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Research Article

Uniform Convergence of Some Extremal Polynomials in Domain with Corners on the Boundary

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The aim of this paper is to investigate approximation properties of some extremal polynomials in A_p^1, $p > 0$ space. We are interested in finding approximation rate of extremal polynomials to Riemann function in A_p^1 and C-norms on domains bounded by piecewise analytic curve.

1. Problem and Main Results

Let G be a finite region with $z_0 \in G$ bounded by Jordan curve $L := \partial G$ and let $w = \varphi(z)$ be the canonical conformal mapping of G onto the disc $D_{r_0} := \{w : |w| < r_0\}$ with $\varphi(z_0) = 0,\ \varphi'(z_0) = 1$, where r_0 is called the conformal radius of $\overline{G}$ with respect to z_0.

Denote by $A_p^1(G)$, $p \in (0, \infty)$ the set of functions $f(z)$ analytic in G with $f(z_0) = 0,\ f'(z_0) = 1$ such that

$$\|f\|_{A_p^1} = \|f'\|_{A_p(G)} := \left(\iint_G |f'(z)|^p d\sigma_z\right)^{1/p} < \infty, \tag{1.1}$$

where $d\sigma_z$ is two-dimensional Lebesgue measure.

Also, let us denote by $\wp_n$ the class of all polynomials $P_n(z)$, $\deg P_n \le n$, with $P_n(z_0) = 0,\ P_n'(z_0) = 1$ and consider following extremal problem:

$$\iint_G |\varphi'(z) - P_n'(z)|^p d\sigma_z \longrightarrow \min, \quad p > 0. \tag{1.2}$$

Using a method given in [1, page 137], it is seen that the solution of the extremal problem in (1.2) exists, and if $p > 1$, the solution is unique [1, page 142]. This unique solution was denoted by $B_{n,p}(z)$ and it was called p-Bieberbach polynomials in [2].

Let us denote the best approximation to f in the class $\wp_n$ by A^1_p-norm and C-norm by

$$E_n\left(f, A^1_p\right) := \inf_{P_n \in \wp_n} \|f - P_n\|_{A^1_p}, \tag{1.3}$$

$$E_n\left(f, \overline{G}\right) := \inf_{P_n \in \wp_n} \|f - P_n\|_C = \inf_{P_n \in \wp_n} \max_{z \in \overline{G}} |f(z) - P_n(z)|, \tag{1.4}$$

respectively.

It is clear from the definition of p-Bieberbach polynomials that

$$E_n\left(\varphi, A^1_p\right) = \|\varphi - B_{n,p}\|_{A^1_p}. \tag{1.5}$$

One of the problem in approximation theory is to calculate $E_n(f, \overline{G})$ through the calculation of $E_n(f, A^1_p)$ for given f. This idea goes back at least as far as in [3, pages 116–141].

The special case $p = 2$ in (1.2) has two important properties. First, $B_{n,2}(z)$ coincides with usual Bieberbach polynomials $B_n(z)$ and it has an explicit representation via orthogonal polynomials [4]. Second, $B_{n,2}(z)$ is a main tool in the construction of Riemann mapping function for the given region.

Especially, approximation properties of Bieberbach polynomials were first investigated by Keldych in 1939 in [5], and then considerable progress in this area has been achieved by Mergelyan [6], Suetin [7], Simonenko [8], Andrievskiĭ [9, 10], Gaier [11, 12], Abdullayev [13–15], Israfilov [16, 17], and the others.

Besides this, approximation properties of $B_{n,p}(z)$ have been investigated only by authors of [2].

In this study, we are going to investigate the problem mentioned above in the region bounded by piecewise analytic curve and consider analytic curve as the image of a segment $[0,1]$ under conformal mapping in a neighborhood of this segment.

Definition 1.1. (a) The curve $L := \partial G$ is called piecewise analytic if it is a union of finite number of analytic arcs and it has $\lambda_j \pi$, $(0 < \lambda_j < 2,\ j = 1, 2, \ldots, m)$ exterior angles with respect to G on the z_j, $j = 1, 2, \ldots, m$ corners where two arcs meet.

(b) One denotes the class of piecewise analytic curve by $A(\lambda)$ where $\lambda := \min_{1 \le j \le m}\{\lambda_j\}$.

(c) One says $G \in A(\lambda)$, $0 < \lambda < 2$, if $L := \partial G \in A(\lambda)$, $0 < \lambda < 2$.

For any λ, $0 < \lambda < 2$ and p, $1 < p < 2/(1 - \lambda_*)$, let one denote

$$\lambda^* := \max\{1, \lambda\}, \quad \lambda_* := \min\{1, \lambda\}, \quad \gamma := \gamma(\lambda; p) = \frac{\lambda(\lambda - 1)}{2 - \lambda} + \frac{2}{p}\lambda, \tag{1.6}$$

$$\alpha(\lambda) := \max\left\{1, \frac{2(1-\lambda)(2-\lambda)}{1 + (1-\lambda)^2}\right\}. \tag{1.7}$$

Theorem 1.2. *Let $G \in A(\lambda)$ for some λ, $0 < \lambda < 2$ and p, $1 < p < 2/(1-\lambda_*)$. Then, for any $n = 1, 2, \ldots$, the p-Bieberbach polynomials $B_{n,p}$ satisfy*

$$\|\varphi - B_{n,p}\|_{A_p^1} \leq \text{const} \cdot n^{-\gamma}, \tag{1.8}$$

where γ is as in (1.6).

Theorem 1.3 (main theorem). *Let $G \in A(\lambda)$ for some λ, $0 < \lambda < 2$. Then, for any $n = 1, 2, \ldots$ the p-Bieberbach polynomials $B_{n,p}$ satisfy*

$$\|\varphi - B_{n,p}\|_C \leq \text{const} \begin{cases} n^{-\gamma}, & 2 < p < \dfrac{2}{1-\lambda_*}, \\ n^{-\gamma} \log n, & p = 2, \\ n^{-\gamma + (2/p - 1)\lambda^*}, & \alpha(\lambda) < p < 2, \end{cases} \tag{1.9}$$

where λ_*, λ^*, and $\alpha(\lambda)$ are defined in (1.6) *and* (1.7), *respectively.*

Corollary 1.4. (a) *If the region is a square, then Theorems 1.2 and 1.3 are true for*

$$\gamma = 3\left(\frac{1}{2} + \frac{1}{p}\right) \tag{1.10}$$

when $1 < p < \infty$.

(b) *If the region is an L-shaped region then Theorems 1.2 and 1.3 are true for*

$$\gamma = -\frac{1}{6} + \frac{1}{p} \tag{1.11}$$

when $6/5 < p < 4$.

Remark 1.5. If we take $p = 2$ in Theorems 1.2 and 1.3, we obtain the result of Gaier in [18].

2. Integral Representation of φ

We are going to follow the analog used by Andrievskiĭ and Gaier in [19]. Let us suppose that τ_i is a conformal mapping in an open neigborhood of $[0,1]$ such that $L_i := \tau_i([0,1])$. Then, there is a symmetric lens-shaped domain S_i whose closure is contained in this open neigborhood of $[0,1]$ (for more information see [19]).

So, we obtain

$$\widetilde{G} := G \cup \left(\bigcup_{i=1}^{m} \tau_i(S_i)\right), \tag{2.1}$$

and φ can be extended into $\tilde{G}$ as follows:

$$\tilde{\varphi}(z) := \begin{cases} \varphi(z), & z \in \overline{G}, \\ \dfrac{r_0^2}{\overline{\overline{\varphi\left[\tau_i\left(\overline{\tau_i^{-1}(z)}\right)\right]}}}, & z \in \tau_i(S_i) \setminus G. \end{cases} \tag{2.2}$$

From the construction of $\tilde{G}$, it is clear that $\partial\tilde{G}$ consists of m analytic arc Γ_i, $i = 1, 2, \ldots, m$, and $z_1, z_2, \ldots, z_m$ are the common end points of L_i and Γ_i.

For an arbitrary small ε, $\varepsilon < 1$, let us choose $R = 1 + cn^{\varepsilon-1}$ such that $1 < R < 2$, the points $z_i^{(j)}$, $i = 1, \ldots, m$, $j = 1, 2$ being the intersection of Γ_i and L_R. So, these points divide Γ_i into three parts such that

$$\Gamma_i = \Gamma_i^1 \cup \Gamma_i^2 \cup \Gamma_i^3, \tag{2.3}$$

where

$$\Gamma_i^1 := \Gamma_i\left(z_{i+1}, z_i^{(2)}\right), \qquad \Gamma_i^2 := \Gamma_i\left(z_i^{(2)}, z_i^{(1)}\right), \qquad \Gamma_i^3 := \Gamma_i\left(z_i^{(1)}, z_i\right), \tag{2.4}$$

so that

$$\partial\tilde{G} = \bigcup_{i=1}^{m} \bigcup_{j=1}^{3} \Gamma_i^j. \tag{2.5}$$

From the Cauchy integral formula, we have for all $z \in \overline{G}$

$$\begin{aligned} \varphi(z) &= \frac{1}{2\pi i} \int_{\partial\tilde{G}} \frac{\varphi(t)}{t - z} dt = \frac{1}{2\pi i} \sum_{i=1}^{m} \sum_{j=1}^{3} \int_{\Gamma_i^j} \frac{\varphi(t)}{t - z} dt \\ &= \sum_{i=1}^{m} \left(J_i^{(1)} + J_i^{(2)} + J_i^{(3)} \right), \end{aligned} \tag{2.6}$$

where

$$J_i^{(1)} := \frac{1}{2\pi i} \int_{\Gamma_i^1} \frac{\varphi(t)}{t - z} dt, \qquad J_i^{(3)} := \frac{1}{2\pi i} \int_{\Gamma_i^3} \frac{\varphi(t)}{t - z} dt, \qquad J_i^{(2)} := \frac{1}{2\pi i} \int_{\Gamma_i^2} \frac{\varphi(t)}{t - z} dt. \tag{2.7}$$

3. Some Auxiliary Results

We will use the notation $a \prec b$ for $a < cb$, where c is a constant independent from n. The following lemma plays central role in proving the main theorem.

Lemma 3.1. *Let $G \in A(\lambda)$ for some λ, $0 < \lambda < 2$ and let $1 < p < 2/(1-\lambda_*)$. Then, for any $n = 1, 2, \ldots$, there is a polynomial $Q_n(z)$ which satisfies both $Q_n(z_0) = 0$ and*

$$\|\varphi - Q_n\|_{A_p^1} \prec \frac{1}{n^\gamma}, \tag{3.1}$$

where

$$\gamma = \lambda\left(\frac{\lambda - 1}{2 - \lambda} + \frac{2}{p}\right). \tag{3.2}$$

Proof. Since "$J_i^{(2)}(z)$, $i = 1, \ldots, m$" is analytic on $\overline{G}$, there exists a polynomial with $\deg p_{n-1} \leq n-1$ [1, page 142] such that

$$\left|\left(J_i^{(2)}(z)\right)' - p_{n-1}(z)\right| \leq \frac{c}{n}, \quad i = 1, 2, \ldots, m, \tag{3.3}$$

where c is a constant independent from n.

Let us define $Q_n(z) := \int_{z_0}^{z} p_{n-1}(t)dt$. Then, $Q_n(z_0) = 0$, and from (2.6) and (3.3) we have

$$\left|\varphi'(z) - Q_n'(z)\right| \leq \frac{cm}{n} + \sum_{i=1}^{m}\left(\left|\left(J_i^{(1)}(z)\right)'\right| + \left|\left(J_i^{(3)}(z)\right)'\right|\right). \tag{3.4}$$

By taking integral over G of the pth power of both sides of (3.4), we obtain

$$\iint_G \left|\varphi'(z) - Q_n'(z)\right|^p d\sigma_z \prec \frac{1}{n^p} + \sum_{i=1}^{m}\left(\iint_G \left|\left(J_i^1(z)\right)'\right|^p d\sigma_z + \iint_G \left|\left(J_i^3(z)\right)'\right|^p d\sigma_z\right). \tag{3.5}$$

$J_i^{(1)}(z)$ and $J_i^{(3)}(z)$ $(i = 1, 2, \ldots, m)$ have the same property in $\overline{G}$, therefore, it is sufficient to show that A_p^1-norms of $J_i^{(1)}(z)$ and $J_i^{(3)}(z)$ tend to zero. So, we can restrict our attention only to the estimate of

$$\iint_G \left|\int_l \frac{\varphi(t)}{(t-z)^2}dt\right|^p d\sigma_z \longrightarrow 0 \tag{3.6}$$

where $l = \Gamma_i^{(1)}$ or $\Gamma_i^{(3)}$, $(i = 1, 2, \ldots, m)$.

To estimate this term, we need to know the behaviour of $\varphi(t)$ in the neigboorhood of the corner. For this, the main tool is the Lehman result.

We have from [20]

$$\left|\varphi(t)\right| \leq C|t - z_i|^{\alpha_i}, \quad t \longrightarrow z_i,\ i = 1, \ldots, m, \tag{3.7}$$

where $\alpha_i = 1/(2 - \lambda_i)$, $i = 1, 2, \ldots, m$.

We conclude from (3.7) and (3.6) that

$$\begin{aligned}\iint_G\left|\int_l \frac{|\varphi(t)|}{|t-z|^2}dt\right|^p d\sigma_z &\prec \iint_G\left(\int_l \frac{|\varphi(t)|}{|t-z|^2}dt\right)^p d\sigma_z \prec \iint_G\left(\int_l \frac{|t-z_i|^{\alpha_i}}{|t-z|^2}dt\right)^p d\sigma_z \\ &\prec \iint_{G_1}\left|\int_l \frac{|t-z_i|^{\alpha_i}}{|t-z|^2}dt\right|^p d\sigma_z + \iint_{G_2}\left(\int_l \frac{|t-z_i|^{\alpha_i}}{|t-z|^2}dt\right)^p d\sigma_z,\end{aligned} \tag{3.8}$$

where

$$\begin{gathered} G_1 := \{z : |z-z_i| \le \delta_R\} \cap G, \qquad G_2 := \{z : |z-z_i| > \delta_R\} \cap G, \\ \delta_R := \left|z_i^{(j)} - z_i\right|, \quad j = 1,2. \end{gathered} \tag{3.9}$$

If $z \in G_1$, we have $|t-z| \sim |t-z_i| + |z-z_i|$. Let us denote $|t-z_i|$ and $|z-z_i|$ with s, r, respectively. So,

$$\begin{aligned}\iint_{G_1}\left(\int_l \frac{|t-z_i|^{\alpha_i}}{|t-z|^2}|dt|\right)^p d\sigma_z &\le c_3\int_0^{\delta_R} r\left|\int_0^{c_4\delta_R} \frac{s^{\alpha_i}}{(s+r)^2}ds\right|^p dr \\ &\le c_3\int_0^{\delta_R} r\left|\int_0^{r} \frac{s^{\alpha_i}}{r^2}ds + \int_r^{c_4\delta_R} s^{\alpha_i-2}ds\right|^p dr \\ &\le c_3\int_0^{\delta_R} r\left(\frac{r^{\alpha_i+1}}{r^2} + c_5\delta_R^{\alpha_i-1} - r^{\alpha_i-1}\right)^p dr \\ &\le \int_0^{\delta_R} r\delta_R^{p(\alpha_i-1)}dr \le c_6\delta_R^{p(\alpha_i-1)+2}\end{aligned} \tag{3.10}$$

for $p(\alpha_i-1)+2>0$.

If $z \in G_2$, we have $|t-z| \sim |z-z_i|$. So,

$$\begin{aligned}\iint_{G_2}\left|\int_l \frac{|t-z_i|^{\alpha_i}}{|t-z|^2}|dt|\right|^p d\sigma_z &\le \iint_{G_2}\left|\int_l \frac{|t-z_i|^{\alpha_i}}{|z-z_i|^2}|dt|\right|^p d\sigma_z \le \iint_{G_2}\frac{\delta_R^{(\alpha_i+1)p}}{|z|^{2p}}d\sigma_z \\ &\le c\delta_R^{(\alpha_i+1)p}\int_{\delta_R}^{\infty} r^{1-2p}dr \le \delta_R^{p(\alpha_i-1)+2}.\end{aligned} \tag{3.11}$$

Substituting (3.10) and (3.11) into (3.6), we obtain

$$\iint_G\left|\int_l \frac{\varphi(t)}{(t-z)^2}dt\right|^p d\sigma_z \le \delta_R^{p(\alpha_i-1)+2}, \tag{3.12}$$

and also from (3.5), we have

$$\|\varphi - Q_n\|_{A_p^1}^p \le \delta_R^{p(\alpha_i-1)+2}. \tag{3.13}$$

If we use Lehman result [20] for $\Psi = \Phi^{-1}$, we obtain

$$\delta_R := \left|z_i^{(j)} - z_i\right| = \left|\Psi\left(\Phi\left(z_i^{(j)}\right)\right) - \Psi(\Phi(z_i))\right| \leq \left|\Phi\left(z_i^{(j)}\right) - \Phi(z_i)\right|^{\lambda_i} \leq n^{(\epsilon-1)\lambda_i}. \tag{3.14}$$

The proof is completed by (3.13) and (3.14). □

Lemma 3.2. *Let $G \in A(\lambda)$, $0 < \lambda < 2$. Then, for all polynomials $P_n(z), \deg P_n(z) \leq n$ with $P_n(z_0) = 0$, $n = 2, 3, \ldots$, one has*

$$\|P_n\|_C \prec \|P_n\|_{A_p^1} \begin{cases} 1, & p > 2, \\ \sqrt{\log n}, & p = 2, \\ n^{(2/p-1)\lambda^*}, & p < 2. \end{cases} \tag{3.15}$$

Proof. We will prove only the case $p < 2$ since the other cases are already given in [10, 21].

Let z be an arbitrary fixed point on the boundary. It is clear from [14, Lemma 2.2] that $l(z_0, z) \subset G$ exists joining z_0, z and satisfying cord arc properties. If $l_1 := \{\xi \in l(z_0, z) : |\xi - z| \leq \varepsilon n^{-\lambda_*}\}$ and $l_2 := l(z_0, z) \setminus l_1$, then we have

$$|P_n(z)| = \left|\int_{l(z_0,z)} P_n'(\xi)d\xi\right| \leq \int_{l_1} \left|P_n'(\xi)\right||d\xi| + \int_{l_2} \left|P_n'(\xi)\right||d\xi|. \tag{3.16}$$

It is well known from [14, Corollary 2.3] that

$$\left\|P_n'\right\|_{C(\overline{G})} \leq c_1 n^{\lambda^*} \cdot \|P_n\|_{C(\overline{G})}. \tag{3.17}$$

At the same time, $\operatorname{mes}(l_1) \leq c_2 \varepsilon n^{-\lambda^*}$ is valid for a positive constant c_2 which is independent from ε. Using the Mean Value property of subharmonic function $|P_n'(\xi)|^p$ (see [22, page 482]), we have for arbitrary point $\xi \in l_2$

$$\left|P_n'(\xi)\right| \leq \frac{1}{\left[\pi d^2(\xi, L)\right]^{1/p}} \|P_n\|_{A_p^1}, \tag{3.18}$$

and after combining (3.18) and (3.16), we obtain

$$\begin{aligned} |P_n(z)| &\leq c_1 n^{\lambda^*} \cdot \|P_n\|_{C(\overline{G})} \int_{l_1} |d\xi| + c_3 \|P_n\|_{A_p^1} \int_{l_2} \frac{|d\xi|}{d^{2/p}(\xi, L)} \\ &\leq c_1 n^{\lambda^*} \cdot \|P_n\|_{C(\overline{G})} . c_2 \varepsilon n^{-\lambda^*} + c_3 \|P_n\|_{A_p^1} \int_{l_2} \frac{|d\xi|}{|\xi - z|^{2/p}} \\ &\leq c_1 c_2 \varepsilon \|P_n\|_{C(\overline{G})} + c_3 \|P_n\|_{A_p^1} \int_{c_2 \varepsilon n^{-\lambda^*}}^{\operatorname{mes}(l)} \frac{dt}{t^{2/p}} \\ &\leq c_1 c_2 \varepsilon \|P_n\|_{C(\overline{G})} + c_3 \|P_n\|_{A_p^1} n^{(2/p-1)\lambda^*}. \end{aligned} \tag{3.19}$$

Using the maximum modulus principle and choosing ε satisfying $c_1c_2\varepsilon < 1$, the proof is obtained. □

Lemma 3.2 shows how we can measure C-norm of polynomials by using its A_p^1-norm.

Lemma 3.3 (see [2]). *Let $G \subset \mathbb{C}$ be a simply connected region so that*

$$\left\|\varphi - B_{n,p}\right\|_{A_p^1} \leq n^{-\eta} \tag{3.20}$$

for each $\mu \in (0,1)$, $n = 1, 2, \ldots$, and

$$\|P_n\|_C \prec n^{\mu} \|P_n\|_{A_p^1} \tag{3.21}$$

for all polynomials $P_n(z)$, $\deg P_n \leq n$ with $P_n(z_0) = 0$. Then,

$$\left\|\varphi - B_{n,p}\right\|_C \leq n^{\mu-\eta}. \tag{3.22}$$

4. Proof of Theorems 1.2 and 1.3

4.1. Proof of Theorem 1.2

Let us set $P_n(z)$ as follows:

$$P_n(z) := Q_n(z) + \left(\varphi'(z_0) - Q_n'(z)\right)(z - z_0), \tag{4.1}$$

where $Q_n(z)$ as in Lemma 3.1 and satisfying $Q_n(z_0) = 0$.

It is clear from the definition of $P_n(z)$ that $P_n(z_0) = 0, P_n'(z_0) = 1$ is satisfying

$$\left|\varphi'(z) - P_n'(z)\right| \leq \left|\varphi'(z) - Q_n'(z)\right| + \left|\varphi'(z_0) - Q_n'(z_0)\right|. \tag{4.2}$$

So, we have

$$\left\|\varphi - P_n\right\|_{A_p^1}^p \leq \delta_R^{p(\alpha_i-1)+2} + \left|\varphi'(z_0) - Q_n'(z_0)\right|, \tag{4.3}$$

and from the Mean Value Theorem in [4] we also have

$$\left|\varphi'(z_0) - Q_n'(z_0)\right| \leq \frac{1}{\pi d^{2/p}(z_0, L)} \left\|\varphi - Q_n\right\|_{A_p^1}. \tag{4.4}$$

So, (4.3), (4.4), and (3.13) give

$$\left\|\varphi - P_n\right\|_{A_p^1}^p \leq n^{-(p(\alpha_i-1)+2))}. \tag{4.5}$$

Using extremal properties of the p-Bieberbach polynomials, the proof is completed.

4.2. Proof of Theorem 1.3

Lemma 3.3 shows that it is enough to choose η, μ in (3.20) and (3.21), respectively. For this, we take η as in Theorem 1.2 and μ as in Lemma 3.2.

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