) \quad \text{for } x, y \in (0, \infty),\ x \neq y. \tag{1.8}$$

Some simple proofs of inequality (1.8) and related results on means of Stolarsky type are given in [6].

The aim of this paper is to prove the exponential convexity of the functions deduced from (1.5) and apply these functions to generalize the means of Stolarsky type, and at last we prove the monotonicity property of these new means.

We review some necessary definitions and preliminary results.

Definition 1.4 (see [7]). A function $f : (a, b) \to \mathbb{R}$ is exponentially convex if it is continuous and

$$\sum_{i,j=1}^{n} \xi_i \xi_j f(x_i + x_j) \geq 0, \tag{1.9}$$

for each $n \in N$ and every $\xi_i \in \mathbb{R}$, $i = 1, \ldots, n$ such that $x_i + x_j \in (a, b)$, $1 \leq i, j \leq n$.

Proposition 1.5 (see [7]). *Let $f : (a, b) \to \mathbb{R}$, be a function. Then f is exponentially convex if and only if f is continuous and*

$$\sum_{i,j=1}^{n} \xi_i \xi_j f\left(\frac{x_i + x_j}{2}\right) \geq 0, \tag{1.10}$$

for all $n \in N$, $\xi_i \in \mathbb{R}$ and $x_i \in (a, b)$, $1 \leq i \leq n$.

Definition 1.6 (see [1]). A function $f : I \to \mathbb{R}^+$, where I is an interval in $\mathbb{R}$, is said to be log-convex if $\log f$ is convex, or equivalently if for all $x, y \in I$ and all $\alpha \in [0, 1]$, one has

$$f(\alpha x + (1 - \alpha)y) \leq f^{\alpha}(x) f^{1-\alpha}(y). \tag{1.11}$$

Corollary 1.7 (see [7]). *If $f : (a, b) \to \mathbb{R}^+$ is exponentially convex then f is log-convex function.*

The following lemma is another way to define convex function [1, page 2].

Lemma 1.8. *If f is a convex on an interval $I \subseteq \mathbb{R}$, then*

$$f(s_1)(s_3 - s_2) + f(s_2)(s_1 - s_3) + f(s_3)(s_1 - s_2) \geq 0 \tag{1.12}$$

holds for each $s_1 < s_2 < s_3$, where $s_1, s_2, s_3 \in I$.

In Section 2, we prove the exponential and logarithmic convexity of the functions deduced from (1.5). We also prove related mean value theorems of Cauchy type.

2. Main Results

The following lemma gives us very important family of convex functions.

Lemma 2.1 (see [7]). *Consider a family of functions $\phi_r : (0, \infty) \to \mathbb{R}$, $r \in \mathbb{R}$ defined as*

$$\phi_r(x) = \begin{cases} \dfrac{x^r}{r(r-1)}, & r \neq 0, 1, \\ -\log x, & r = 0, \\ x \log x, & r = 1. \end{cases} \tag{2.1}$$

Then ϕ_r is convex on $(0, \infty)$ for each $r \in \mathbb{R}$.

Theorem 2.2. *Let p, q, a, b, A, and y be positive real numbers such that*

$$\begin{gathered} a < b, \quad A = \frac{pa + qb}{p + q}, \quad y \leq \frac{b - a}{p + q} \min\{p, q\}, \\ \Im^i(r) := F^i(\phi_r; p, q; a, b, y), \quad i = 1, 2, 3, \end{gathered} \tag{2.2}$$

where ϕ_r is defined in Lemma 2.1. Then

(i) *matrix $[\Im^i((r_j + r_k)/2)]_{j,k=1}^n$ is positive semidefinite for each $n \in N$ and $r_1, \ldots, r_n \in \mathbb{R}$; particularly,*

$$\det\left[\Im^i\left(\frac{r_j + r_k}{2}\right)\right]_{j,k=1}^p \geq 0 \quad \text{for } 1 \leq p \leq n; \tag{2.3}$$

(ii) *the function $r \mapsto \Im^i(r)$ is exponentially convex on $\mathbb{R}$;*

(iii) *if* $\mathfrak{I}^i(r) > 0$, *then the function* $r \mapsto \mathfrak{I}^i(r)$ *is a log-convex on* $\mathbb{R}$ *and the following inequality holds for* $r, s, t \in \mathbb{R}$ *such that* $r < s < t$;

$$\left(\mathfrak{I}^i(s)\right)^{t-r} \leq \left(\mathfrak{I}^i(r)\right)^{t-s}\left(\mathfrak{I}^i(t)\right)^{s-r}. \tag{2.4}$$

Proof. (i) Consider the function

$$\mu(x) = \sum_{j,k=1}^{n} u_j u_k \phi_{r_{jk}}(x) \tag{2.5}$$

for $1 \leq p \leq n$, $x > 0 u_j \in \mathbb{R}$, where u_j is not identically zero and $r_{jk} = (r_j + r_k)/2$

$$\begin{aligned} \mu''(x) &= \sum_{j,k=1}^{n} u_j u_k x^{r_{jk}-2} \\ &= \left(\sum_{j=1}^{n} u_j x^{(r_j/2)-1}\right)^2 \geq 0, \quad x > 0. \end{aligned} \tag{2.6}$$

This shows that μ is a convex function for $x > 0$. By setting $f = \mu$ in (1.5), respectively and from Remark 1.3, we get

$$\begin{gathered} \sum_{j,k=1}^{n} u_j u_k \left(\frac{p\phi_{r_{jk}}(a) + q\phi_{r_{jk}}(b)}{p+q} - \frac{1}{2y}\int_m^M \phi_{r_{jk}}(x)dx\right) \geq 0, \\ \sum_{j,k=1}^{n} u_j u_k \left(\frac{1}{2y}\int_m^M \phi_{r_{jk}}(x)dx - \phi_{r_{jk}}\left(\frac{pa+qb}{p+q}\right)\right) \geq 0, \\ \sum_{j,k=1}^{n} u_j u_k \left(\frac{p\phi_{r_{jk}}(a) + q\phi_{r_{jk}}(b)}{p+q} + \phi_{r_{jk}}\left(\frac{pa+qb}{p+q}\right) - \frac{1}{y}\int_m^M \phi_{r_{jk}}(x)dx\right) \geq 0, \end{gathered} \tag{2.7}$$

or equivalently

$$\sum_{j,k=1}^{n} u_j u_k \mathfrak{I}^i(r_{jk}) \geq 0. \tag{2.8}$$

Therefore the given matrix is a positive semidefinite. By using well-known Sylvester criterion, we have

$$\det\left[\Im^i\left(\frac{r_j+r_k}{2}\right)\right]_{j,k=1}^{p} \geq 0 \quad \text{for each } 1 \leq p \leq n. \tag{2.9}$$

(ii) Since $\lim_{r\to l}\Im^i(r) = \Im^i(l)$ for $l = 0, 1$, it follows that $\Im^i$ is continuous on $\mathbb{R}$. Therefore, by Proposition 1.5 for $f = \Im^i$, we get exponential convexity of $\Im^i$ on $\mathbb{R}$.

(iii) Let $\Im^i(r) > 0$, then the log-convexity of $\Im^i$ is a simple consequence of Corollary 1.7. By setting $f = \log\,\Im^i$, $s_1 = r$, $s_2 = s$, $s_3 = t$ in Lemma 1.8, we have

$$(t-r)\log\Im^i(s) \leq (t-s)\log\Im^i(r) + (s-r)\Im^i(t), \tag{2.10}$$

which implies (2.4). □

We will use the following lemma in the proof of mean value theorem.

Lemma 2.3 (see [1, page 4]). *Let $f \in C^2([a,b])$ such that*

$$\alpha \leq f''(x) \leq \beta \quad \forall x \in [a,b]. \tag{2.11}$$

If one considers the functions h_1, h_2, defined by

$$h_1(x) = \frac{\alpha x^2}{2} - f(x), \qquad h_2(x) = f(x) - \frac{\beta x^2}{2}, \tag{2.12}$$

then h_1 and h_2 are convex on $[a,b]$.

Proof. Therefore

$$h_1''(x) = \alpha - f''(x) \geq 0, \qquad h_2''(x) = f''(x) - \beta \geq 0, \tag{2.13}$$

that is, h_j for $j = 1,2$ are convex on $[a,b]$. □

Theorem 2.4. *Let p, q, a, b, A, and y be real numbers as given in Theorem 1.1. If $f \in C^2([a,b])$ then there exists $\xi \in [a,b]$ such that*

$$F^i(f;p,q;a,b,y) = \frac{f''(\xi)}{2}F^i\left(x^2;p,q;a,b,y\right) \quad \text{for } i = 1,2,3. \tag{2.14}$$

Proof. Since $f \in C^2([a,b])$, we can take that $\alpha \le f'' \le \beta$. Now in Remark 1.3, replacing f by h_j, $j = 1,\ 2$ defined in Lemma 2.3, we have

$$F^i(h_j; p, q; a, b, y) \ge 0 \quad \text{for } j = 1, 2. \tag{2.15}$$

This gives

$$\begin{aligned} F^i(f(x); p, q : a, b, y) &\le \frac{\beta}{2} F^i\left(x^2; p, q; a, b, y\right), \\ \frac{\alpha}{2} F^i\left(x^2; p, q; a, b, y\right) &\le F^i(f(x); p, q; a, b, y). \end{aligned} \tag{2.16}$$

Combining (2.16) and (14), we get

$$\frac{\alpha}{2} F^i\left(x^2; p, q; a, b, y\right) \le F^i(f(x); p, q; a, b, y) \le \frac{\beta}{2} F^i\left(x^2; p, q; a, b, y\right). \tag{2.17}$$

By using Remark 1.2

$$F^i\left(x^2; p, q; a, b, y\right) > 0, \tag{2.18}$$

therefore

$$\alpha \le \frac{2F^i(f(x); p, q; a, b, y)}{F^i(x^2; p, q; a, b, y)} \le \beta. \tag{2.19}$$

We get the required result. □

Theorem 2.5. *Let p, q, a, b, A, and y be real numbers as given in Theorem 1.1. If $f, g \in C^2([a,b])$ such that $g''(x)$ do not vanish for any $x \in [a,b]$, then there exits $\xi \in [a,b]$ such that*

$$\frac{F^i(f; p, q; a, b, y)}{F^i(g; p, q; a, b, y)} = \frac{f''(\xi)}{g''(\xi)} \quad \text{for } i = 1, 2, 3. \tag{2.20}$$

Proof. Define functions $\phi^i \in C^2([a,b])$, $i = 1, 2, 3$ by

$$\phi^i = c_1^i g - c_2^i f, \tag{2.21}$$

where

$$c_1^i = F^i(f; p, q; a, b, y),$$
$$c_2^i = F^i(g; p, q; a, b, y). \tag{2.22}$$

Then using Theorem 2.4 for $f = \phi^i$, we have

$$0 = \left(c_1^i \frac{g''(\xi)}{2} - c_2^i \frac{f''(\xi)}{2}\right) F^i\left(x^2; p, q; a, b, y\right). \tag{2.23}$$

Using Remark 1.2

$$F^i\left(x^2; p, q; a, b, y\right) > 0, \tag{2.24}$$

therefore

$$\frac{c_1^i}{c_2^i} = \frac{f''(\xi)}{g''(\xi)}, \tag{2.25}$$

which is clearly (2.20). □

Corollary 2.6. *If p, q, a, b, A, and y are real numbers as defined in Theorem 1.1 then for $-\infty < r$, $t < \infty$, $r \neq t$, $r \neq 0, 1$ and there exists $\xi \in [a, b]$ such that*

$$\xi^{r-t} = \frac{t(t-1)F^i(x^r; p, q; a, b, y)}{r(r-1)F^i(x^t; p, q; a, b, y)} \quad \text{for } i = 1, 2, 3. \tag{2.26}$$

Remark 2.7. If the inverse of f''/g'' exists, then from (2.20) we get

$$\xi = \left(\frac{f''}{g''}\right)^{-1} \left(\frac{F^i(f; p, q; a, b, y)}{F^i(g; p, q; a, b, y)}\right) \quad \text{for } i = 1, 2, 3. \tag{2.27}$$

3. Means of Stolarsky Type

Expression (2.27) gives the means. We can consider

$$E^{i}_{r,t}(p,q;a,b,y)=\left(\frac{F^{i}(\phi_r;p,q;a,b,y)}{F^{i}(\phi_t;p,q;a,b,y)}\right)^{1/(r-t)},\quad r\neq t,\ \text{for } i=1,2,3 \tag{3.1}$$

as a means in the broader sense. Moreover we can extend these means in other cases. Consider the following functions to cover all continuous extensions of (3.1):

$$\mathfrak{I}^{1}(r)=\begin{cases}\dfrac{1}{r(r-1)}\left[\dfrac{pa^{r}+qb^{r}}{p+q}-\dfrac{M^{r+1}-m^{r+1}}{2y(r+1)}\right], & r\neq -1,0,1,\\ \dfrac{pb+qa}{2ab(p+q)}-\dfrac{\log M-\log m}{4y}, & r=-1,\\ \dfrac{M[\log M-1]-m[\log m-1]}{2y}-\dfrac{p\log a+q\log b}{p+q}, & r=0,\\ \dfrac{pa\ \log\ a+qb\ \log\ b}{p+q}-\Gamma, & r=1,\end{cases}$$

$$\mathfrak{I}^{2}(r)=\begin{cases}\dfrac{1}{r(r-1)}\left[\dfrac{M^{r+1}-m^{r+1}}{2y(r+1)}-\left(\dfrac{pa+qb}{p+q}\right)^{r}\right], & r\neq -1,0,1,\\ \dfrac{\log M-\log m}{4y}-\dfrac{p+q}{2(pa+qb)}, & r=-1,\\ \log\left(\dfrac{pa+qb}{p+q}\right)-\dfrac{M[\log M-1]-m[\log m-1]}{2y}, & r=0,\\ \Gamma-\dfrac{pa+qb}{p+q}\log\left(\dfrac{pa+qb}{p+q}\right), & r=1,\end{cases}$$

$$\mathfrak{I}^{3}(r)=\begin{cases}\dfrac{1}{r(r-1)}\left[\dfrac{pa^{r}+qb^{r}}{p+q}+\left(\dfrac{pa+qb}{p+q}\right)^{r}-\dfrac{M^{r+1}-m^{r+1}}{y(r+1)}\right], & r\neq -1,0,1,\\ \dfrac{pb+qa}{2ab(p+q)}+\dfrac{p+q}{2(pa+qb)}-\dfrac{\log M-\log m}{2y}, & r=-1,\\ \dfrac{M[\log M-1]-m[\log m-1]}{y}-\dfrac{p\ \log\ a+q\ \log\ b}{p+q}-\log\left(\dfrac{pa+qb}{p+q}\right), & r=0,\\ \dfrac{pa\ \log\ a+qb\ \log\ b}{p+q}+\dfrac{pa+qb}{p+q}\left[\log\left(\dfrac{pa+qb}{p+q}\right)-1\right]-2\Gamma, & r=1,\end{cases} \tag{3.2}$$

where $\Gamma=(M^{2}[2\log M-1]-m^{2}[2\log m-1])/8y$.

We have

$$E^i_{r,t}(p,q;a,b,y) = \begin{cases} \left(\dfrac{F^i(\phi_r;p,q;a,b,y)}{F^i(\phi_t;p,q;a,b,y)}\right)^{1/(r-t)}, & r \neq t, \\ \exp\left(\dfrac{1-2r}{r(r-1)} - \dfrac{F^i(\phi_0\phi_r;p,q;a,b,y)}{F^i(\phi_r;p,q;a,b,y)}\right), & r = t \neq -1,0,1, \\ \exp\left(\dfrac{3}{2} - \dfrac{F^i(\phi_0\phi_{-1};p,q;a,b,y)}{F^i(\phi_{-1};p,q;a,b,y)}\right), & r = t = -1, \\ \exp\left(1 - \dfrac{F^i(\phi_0^2;p,q;a,b,y)}{2F^i(\phi_0;p,q;a,b,y)}\right), & r = t = 0, \\ \exp\left(-1 - \dfrac{F^i(\phi_0\phi_1;p,q;a,b,y)}{2F^i(\phi_1;p,q;a,b,y)}\right), & r = t = 1, \end{cases} \tag{3.3}$$

for $i = 1,2,3$. We will use the following lemma to prove the monotonicity of Stolarsky type means.

Lemma 3.1. *Let f be log-convex function, and if $r \leq u$, $t \leq v$, $r \neq t$, $u \neq v$, then the following inequality is valid:*

$$\left(\frac{f(r)}{f(t)}\right)^{1/(r-t)} \leq \left(\frac{f(u)}{f(v)}\right)^{1/(u-v)}. \tag{3.4}$$

The proof of this Lemma is given in [1].

Theorem 3.2. *Let p, q, a, b, A, and y be real numbers as defined in Theorem 1.1 and let $r,t,u,v \in \mathbb{R}$ such that $r \leq u$, $t \leq v$, then the following inequality is valid:*

$$E^i_{r,t}(p,q;a,b,y) \leq E^i_{u,v}(p,q;a,b,y) \quad \text{for } i = 1,2,3. \tag{3.5}$$

Proof. For a convex function ϕ, a simple consequence of the definition of convex function is the following inequality [1, page 2]:

$$\frac{\phi(x_1)-\phi(x_2)}{x_2-x_1} \leq \frac{\phi(y_2)-\phi(y_1)}{y_2-y_1}, \quad \text{with } x_1 \leq y_1,\ x_2 \leq y_2,\ x_1 \neq x_2,\ y_1 \neq y_2. \tag{3.6}$$

As $\mathfrak{I}^i$ is log-convex we set $\phi(r) = \log \mathfrak{I}^i(r)$, $x_1 = r$, $x_2 = t$, $y_1 = v$, $y_2 = u$ in the above inequality and get

$$\frac{\log \mathfrak{I}^i(r) - \log \mathfrak{I}^i(t)}{r-t} \leq \frac{\log \mathfrak{I}^i(u) - \log \mathfrak{I}^i(v)}{u-v}, \tag{3.7}$$

which is equivalent to (3.5) for $t \neq r$, $u \neq v$. By continuity of $\mathfrak{I}^i$, (3.5) is valid for $t = r$, $u = v$. □

Remark 3.3. If we substitute $p = q = 1$ and replace $r \to r-1$ and $t \to t-1$ in $E^i_{r,t}(p,q;a,b,y)$, for $i = 1,2,3$, then means of Stolarsky type and related results given in [6] are obtained.

4. Generalized Means of Stolarsky Type

By substiting $a \to a^s, b \to b^s, y \to y^s, r \to r/s, t \to t/s, \xi \to \xi^{1/s}$ in (2.26), we get

$$\xi^{r-t} = \frac{t(t-s)F^i(x^{r/s};p,q;a^s,b^s,y^s)}{r(r-s)F^i(x^{t/s};p,q;a^s,b^s,y^s)}, \quad s \neq 0,\ t \neq r \text{ for } i = 1,2,3. \tag{4.1}$$

It follows that

$$E^i_{r,t;s}(p,q;a^s,b^s,y^s) = \left(\frac{F^i(\phi_{r/s};p,q;a^s,b^s,y^s)}{F^i(\phi_{t/s};p,q;a^s,b^s,y^s)}\right)^{1/(r-t)}, \quad s \neq 0, t \neq r \text{ for } i = 1,2,3. \tag{4.2}$$

To get all continuous extension of (4.2), we consider

$$A = \begin{cases} \left(\dfrac{pa^s + qb^s}{p+q}\right)^{1/s}, & s \neq 0, \\ (a^p b^q)^{1/(p+q)}, & s = 0, \end{cases}$$
$$y \leq \begin{cases} \left(\dfrac{b^s - a^s}{p+q}\min\{p,q\}\right)^{1/s}, & s \neq 0, \\ \left(\dfrac{b}{a}\right)^{1/(p+q)\min\{p,q\}}, & s = 0. \end{cases} \tag{4.3}$$

For $s \neq 0$, we define

$$\Im^i_s(r) = F^i(\phi_{r/s};p,q;a^s,b^s,y^s) \quad \text{for } i = 1,2,3, \tag{4.4}$$

where $\{\phi_r; r \in \mathbb{R}\}$ is the family of functions defined in Lemma 2.1. Here we have $F^i(f;p,q;a^s,b^s,y^s)$ defined as

$$F^1(f;p,q;a^s,b^s,y^s) = \frac{pf(a^s) + qf(b^s)}{p+q} - \frac{1}{2y^s}\int_{m_s}^{M_s} f(x)dx,$$
$$F^2(f;p,q;a^s,b^s,y^s) = \frac{1}{2y^s}\int_{m_s}^{M_s} f(x)dx - f\left(\frac{pa^s + qb^s}{p+q}\right), \tag{4.5}$$
$$F^3(f;p,q;a^s,b^s,y^s) = f\left(\frac{pa^s + qb^s}{p+q}\right) + \frac{pf(a^s) + qf(b^s)}{p+q} - \frac{1}{y^s}\int_{m_s}^{M_s} f(x)dx,$$

where $i = 1,2,3$, $m_s = A^s - y^s$, and $M_s = A^s + y^s$.

We have

$$\mathfrak{I}_s^1(r)=\begin{cases}\dfrac{s^2}{r(r-s)}\left[\dfrac{pa^r+qb^r}{p+q}-\dfrac{s}{r+s}\dfrac{M_s^{(r+s)/s}-m_s^{(r+s)/s}}{2y^s}\right], & r\neq -s,0,s,\\[2ex] \dfrac{pb^s+qa^s}{2a^sb^s(p+q)}-\dfrac{\log M_s-\log m_s}{4y^s}, & r=-s,\\[2ex] \dfrac{(M_s)\left[\log M_s-1\right]-m_s\left[\log m_s-1\right]}{2y^s}-s\dfrac{p\ \log\ a+q\ \log\ b}{p+q}, & r=0,\\[2ex] s\dfrac{pa^s\log\ a+qb^s\log\ b}{p+q}-\Gamma_s, & r=s,\end{cases}$$

$$\mathfrak{I}_s^2(r)=\begin{cases}\dfrac{s^2}{r(r-s)}\left[\dfrac{s}{r+s}\dfrac{{M_s}^{(r+s)/s}-{m_s}^{(r+s)/s}}{2y^s}-\left(\dfrac{pa^s+qb^s}{p+q}\right)^{r/s}\right], & r\neq -s,0,s,\\[2ex] \dfrac{\log M_s-\log m_s}{4y^s}-\left(\dfrac{p+q}{2(pa^s+qb^s)}\right)^s, & r=-s,\\[2ex] \dfrac{M_s\left[\log M_s-1\right]-m_s\left[\log m_s-1\right]}{2y^s}-s\ \log\dfrac{pa^s+qb^s}{p+q}, & r=0,\\[2ex] \Gamma_s-s\left(\dfrac{pa^s+qb^s}{p+q}\right)^s\log\left(\dfrac{pa^s+qb^s}{p+q}\right), & r=s,\end{cases}\tag{4.6}$$

$$\mathfrak{I}_s^3(r)=\begin{cases}\dfrac{s^2}{r(r-s)}\left[\dfrac{pa^r+qb^r}{p+q}+\left(\dfrac{pa^s+qb^s}{p+q}\right)^{r/s}-\dfrac{s}{r+s}\dfrac{M_s^{(r+s)/s}-m_s^{(r+s)/s}}{y^s}\right], & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad r\neq -s,0,s; & \\[2ex] \dfrac{pb^s+qa^s}{2a^sb^s(p+q)}+\left(\dfrac{p+q}{2(pa^s+qb^s)}\right)^s-\dfrac{\log M_s-\log m_s}{2y^s}, & r=-s,\\[2ex] \dfrac{M_s\left[\log M_s-1\right]-m_s\left[\log m_s-1\right]}{y^s}-s\log\left(\dfrac{pa^s+qb^s}{p+q}\right) & \\ \quad -s\dfrac{p\log\ a+q\log\ b}{p+q}, & r=0,\\[2ex] s\dfrac{pa\log\ a+qb\log\ b}{p+q}+s\left(\dfrac{pa^s+qb^s}{p+q}\right)^s\log\left(\dfrac{pa^s+qb^s}{p+q}\right)-2\Gamma_s, & r=s,\end{cases}$$

where $\Gamma_s=({M_s}^2[2\log M_s-1]-{m_s}^2[2\log m_s-1])/8y^s$.

For $s = 0$, we consider a family of convex functions $\{\psi_r : r \in \mathbb{R}\}$ defined on $\mathbb{R}$ by

$$\psi_r(x) = \begin{cases} \dfrac{1}{r^2}e^{rx}, & r \neq 0, \\ \dfrac{1}{2}x^2, & r = 0. \end{cases} \tag{4.7}$$

We have $\widehat{F}^i(f; p, q; \log a, \log b, \log y)$, $i = 1, 2, 3$ defined as

$$\begin{aligned} \widehat{F}^1(f; p, q; \log a, \log b, \log y) &= \frac{pf(\log a) + qf(\log b)}{p+q} - \frac{1}{2\log y}\int_{\log m}^{\log M} f(x)dx, \\ \widehat{F}^2(f; p, q; \log a, \log b, \log y) &= \frac{1}{2\log y}\int_{\log m}^{\log M} f(x)dx - f\left(\frac{p(\log a) + q(\log b)}{p+q}\right), \\ \widehat{F}^3(f; p, q; \log a, \log b, \log y) &= f\left(\frac{p\log a + q\log b}{p+q}\right) + \frac{pf(\log a) + qf(\log b)}{p+q} \\ &\quad - \frac{1}{\log y}\int_{\log m}^{\log M} f(x)dx, \end{aligned} \tag{4.8}$$

where $\log m = \log((a^p b^q)^{1/(p+q)}/y)$, $\log M = \log(y(a^p b^q)^{1/(p+q)})$. Now for

$$\begin{aligned} \mathfrak{I}_0^i(r) &= \widehat{F}^i(\psi_r; p, q; \log a, \log b, \log y) \quad \text{for } i = 1, 2, 3, \\ \mathfrak{I}_0^1(r) &= \begin{cases} \dfrac{1}{r^2}\left[\dfrac{pa^r + qb^r}{p+q} - \dfrac{(a^p b^q)^{r/(p+q)}(y^{2r} - 1)}{2ry^r \log y}\right], & r \neq 0, \\ \dfrac{1}{2}\left[\dfrac{p\log^2 a + q\log^2 b}{p+q} - \log^2 (a^p b^q)^{1/(p+q)} - \dfrac{1}{3}\log^2 y\right], & r = 0, \end{cases} \\ \mathfrak{I}_0^2(r) &= \begin{cases} \dfrac{1}{r^2}\left[\dfrac{(a^p b^q)^{r/(p+q)}(y^{2r} - 1)}{2ry^r \log y} - (a^p b^q)^{r/(p+q)}\right], & r \neq 0, \\ \dfrac{1}{6}\log^2 y, & r = 0, \end{cases} \\ \mathfrak{I}_0^3(r) &= \begin{cases} \dfrac{1}{r^2}\left[\dfrac{pa^r + qb^r}{p+q} + (a^p b^q)^{r/(p+q)} - \dfrac{(a^p b^q)^{r/(p+q)}(y^{2r} - 1)}{ry^r \log y}\right], & r \neq 0, \\ \dfrac{1}{2}\left[\dfrac{p\log^2 a + q\log^2 b}{p+q} - \log^2 (a^p b^q)^{1/(p+q)} - \dfrac{2}{3}\log^2 y\right], & r = 0. \end{cases} \end{aligned} \tag{4.9}$$

We get means

$$E^i_{r,t;s}(p,q;a^s,b^s,y^s)=\begin{cases}\left(\dfrac{F^i(\phi_{r/s};p,q;a^s,b^s,y^s)}{F^i(\phi_{t/s};p,q;a^s,b^s,y^s)}\right)^{1/(r-t)} & r\neq t,\ s\in\mathbb{R}\setminus\{0\},\\ \exp\left(\dfrac{r-2s}{r(r-s)}-\dfrac{F^i(\phi_0\phi_{r/s};p,q;a^s,b^s,y^s)}{sF^i(\phi_{r/s};p,q;a^s,b^s,y^s)}\right) & r=t,\ r^2-rs\neq 0,\\ \exp\left(\dfrac{3}{2s}-\dfrac{F^i(\phi_0\phi_{-1};p,q;a^s,b^s,y^s)}{sF^i(\phi_{-1};p,q;a^s,b^s,y^s)}\right) & r=t=-s,\ s\neq 0,\\ \exp\left(\dfrac{1}{s}-\dfrac{F^i(\phi_0^2;p,q;a^s,b^s,y^s)}{2sF^i(\phi_0;p,q;a^s,b^s,y^s)}\right) & r=t=0,\ s\neq 0,\\ \exp\left(-\dfrac{1}{s}-\dfrac{F^i(\phi_0\phi_1;p,q;a^s,b^s,y^s)}{2sF^i(\phi_1;p,q;a^s,b^s,y^s)}\right) & r=t=s,\ s\neq 0,\\ \exp\left(-\dfrac{2}{r}-\dfrac{\widehat{F}^i(x\psi_r;p,q;\log a,\log b,\log y)}{\widehat{F}^i(\psi_r;p,q;\log a,\log b,\log y)}\right) & r=t\neq 0,\ s=0,\\ \exp\left(\dfrac{\widehat{F}^i(x\psi_0;p,q;\log a,\log b,\log y)}{3\widehat{F}^i(\psi_0;p,q;\log a,\log b,\log y)}\right) & r=t=s=0,\end{cases}\tag{4.10}$$

for $i=1,2,3$.

Theorem 4.1. *Theorem 2.2 is still valid if one sets* $\phi_r=\psi_r$.

Proof. The proof is similar to the proof of Theorem 2.2. □

Theorem 4.2. *Let* p, q, a, b, A, *and* y *are real numbers as defined in Theorem 1.1 also let* $r,t,u,v\in\mathbb{R}$ *such that* $r\leq u$, $t\leq v$, *then the following inequality is valid:*

$$E^i_{r,t;s}(p,q;a^s,b^s,y^s)\leq E^i_{u,v;s}(p,q;a^s,b^s,y^s)\quad\text{for } i=1,2,3.\tag{4.11}$$

Proof. For $s\neq 0$, in this case we use Lemma 3.1 for $f=\Im^i$, and we have that

$$\left(\frac{\Im^i(r)}{\Im^i(t)}\right)^{1/(r-t)}\leq\left(\frac{\Im^i(u)}{\Im^i(v)}\right)^{1/(u-v)},\tag{4.12}$$

for $t, r, u, v \in \mathbb{R}$, $r \le u$, $t \le v$, $r \ne t$, $u \ne v$. For $s > 0$, by substituting $a \to a^s$, $b \to b^s$, $r \to r/s$, $t \to t/s$, $u \to u/s$, $v \to v/s$, such that $r/s \le u/s$, $t/s \le v/s$, $t \ne r$, $u \ne v$ in (4.12), we get

$$\left(\frac{\mathfrak{I}_s^i(r)}{\mathfrak{I}_s^i(t)}\right)^{s/(r-t)} \le \left(\frac{\mathfrak{I}_s^i(u)}{\mathfrak{I}_s^i(v)}\right)^{s/(u-v)}. \tag{4.13}$$

For $s < 0$, by substituting $\mathfrak{I}^i(r) = \mathfrak{I}_s^i(r)$, $a \to a^s$, $b \to b^s$, $r \to r/s$, $t \to t/s$, $u \to u/s$, $v \to v/s$, such that $u/s \le r/s$, $v/s \le t/s$, in (4.12) we have

$$\left(\frac{\mathfrak{I}_s^i(u)}{\mathfrak{I}_s^i(v)}\right)^{s/(u-v)} \le \left(\frac{\mathfrak{I}_s^i(r)}{\mathfrak{I}_s^i(t)}\right)^{s/(r-t)}, \tag{4.14}$$

By raising power $1/s$, to (4.13) and $-(1/s)$, to (4.14), we get (4.11) for $t \ne r$, $u \ne v$.

For $s = 0$, since $\mathfrak{I}_0^i(r)$ is log-convex function, therefore Lemma 3.1 implies that for $r \le u$, $t \le v$, $t \ne r$, $u \ne v$, we have

$$E_{r,t;0}^i(p, q; \log a, \log b, \log y) \le E_{u,v;0}^i(p, q; \log a, \log b, \log y), \tag{4.15}$$

which completes the proof. □

Remark 4.3. If we substitute $p = q = 1$, $s \to s - 1$, and $t \to t - 1$ in the above results, then the results of generalized Stolarsky type means proved in [6] are recaptured.

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