Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 439682, 18 pages
doi:10.1155/2010/439682

Research Article

Application of the Banach Fixed-Point Theorem to the Scattering Problem at a Nonlinear Three-Layer Structure with Absorption

V. S. Serov,[1] H. W. Schüermann,[2] and E. Svetogorova[2]

[1] *Department of Mathematical Sciences, University of Oulu, P.O. Box 3000, Oulu 90014, Finland*
[2] *Department of Physics, University of Osnabrück, Barbarastraße 7, D-49069 Osnabrück, Germany*

Correspondence should be addressed to V. S. Serov, vserov@cc.oulu.fi

Received 29 August 2009; Accepted 23 April 2010

Academic Editor: Massimo Furi

A method based on the Banach fixed-point theorem is proposed for obtaining certain solutions (TE-polarized electromagnetic waves) of the Helmholtz equation describing the reflection and transmission of a plane monochromatic wave at a nonlinear lossy dielectric film situated between two lossless linear semiinfinite media. All three media are assumed to be nonmagnetic and isotropic. The permittivity of the film is modelled by a continuously differentiable function of the transverse coordinate with a saturating Kerr nonlinearity. It is shown that the solution of the Helmholtz equation exists in form of a uniformly convergent sequence of iterations of the equivalent Volterra integral equation. Numerical results are presented.

1. Introduction

Scattering of transverse-electric (TE) electromagnetic waves at a single nonlinear homogeneous, isotropic, nonmagnetic layer situated between two homogeneous, semiinfinite media has been the subject of intense theoretical and experimental investigations in recent years. In particular, the Kerr-like nonlinear dielectric film has been the focus of a number of studies [1–6].

Exact analytical solutions have been obtained for the scattering of plane TE-waves with Kerr-nonlinear films [7, 8]. As far as exact analytical solutions were considered in those papers, absorbtion was excluded, at most it was treated numerically [9–11].

As Chen and Mills have pointed out, it is a nontrivial extension of the usual scattering theory to include absorption [3] and it seems (to the best of our knowledge) that the problem was not solved till now. In the following we propose a solution based on the Banach fixed-point theorem (contraction principle) of the functional analysis [12]. To demonstrate the broad range of applicability of this theorem we consider a nonlinear lossy dielectric film with spatially varying saturating permittivity. In Section 2 we reduce Maxwell's equations to a Volterra integral equation (2.12) for the intensity of the electric field $E(y)$ and give a solution in form of a uniform convergent sequence of iterate functions. Using these solutions, we determine the phase function $\vartheta(y)$ of the electric field, and, evaluating the boundary conditions in Section 3, we derive analytical expressions for reflectance, transmittance, absorptance, and phase shifts on reflection and transmission.

It should be emphasised that the contraction principle includes the proof of the existence of the exact bounded (i.e., "physical") solution of the problem and additionally yields approximate analytical solution by iterations. Furthermore, the rate of convergence of the iterative procedure and the error estimate can be evaluated [12]. Thus this approach is useful for physical applications.

Referring to Figure 1, we consider the reflection and transmission of an electromagnetic plane wave at a dielectric film between two linear semiinfinite media (substrate and cladding). All media are assumed to be homogeneous in x- and z-direction, isotropic, and nonmagnetic. The film is assumed to be absorbing and characterized by a complex valued permittivity function $\varepsilon_f(y)$.

A plane wave of frequency ω_0 and intensity E_0^2, with electric vector $\mathbf{E}_0$ parallel to the z-axis (TE), is incident on the film of thickness d. Since the geometry is independent of the z-coordinate and because of the supposed TE-polarization, fields are parallel to the z-axis ($\mathbf{E} = (0, 0, E_z)$). More precisely, we look for solution $\mathbf{E}$ of Maxwell's equations

$$\begin{aligned} \operatorname{rot}\mathbf{H} &= -i\omega_0\varepsilon\mathbf{E}, \\ \operatorname{rot}\mathbf{E} &= i\omega_0\mu_0\mathbf{H} \end{aligned} \tag{1.1}$$

that satisfy the boundary conditions (continuity of E_z and $\partial E_z/\partial y$ at interfaces $y \equiv 0$ and $y \equiv d$) and where (due to TE-polarization) $\mathbf{H} = (H_x, H_y, 0)$. Due to the requirement of the translational invariance in x-direction and partly satisfying the boundary conditions, the fields tentatively are written as ($\hat{z}$ denotes the unit vector in z-direction)

$$\mathbf{E}(x,y,t) = \begin{cases} \hat{z}\left[E_0 e^{i(px - q_c\cdot(y-d) - \omega_0 t)} + E_r e^{i(px + q_c\cdot(y-d) - \omega_0 t)}\right], & y > d, \\ \hat{z}\left[E(y) e^{i(px + \vartheta(y) - \omega_0 t)}\right], & 0 < y < d, \\ \hat{z}\left[E_3 e^{i(px - q_s y - \omega_0 t)}\right], & y < 0, \end{cases} \tag{1.2}$$

where $E(y)$, $p = \sqrt{\varepsilon_c}k_0 \sin\varphi$, $k_0 = \omega_0\sqrt{\varepsilon_0\mu_0}$, q_c, q_s, and $\vartheta(y)$ are real and $E_r = |E_r|\exp(\mathrm{i}\delta_r)$ and $E_3 = |E_3|\exp(\mathrm{i}\delta_t)$ are independent of y.

We assume a permittivity $\varepsilon(y)$ of the three-layer system modelled by

$$\frac{\varepsilon(y)}{\varepsilon_0} = \begin{cases} \varepsilon_c, & y > d, \\ \varepsilon_f(y) = \varepsilon_f^0 + \varepsilon_R(y) + i\varepsilon_I(y) + \dfrac{aE^2(y)}{1 + arE^2(y)}, & 0 < y < d, \\ \varepsilon_s, & y < 0, \end{cases} \tag{1.3}$$

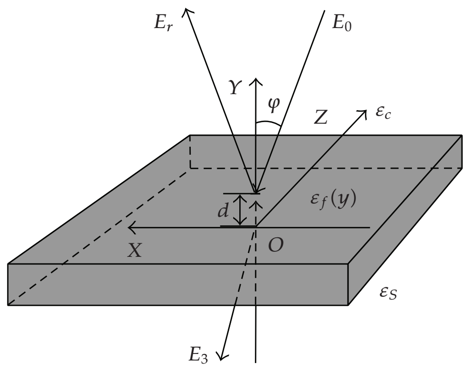

Figure 1: Configuration considered in this paper. A plane wave is incident to a nonlinear slab (situated between two linear media) to be reflected and transmitted.

with real constants ε_c, ε_s, ε_f^0, a, r and real-valued continuously differentiable functions $\varepsilon_R(y)$, $\varepsilon_I(y)$ on $[0,d]$.

2. Reduction of the Scattering Problem to the Solution of a Volterra Integral Equation

By inserting (1.2) and (1.3) into Maxwell's equations we obtain the nonlinear Helmholtz equations, valid in each of the three media ($j = s, f, c$),

$$\frac{\partial^2 \widetilde{E}_j(x,y)}{\partial x^2} + \frac{\partial^2 \widetilde{E}_j(x,y)}{\partial y^2} + k_0^2 \frac{\varepsilon(y)}{\varepsilon_0} \widetilde{E}_j(x,y) = 0, \quad j = s, f, c, \tag{2.1}$$

where $\widetilde{E}_j(x,y)$ denotes the time-independent part of $\mathbf{E}(x,y,t)$.

Scaling x, y, z, p, q_c, q_s by k_0 and using the definition of $\varepsilon/\varepsilon_0$ in (1.3), (2.1) reads

$$\frac{\partial^2 \widetilde{E}_j(x,y)}{\partial x^2} + \frac{\partial^2 \widetilde{E}_j(x,y)}{\partial y^2} + \varepsilon_j(y) \widetilde{E}_j(x,y) = 0, \quad j = s, f, c, \tag{2.2}$$

where the same symbols have been used for unscaled and scaled quantities. Using ansatz (1.2) in (2.2), we get for the semiinfinite media

$$q_j^2 = \varepsilon_j - p^2, \quad j = s, c. \tag{2.3}$$

For the film ($j = f$), we obtain, omitting tildes,

$$\frac{\mathrm{d}^2 E(y)}{\mathrm{d}y^2} - E(y)\left(\frac{\mathrm{d}\vartheta(y)}{\mathrm{d}y}\right)^2 + \left(\varepsilon_f^0 + \varepsilon_R(y) - p^2 + \frac{aE^2(y)}{1 + arE^2(y)}\right)E(y) = 0, \tag{2.4}$$

$$E(y)\frac{\mathrm{d}^2\vartheta(y)}{\mathrm{d}y^2} + 2\frac{\mathrm{d}\vartheta(y)}{\mathrm{d}y}\frac{\mathrm{d}E(y)}{\mathrm{d}y} + \varepsilon_I(y)E(y) = 0. \tag{2.5}$$

Equation (2.5) can be integrated leading to

$$E^2(y)\frac{\mathrm{d}\vartheta(y)}{\mathrm{d}y} = c_1 - \int_0^y \varepsilon_I(\tau)E^2(\tau)\mathrm{d}\tau, \tag{2.6}$$

where c_1 is a constant that is determined by means of the boundary conditions:

$$c_1 = E^2(0)\frac{\mathrm{d}\vartheta(0)}{\mathrm{d}y} = -q_s E^2(0). \tag{2.7}$$

Insertion of $\mathrm{d}\vartheta(y)/\mathrm{d}y$ according to equation (2.6) into (2.4) leads to

$$\frac{d^2E(y)}{dy^2} + \left(q_f^2(y) - p^2\right)E(y) + \frac{aE^3(y)}{1 + arE^2(y)} - \frac{\left(c_1 - \int_0^y \varepsilon_I(t)E^2(t)\mathrm{d}t\right)^2}{E^3(y)} = 0, \tag{2.8}$$

with

$$q_f^2(y) = \varepsilon_f^0 + \varepsilon_R(y). \tag{2.9}$$

As for real permittivity, real q_s (transmission) implies that $c_1 \neq 0$.

Setting $I(y) = aE^2(y)$, multiplying (2.8) by $4E^3(y)$, and differentiating the result with respect to y, we obtain

$$\begin{aligned}\frac{\mathrm{d}^3I(y)}{\mathrm{d}y^3} + 4\frac{\mathrm{d}\left(\left(q_f^2(y) - p^2\right)I(y)\right)}{\mathrm{d}y} &= 2\frac{\mathrm{d}\left(q_f^2(y)\right)}{\mathrm{d}y}I(y) - \frac{2I(y)(\mathrm{d}I(y)/\mathrm{d}y)(3 + 2rI(y))}{(1 + rI(y))^2}\\ &\quad - 4\varepsilon_I(y)\left(ac_1 - \int_0^y \varepsilon_I(t)I(t)\mathrm{d}t\right).\end{aligned} \tag{2.10}$$

Equation (2.10) can be integrated with respect to $I(y)$ to yield

$$\begin{aligned}\frac{\mathrm{d}^2I(y)}{\mathrm{d}y^2} + 4\kappa^2 I(y) &= -4\varepsilon_R(y)I(y) + 2\int_0^y \frac{\mathrm{d}\varepsilon_R(t)}{\mathrm{d}t}I(t)\mathrm{d}t\\ &\quad - \frac{2}{r^2}\left(2rI(y) + \frac{1}{1 + rI(y)} - \ln(1 + rI(y))\right)\\ &\quad + 4\int_0^y \varepsilon_I(t)\left(\int_0^t \varepsilon_I(z)I(z)\mathrm{d}z\right)\mathrm{d}t - 4ac_1\int_0^y \varepsilon_I(t)\mathrm{d}t + c_2,\end{aligned} \tag{2.11}$$

where $\kappa^2 = \varepsilon_f^0 - p^2$ and c_2 is a constant of integration.

The homogeneous equation $d^2I(y)/dy^2 + 4\kappa^2 I(y) = 0$ has the solution

$$\widetilde{I}_0(y) = a|E_3|^2 \cos(2\kappa y) \tag{2.12}$$

so that the general solution of (2.11) reads [13]

$$\begin{aligned} I(y) = \widetilde{I}_0(y) + \int_0^y \mathrm{d}t \frac{\sin 2\kappa(y-t)}{2\kappa} \\ \cdot \left(-4\,\varepsilon_R(t)I(t) + 2\int_0^t \frac{\mathrm{d}\varepsilon_R(\tau)}{\mathrm{d}\tau} I(\tau)\mathrm{d}\tau - \frac{2}{r^2}\left(2rI(t) + \frac{1}{1+rI(t)} - \ln(1+rI(t))\right) \right. \\ \left. +4\int_0^t \varepsilon_I(\tau)\left(\int_0^\tau \varepsilon_I(z)I(z)\mathrm{d}z\right)\mathrm{d}\tau - 4ac_1\int_0^t \varepsilon_I(\tau)\mathrm{d}\tau + c_2 \right), \end{aligned} \tag{2.13}$$

where the constant c_2 must be determined by the boundary conditions.

The Volterra equation (2.13) is equivalent to (2.1) for $0 < y < d$. According to (2.13) $I(y)$ and $\widetilde{I}_0(y)$ satisfy the boundary conditions at $y = 0$. Evaluating some of the integrals on the righthand side, (2.13) can be written as

$$\begin{aligned} I(y) = I_0(y) + \frac{1}{\kappa^2}\int_0^y \sin^2\kappa(y-\tau)\frac{d\varepsilon_R(\tau)}{d\tau} I(\tau)\mathrm{d}\tau \\ - \frac{2}{\kappa}\int_0^y \sin 2\kappa(y-\tau)\varepsilon_R(\tau)I(\tau)\mathrm{d}\tau \\ - \frac{2}{r\kappa}\int_0^y \sin 2\kappa(y-\tau)I(\tau)\mathrm{d}\tau \\ - \frac{1}{\kappa r^2}\int_0^y \sin 2\kappa(y-\tau)\frac{1}{1+rI(\tau)}\mathrm{d}\tau \\ + \frac{1}{\kappa r^2}\int_0^y \sin 2\kappa(y-\tau)\ln(1+rI(\tau))\mathrm{d}\tau \\ + 4\int_0^y \varepsilon_I(z)I(z)\mathrm{d}z \int_z^y \frac{\sin 2\kappa(y-t)}{2\kappa}\left\{\int_z^t \varepsilon_I(\tau)\mathrm{d}\tau\right\}\mathrm{d}t, \end{aligned} \tag{2.14}$$

with (on the evaluation of c_2 see Appendix A)

$$I_0(y) = \widetilde{I}_0(y) + \frac{c_2 \sin^2\kappa y}{2\kappa^2} - 4ac_1\int_0^y \frac{\sin 2\kappa(y-t)}{2\kappa}\int_0^t \varepsilon_I(z)\mathrm{d}z\,\mathrm{d}t, \tag{2.15}$$

$$ac_1 = -q_s I(0), \tag{2.16}$$

$$c_2 = 2I(0)\left(q_s^2 + q_f^2(0) - p^2\right) - \frac{2I^2(0)}{1+rI(0)} + \frac{2}{r^2}\left(2rI(0) + \frac{1}{1+rI(0)} - \ln(1+rI(0))\right), \tag{2.17}$$

where $\widetilde{I}_0(y)$ is given by (2.12).

Let us introduce in the Banach space $C[0,d]$ bounded integral operators $N_1, N_2, N_3, N_4, N_5, N_c$ by

$$
\begin{aligned}
N_1(I) &= \frac{1}{\kappa^2}\int_0^y \sin^2\kappa(y-\tau)\frac{d\varepsilon_R(\tau)}{d\tau}I(\tau)\mathrm{d}\tau,\\
N_2(I) &= -\frac{2}{\kappa}\int_0^y \sin 2\kappa(y-\tau)\varepsilon_R(\tau)I(\tau)\mathrm{d}\tau,\\
N_3(I) &= -\frac{2}{\kappa}\int_0^y \sin 2\kappa(y-\tau)I(\tau)\mathrm{d}\tau,\\
N_4(I) &= -\frac{1}{\kappa}\int_0^y \sin 2\kappa(y-\tau)\frac{1}{1+rI(\tau)}\mathrm{d}\tau,\\
N_5(I) &= \frac{1}{\kappa}\int_0^y \sin 2\kappa(y-\tau)\ln(1+rI(\tau))\mathrm{d}\tau,\\
N_c(I) &= 4\int_0^y \varepsilon_I(s)\varphi(y,s)I(s)\mathrm{d}s,
\end{aligned}
\tag{2.18}
$$

where constant c_2 is given by (A.3) and

$$
\varphi(y,s) = \int_s^y \frac{\sin 2\kappa(y-t)}{2\kappa}\left\{\int_s^t \varepsilon_I(\tau)\mathrm{d}\tau\right\}\mathrm{d}t \tag{2.19}
$$

with the values $\|N_1\|$, $\|N_2\|$, $\|N_3\|$, $\|N_4\|$, $\|N_5\|$, $\|N_c\|$, and $\|I_{0s}\|$ which are defined as

$$
\begin{aligned}
\|N_1\| &= \frac{1}{\kappa^2}\max_{0\le y\le d}\int_0^y \left|\sin^2\kappa(y-\tau)\right|\cdot\left|\frac{\mathrm{d}\varepsilon_R(\tau)}{\mathrm{d}\tau}\right|\mathrm{d}\tau,\\
\|N_2\| &= \frac{2}{\kappa}\max_{0\le y\le d}\int_0^y \left|\sin 2\kappa(y-\tau)\right|\cdot|\varepsilon_R(\tau)|\mathrm{d}\tau,\\
\|N_3\| &= \frac{2}{\kappa}\max_{0\le y\le d}\int_0^y \left|\sin 2\kappa(y-\tau)\right|\mathrm{d}\tau,\\
\|N_4\| &= \frac{1}{\kappa}\max_{0\le y\le d}\int_0^y \left|\sin 2\kappa(y-\tau)\right|\mathrm{d}\tau,\\
\|N_5\| &= \frac{1}{\kappa}\max_{0\le y\le d}\int_0^y \left|\sin 2\kappa(y-\tau)\right|\mathrm{d}\tau = \|N_4\|,\\
\|N_c\| &= 4\max_{0\le y\le d}\int_0^y |\varepsilon_I(z)|\cdot\left|\varphi(y,z)\right|\mathrm{d}z,\\
\|I_0\| &= \max_{0\le y\le d}|I_0|.
\end{aligned}
\tag{2.20}
$$

The following main result holds.

Theorem 2.1. *If*

$$\|N_1\| + \|N_2\| + \|N_c\| + \frac{\|N_3\| + 2\|N_4\|}{r} < 1, \tag{2.21}$$

$$\frac{\|I_{0s}\| + (1/r^2)\|N_4\|}{1 - (\|N_1\| + \|N_2\| + \|N_c\| + (\|N_3\| + \|N_4\|)/r)} < \rho, \tag{2.22}$$

then in any ball $S_\rho(0)$ there exists a unique solution of the nonlinear integral equation (2.14) *and this solution can be obtained as a uniform limit*

$$I(y) = \lim_{j\to\infty} I_j(y) \tag{2.23}$$

of the iterations of (2.14).

Proof. Let us introduce the iterations of (2.14) as follows:

$$\begin{aligned} I_j(y) = I_0(y) &+ \frac{1}{\kappa^2}\int_0^y \sin^2\kappa(y-\tau)\frac{d\varepsilon_R(\tau)}{d\tau} I_{j-1}(\tau)\mathrm{d}\tau \\ &- \frac{2}{\kappa}\int_0^y \sin 2\kappa(y-\tau)\varepsilon_R(\tau)I_{j-1}(\tau)\mathrm{d}\tau \\ &- \frac{2}{r\kappa}\int_0^y \sin 2\kappa(y-\tau)I_{j-1}(\tau)\mathrm{d}\tau \\ &- \frac{1}{\kappa r^2}\int_0^y \sin 2\kappa(y-\tau)\frac{1}{1+rI_{j-1}(\tau)}\mathrm{d}\tau \\ &+ \frac{1}{\kappa r^2}\int_0^y \sin 2\kappa(y-\tau)\ln(1+rI_{j-1}(\tau))\mathrm{d}\tau \\ &+ 4\int_0^y \varepsilon_I(z)I_{j-1}(z)\mathrm{d}z \int_z^y \frac{\sin 2\kappa(y-t)}{2\kappa}\left\{\int_z^t \varepsilon_I(\tau)\mathrm{d}\tau\right\}\mathrm{d}t, \end{aligned} \tag{2.24}$$

where $j = 1, 2, \ldots,$ and $I_0(y)$ is given by (2.15). In order to prove that the sequence (2.24) is uniformly convergent to the solution of (2.14) it suffices to check that all conditions of the Banach fixed-point theorem [12] are fulfilled.

We consider the nonlinear operator F as

$$F(I) := I_0(y) + N_1(I) + N_2(I) + \frac{1}{r}N_3(I) + \frac{1}{r^2}N_4(I) + \frac{1}{r^2}N_5(I) + N_c(I). \tag{2.25}$$

Then (2.14) can be rewritten in operator form

$$I(y) = F(I)(y). \tag{2.26}$$

We consider ρ such that $\|I\| = \max_{0\le y\le d} I(y) \le \rho$. First we must check whether this operator F maps the ball $S_\rho(0)$ to itself. Indeed, if $I(y) \in S_\rho(0)$, then

$$\begin{aligned}\|F(I)\| &\le \|I_0\| + \|N_1\|\cdot\|I\| + \|N_2\|\cdot\|I\| + \|N_c\|\cdot\|I\| + \frac{1}{r}\|N_3\|\cdot\|I\| \\ &\quad + \frac{1}{r^2}\|N_4\|\cdot\frac{1}{1+r\min_{0\le y\le d} I(y)} + \frac{1}{r^2}\|N_4\|\cdot r\|I\| \\ &\le \|I_0\| + \|N_1\|\cdot\rho + \|N_2\|\cdot\rho + \|N_c\|\cdot\rho + \frac{1}{r}\|N_3\|\cdot\rho \\ &\quad + \frac{1}{r^2}\|N_4\| + \frac{1}{r}\|N_4\|\cdot\rho.\end{aligned} \tag{2.27}$$

Thus, the following inequality must be valid:

$$\|I_0\| + \frac{1}{r^2}\|N_4\| + \left(\|N_1\| + \|N_2\| + \|N_c\| + \frac{1}{r}(\|N_3\| + \|N_4\|)\right)\cdot\rho < \rho. \tag{2.28}$$

This inequality holds if

$$\frac{\|I_0\| + (1/r^2)\|N_4\|}{1 - (\|N_1\| + \|N_2\| + \|N_c\| + (\|N_3\| + \|N_4\|)/r)} < \rho, \tag{2.29}$$

and, thus, if

$$\|N_1\| + \|N_2\| + \|N_c\| + \frac{\|N_3\| + \|N_4\|}{r} < 1. \tag{2.30}$$

It means that for this value of ρ (2.29) continuous map F transfers ball $S_\rho(0)$ in itself. Hence, (2.14) has at least one solution inside $S_\rho(0)$. For uniqueness of this solution it remains to prove that F is contractive [12]. To prove the contraction of F we consider

$$\begin{aligned}F(I_1) - F(I_2) &= N_1(I_1 - I_2) + N_2(I_1 - I_2) + N_c(I_1 - I_2) \\ &\quad + \frac{1}{r}N_3(I_1 - I_2) + \frac{1}{r^2}(N_4(I_1) - N_4(I_2)) + \frac{1}{r^2}(N_5(I_1) - N_5(I_2)).\end{aligned} \tag{2.31}$$

Hence

$$\begin{aligned}\|F(I_1) - F(I_2)\| &\le \|N_1\|\|I_1 - I_2\| + \|N_2\|\|I_1 - I_2\| + \|N_c\|\|I_1 - I_2\| \\ &\quad + \|N_3\|\left\|\frac{1}{r}(I_1 - I_2)\right\| + \left\|\frac{1}{r^2}(N_4(I_1) - N_4(I_2))\right\| \\ &\quad + \left\|\frac{1}{r^2}(N_5(I_1) - N_5(I_2))\right\|.\end{aligned} \tag{2.32}$$

The following estimations hold:

(i)

$$\left\|\frac{1}{r^2}(N_4(I_1)-N_4(I_2))\right\| \le \max_{0\le y\le d}\frac{1}{\kappa r^2}\int_0^y \left|\sin 2\kappa(y-\tau)\right|\cdot\left|\frac{1}{1+rI_1(\tau)}-\frac{1}{1+rI_2(\tau)}\right|\mathrm{d}\tau$$
$$=\max_{0\le y\le d}\frac{1}{\kappa r^2}\int_0^y \left|\sin 2\kappa(y-\tau)\right|\cdot\left|\frac{rI_2(\tau)-rI_1(\tau)}{(1+rI_1(\tau))(1+rI_2(\tau))}\right|\mathrm{d}\tau$$
$$\le\frac{1}{r}\max_{0\le y\le d}\frac{1}{\kappa}\int_0^y \left|\sin 2\kappa(y-\tau)\right|\mathrm{d}\tau\cdot\|I_1-I_2\|, \tag{2.33}$$

hence

$$\left\|\frac{1}{r^2}(N_4(I_1)-N_4(I_2))\right\| \le \frac{\|N_4\|}{r}\cdot\|I_1-I_2\|. \tag{2.34}$$

And

(ii)

$$\left\|\frac{1}{r^2}(N_5(I_1)-N_5(I_2))\right\| \le \max_{0\le y\le d}\frac{1}{\kappa r^2}\int_0^y \left|\sin 2\kappa(y-\tau)\right|\cdot|\ln(1+rI_1(\tau))-\ln(1+rI_2(\tau))|\mathrm{d}\tau. \tag{2.35}$$

Using

$$|\ln(1+rI_1)-\ln(1+rI_2)| = \left|\ln\frac{1+rI_1}{1+rI_2}\right| = \left|\ln\left(1+\frac{r(I_1-I_2)}{1+rI_2}\right)\right| \le r\|I_1-I_2\|, \tag{2.36}$$

equation (2.35) yields

$$\left\|\frac{1}{r^2}(N_5(I_1)-N_5(I_2))\right\| \le \frac{1}{r^2}\cdot\|N_4\|\cdot r\cdot\|I_1-I_2\| = \frac{\|N_4\|}{r}\|I_1-I_2\|. \tag{2.37}$$

Thus, from (2.32), one obtains

$$\|F(I_1)-F(I_2)\| \le \left(\|N_1\|+\|N_2\|+\|N_c\|+\frac{\|N_3\|}{r}+\frac{\|N_4\|}{r}+\frac{\|N_4\|}{r}\right)\cdot\|I_1-I_2\|$$
$$=\left(\|N_1\|+\|N_2\|+\frac{\|N_3\|+2\|N_4\|}{r}\right)\cdot\|I_1-I_2\|, \tag{2.38}$$

so that F is contractive if

$$\|N_1\| + \|N_2\| + \|N_c\| + \frac{\|N_3\| + 2\|N_4\|}{r} < 1. \tag{2.39}$$

Thus, the theorem is proved. □

Corollary 2.2. *If one denotes by m the left-hand side of inequality* (2.39) *(*(2.21)*), the solution $I(y)$ of* (2.14) *can be approximated by the iterations $I_j(y)$ as follows:*

$$\begin{aligned}\|I - I_j\| &\le \frac{m^j}{1-m}\|I_1 - I_0\| \\ &\le \frac{m^j}{1-m}\left(\frac{1}{\kappa r^2}\max_{0\le y\le d}\int_0^y \left|\sin 2\kappa(y-\tau)\right| \mathrm{d}\tau + m\|I_0\|\right) \\ &\le \frac{m^j}{1-m}\left(\frac{d^2}{r^2} + m\|I_0\|\right),\end{aligned} \tag{2.40}$$

where $j = 0, 1, 2, \ldots$ and I_0 is defined in (2.15).

Proof. See [12]. □

Remark 2.3. For the sufficient condition (2.39) ((2.21)) to hold parameters must be chosen such that (2.39) ((2.21)) holds even if r is small (Equation (2.14) represents the exact solution $I(y)$ if (2.21) and (2.22) are satisfied. $I(y)$ can be approximated by the first iteration $I_1(y)$ with the error $(d^2/r^2)(m/(1-m)) + (m^2/(1-m))\|I_0\|$, where m denotes the left-hand side of (2.21). Condition (2.21) must hold for a particular $r > 0$. In the limit $r \to 0$ (and lossless medium) (2.14) transforms to (2.41) in [14]. In order to obtain a condition of the type of (2.21) for all $0 < r < 1$ (uniformly), combination of N_3, N_4, N_5 and part of I_{0s} within the estimations is necessary. It seems impossible to obtain a condition of the type of (2.21) uniformly with respect to all nonnegative r. It is possible only to obtain such kind of condition uniformly for $0 < r < 1$ or for $1 < r < \infty$ independently. In this respect, some mathematical complications arise that are not the main point of this paper).

Remark 2.4. Estimation of $\|I_0\|$ (cf., Appendix B) gives

$$\|I_0\| \le I(0) + \frac{1}{2}|c_2|d^2 + \frac{2}{3}a|c_1|\|\varepsilon_I\|d^3, \tag{2.41}$$

where constants c_1 and c_2 are defined by (2.16), respectively. Combining (2.40) and (2.41), we obtain the error of approximation

$$\|I - I_j\| \le \frac{d^2}{r^2}\cdot\frac{m^j}{1-m} + \frac{m^{j+1}}{1-m}\left(I(0) + \frac{1}{2}|c_2|d^2 + \frac{2}{3}a|c_1|\|\varepsilon_I\|d^3\right) := R_j, \tag{2.42}$$

where $j = 0, 1, 2, \ldots$.

3. Reflectance, Transmittance, Absorptance, and Phase Shifts

Conservation of energy requires that absorptance A, transmittance T, and reflectance R be related by

$$A = 1 - R - T, \tag{3.1}$$

with

$$T = \frac{q_s}{q_c}\frac{I(0)}{aE_0^2}, \tag{3.2}$$

$$R = \frac{|E_r|^2}{E_0^2}. \tag{3.3}$$

Due to the continuity conditions at $y = d$,

$$E_0 + |E_r|e^{i\delta_r} = E(d)e^{i\vartheta(d)}, \tag{3.4}$$

$$2E_0 e^{-i\vartheta(d)} = \frac{i}{q_c}\frac{\mathrm{d}E(y)}{\mathrm{d}y}\bigg|_{y=d} + E(d)\left(1 - \frac{1}{q_c}\frac{\mathrm{d}\vartheta(y)}{\mathrm{d}y}\bigg|_{y=d}\right), \tag{3.5}$$

reflectance, transmittance, absorptance, and the phase shift on reflection, δ_r, and on transmission, δ_t, can be determined. Combination of (3.4) and (3.5) yields

$$aE_0^2 = \frac{1}{4}\left\{\frac{\left(\mathrm{d}I(y)/\mathrm{d}y\big|_{y=d}\right)^2}{4q_c^2 I(d)} + I(d)\left(1 + \frac{q_s I(0) + \int_0^d \varepsilon_I(\tau)I(\tau)\mathrm{d}\tau}{q_c I(d)}\right)^2\right\}, \tag{3.6}$$

$$a|E_r|^2 = \frac{1}{4}\left\{\frac{\left(\mathrm{d}I(y)/\mathrm{d}y\big|_{y=d}\right)^2}{4q_c^2 I(d)} + I(d)\left(1 - \frac{q_s I(0) + \int_0^d \varepsilon_I(\tau)I(\tau)\mathrm{d}\tau}{q_c I(d)}\right)^2\right\}. \tag{3.7}$$

Inserting equations (3.6) and (3.7) into (3.1) and using (3.2) and (3.3), we obtain

$$A = \frac{1}{q_c aE_0^2}\int_0^d \varepsilon_I(\tau)I(\tau)\mathrm{d}\tau. \tag{3.8}$$

The continuity conditions (3.4), (3.5) and (3.2), (3.6) imply that

$$\delta_r = -\arcsin\frac{\mathrm{d}I(y)/\mathrm{d}y\big|_{y=d}}{4q_c aE_0^2\sqrt{1 - T - A}}, \tag{3.9}$$

for the phase shift on reflection, and

$$\delta_t = \vartheta(0) = \int_0^d \frac{q_s I(0) + q_c a E_0^2 \widetilde{A}(\tau)}{I(\tau)} \mathrm{d}\tau + \arcsin\left(-\frac{\mathrm{d}I(y)/\mathrm{d}y\big|_{y=d}}{4q_c\sqrt{aE_0^2 I(d)}} \right), \tag{3.10}$$

with

$$\widetilde{A}(u) := \frac{1}{q_c a E_0^2} \int_0^\tau \varepsilon_I(u) I(u) \mathrm{d}u, \tag{3.11}$$

for the phase shift on transmission.

4. Numerical Evaluations

A numerical evaluation of the foregoing quantities is straightforward. It is useful to apply a parametric-plot routine using the first approximation $I_1(y)$. If the parameters satisfy the convergence conditions (2.21) and (2.22), the results obtained for $I_1(y)$ are in good agreement with the purely numerical solution of (2.11) (cf., Figures 2 and 3).

Iterating equation (2.24) once by inserting $I_0(\tau)$ according to (2.15), (2.16) for $I_{j-1}(\tau)$, the first approximation $I_1(y)$ is given by

$$\begin{aligned} I_1(y) = I_0(y) &+ \frac{1}{\kappa^2} \int_0^y \sin^2\kappa(y-\tau) \frac{d\varepsilon_R(\tau)}{d\tau} I_0(\tau)\mathrm{d}\tau \\ &- \frac{2}{\kappa} \int_0^y \sin 2\kappa(y-\tau)\varepsilon_R(\tau) I_0(\tau)\mathrm{d}\tau \\ &- \frac{2}{r\kappa} \int_0^y \sin 2\kappa(y-\tau) I_0(\tau)\mathrm{d}\tau \\ &- \frac{1}{\kappa r^2} \int_0^y \sin 2\kappa(y-\tau) \frac{1}{1+rI_0(\tau)}\mathrm{d}\tau \\ &+ \frac{1}{\kappa r^2} \int_0^y \sin 2\kappa(y-\tau) \ln(1+rI_0(\tau))\mathrm{d}\tau \\ &+ 4\int_0^y \varepsilon_I(z) I_0(z)\mathrm{d}z \int_z^y \frac{\sin 2\kappa(y-t)}{2\kappa} \left\{ \int_z^t \varepsilon_I(\tau)\mathrm{d}\tau \right\} \mathrm{d}t. \end{aligned} \tag{4.1}$$

For the numerical evaluations the following steps can be performed.

(i) Prescribe the parameters of the problem such that (2.21) and (2.22) are satisfied.

(ii) Prescribe a certain upper bound (accuracy) of the right-hand side R_j of (2.42) and perform a parametric plot of R_j (with $I(0)$ as parameter) with $j = 1$. If $R_1(aE_0^2)$ is smaller (or equal) than (to) the prescribed accuracy for all aE_0^2 of a certain interval, accept $I_1(y)$ as a suitable approximation.

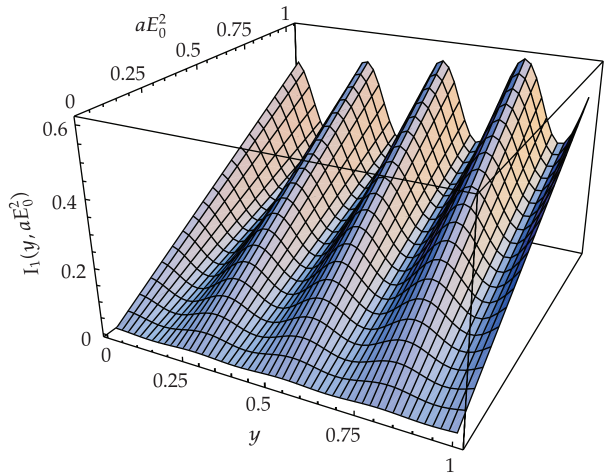

Figure 2: Dependence of the field intensity $I_1(y, aE_0^2)$ inside the slab on the transverse coordinate y and aE_0^2 for $r = 1000$, $\varepsilon_I = 0.1$, $\varepsilon_c = 1$, $\varepsilon_s = 1.7$, $\varepsilon_f^0 = 3.5$, $\varphi = 1.107$, $d = 1$, $\gamma = 0.033$, and $b = 0.1$.

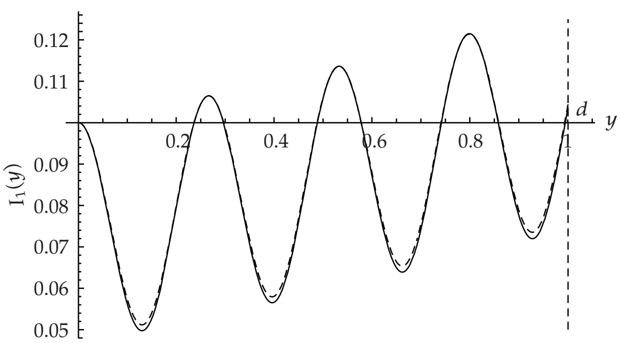

Figure 3: Dependence of the field intensity $I_1(y, aE_0^2)$ inside the slab on the transverse coordinate y for $a|E_3|^2 = 0.1$. The other parameters are as in Figure 2. Solid curve corresponds to the first iteration of (2.24) and dashed curve to the numerical solution of the system of differential equations (2.4), (2.5).

(iii) If $R_1(aE_0^2)$ exceeds the prescribed accuracy, calculate $I_2(y)$ according to (2.24) and check again according to step (ii) or enlarge the accuracy so that $R_1(aE_0^2)$ is smaller (or equal) than (to) the prescribed accuracy.

The reason for the satisfactory agreement between the exact numerical solution and the first approximation $I_1(y)$ (cf., Figure 3) is due to the foregoing explanation.

If $a|E_3|^2$ is fixed (as in the numerical example below), and thus aE_0^2 according to (3.6), inequality (2.42) can be used to optimize the iteration approach with respect to another free parameter, for example, d or r or p, as indicated.

Using the first approximation, the phase function can be evaluated according to (2.6) as

$$\vartheta_1(y) = \arcsin \vartheta_1(d) - q_s I(0) \int_d^y \frac{d\tau}{I_1(\tau)} - \int_d^y \frac{d\tau}{I_1(\tau)} \int_0^\tau \varepsilon_I(\xi) I_1(\xi) d\xi, \tag{4.2}$$

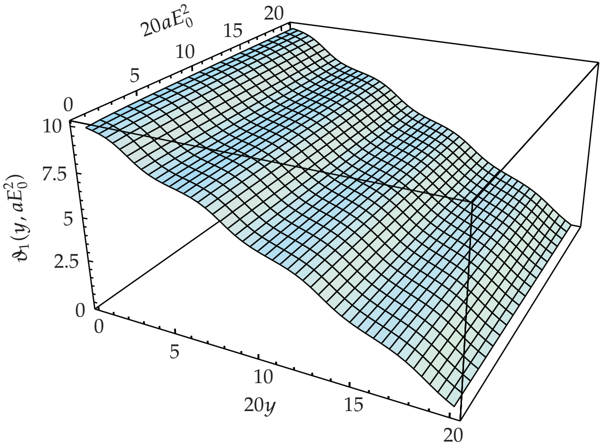

Figure 4: Phase function $\vartheta_1(y, aE_0^2)$ according to (4.2) inside the slab. Parameters are as in Figure 3.

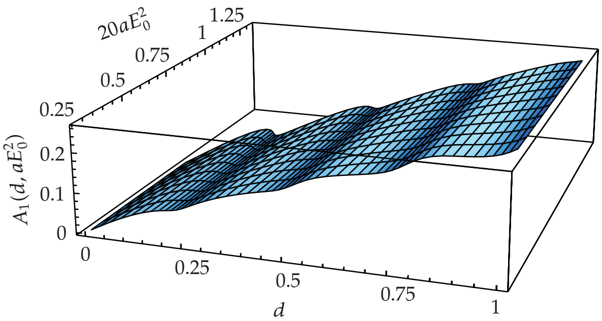

Figure 5: Absorptance A_1 depending on the layer thickness d and on the incident field intensity aE_0^2 for the same parameters as in Figure 3.

where

$$\sin \vartheta_1(d) = -\frac{\mathrm{d}I_1(y)/\mathrm{d}y\big|_{y=d}}{4q_c\sqrt{aE_0^2 I_1(d)}}. \tag{4.3}$$

Thus, the approximate solution of the problem is represented by (4.1) and (4.2). The appropriate parameter is $I(0) = aE^2(0)$, since E_0 in (4.3) can be expressed im terms of $I(0)$ as shown in (3.6).

For illustration we assume a permittivity according to

$$\varepsilon_f(y) = \varepsilon_f^0 + \varepsilon_R(y) + i\varepsilon_I + \frac{I(y)}{1 + rI(y)}, \tag{4.4}$$

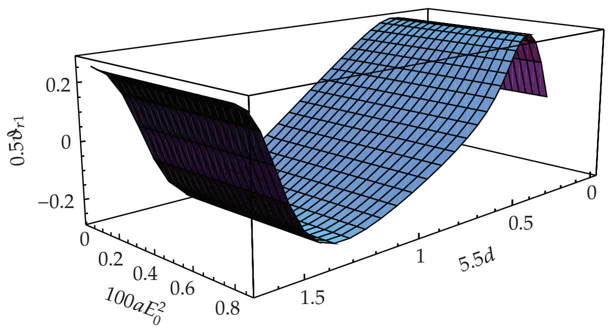

Figure 6: Phase shift on reflection δ_{r1} depending on the layer thickness d and on the incident field intensity aE_0^2 for the same parameters as in Figure 3.

with

$$\varepsilon_R(y) = \gamma \cos^2 \frac{by}{d}, \tag{4.5}$$

where $\varepsilon_f^0, \gamma, b, d, r$ are real constants. For simplicity, ε_I is also assumed to be constant. Results for the first iterate solution $I_1(y, aE_0^2)$ are depicted in Figures 2 and 3. Using $I_1(y, aE_0^2)$, the phase function $\vartheta_1(y, aE_0^2)$, absorptance $A_1(d, aE_0^2)$, and phase shift on reflection $\delta_{r1}(y, aE_0^2)$ are shown in Figures 4, 5, and 6, respectively. The left-hand side of condition (2.21) ((2.39)) is 0.572 for the parameters selected in this example. Results for R, T and the phase shift on transmission can be obtained similarly.

5. Summary

Based on known mathematics, we have proposed an iterative approach to the scattering of a plane TE-polarized optical wave at a dielectric film with permittivities modelled by a complex continuously differentiable function of the transverse coordinate.

The result is an approximate analytical expression for the field intensity inside the film that can be used to express the physical relevant quantities (reflectivity, transmissivity, absorptance, and phase shifts). Comparison with exact numerical solutions shows satisfactory agreement.

It seems appropriate to explain the benefits of the present approach as follows.

(i) The approach yields (approximate) solutions in cases where the usual methods (cf., References [1–6]) fail or could not be applied till now.

(ii) The quality of the approximate solutions can be estimated in dependence on the parameters of the problem.

On the other hand, the conditions of convergence explicitly depend on the permittivity functions in question and thus have to be derived for every permittivity anew (cf., [14] and (2.40)).

Appendices

Appendix A

The constant of integration c_2 is determined by (2.11) with $y = 0$ as

$$c_2 = \left.\frac{\mathrm{d}^2 I(y)}{\mathrm{d}y^2}\right|_{y=0} + 4\left(q_f^2(0) - p^2\right)I(0) + \frac{2}{r^2}\left(2rI(0) + \frac{1}{1+rI(0)} - \ln(1 + rI(0))\right). \tag{A.1}$$

According to (2.7), the second derivative of the field intensity $I(y)$ at $y = 0$ is given by

$$\left.\frac{\mathrm{d}^2 I(y)}{\mathrm{d}y^2}\right|_{y=0} = 2q_s^2 I(0) - 2\left(q_f^2(0) - p^2\right)I(0) - \frac{2I^2(0)}{1+rI(0)}, \tag{A.2}$$

leading to, taking into account boundary conditions, $E(0) = E_3 e^{-i\vartheta(0)}$ and $\mathrm{d}E(y)/\mathrm{d}y|_{y=0} = 0$,

$$c_2 = 2q_s^2 I(0) + 2\left(q_f^2(0) - p^2\right)I(0) - \frac{2I^2(0)}{1+rI(0)} + \frac{2}{r^2}\left(2rI(0) + \frac{1}{1+rI(0)} - \ln(1 + rI(0))\right). \tag{A.3}$$

Appendix B

With $\varepsilon_R(x) \in C^1[0,d]$ and $\varepsilon_I(x) \in C[0,d]$ one obtains

$$\begin{aligned}
\|N_1\| &= \frac{1}{\kappa^2}\max_{0\le y\le d}\int_0^y \left|\sin^2\kappa(y-\tau)\right| \cdot \left|\varepsilon_R'(\tau)\right|\mathrm{d}\tau \\
&\le \max_{0\le y\le d}\int_0^y (y-\tau)^2\mathrm{d}\tau \cdot \|\varepsilon_R'\| = \frac{1}{3}d^3\|\varepsilon_R'\|, \\
\|N_2\| &= \frac{2}{\kappa}\max_{0\le y\le d}\int_0^y \left|\sin 2\kappa(y-\tau)\right| \cdot |\varepsilon_R(\tau)|\mathrm{d}\tau \\
&\le 4\max_{0\le y\le d}\int_0^y (y-\tau)\mathrm{d}\tau \cdot \|\varepsilon_R\| = 2d^2\|\varepsilon_R\|, \\
\|N_3\| &= \frac{2}{\kappa}\max_{0\le y\le d}\int_0^y \left|\sin 2\kappa(y-\tau)\right|\mathrm{d}\tau \le 4\max_{0\le y\le d}\int_0^y (y-\tau)\mathrm{d}\tau = 2d^2,
\end{aligned}$$

$$\begin{aligned}
\|N_4\| = \|N_5\| &= \frac{1}{\kappa}\max_{0\le y\le d}\int_0^y \left|\sin 2\kappa(y-\tau)\right| \mathrm{d}\tau \\
&\le 2\max_{0\le y\le d}\int_0^y (y-\tau)\,\mathrm{d}\tau = d^2, \\
\|N_c\| &= 4\max_{0\le y\le d}\int_0^y |\varepsilon_I(z)|\cdot\left|\psi(y,z)\right|\mathrm{d}z \\
&\le 4\|\varepsilon_I\|\max_{0\le y\le d}\int_0^y\int_z^y \frac{\left|\sin 2\kappa(y-\tau)\right|}{2\kappa}\int_z^t |\varepsilon_I(\tau)|\mathrm{d}\tau\,\mathrm{d}t\,\mathrm{d}z \\
&\le 4\|\varepsilon_I\|^2\max_{0\le y\le d}\int_0^y\int_z^y (y-t)(t-z)\,\mathrm{d}t\,\mathrm{d}z = \frac{d^4}{6}\|\varepsilon_I\|^2, \\
\|I_0\| &\le \max_{0\le y\le d} a|E_3|^2\cdot\left|\cos 2\kappa y\right| + |c_2|\max_{0\le y\le d}\frac{\left|\sin^2\kappa y\right|}{2\kappa^2} \\
&\quad + 4a|c_1|\max_{0\le y\le d}\int_0^y \frac{\left|\sin 2\kappa(y-t)\right|}{2\kappa}\int_0^t |\varepsilon_I(\tau)|\mathrm{d}\tau\,\mathrm{d}t \\
&\le a|E_3|^2 + \frac{1}{2}|c_2|d^2 + 4a|c_1|\cdot\|\varepsilon_I\|\max_{0\le y\le d}\int_0^y (y-t)t\,\mathrm{d}t \\
&= a|E_3|^2 + \frac{1}{2}|c_2|d^2 + \frac{2}{3}a|c_1|\cdot\|\varepsilon_I\|d^3.
\end{aligned} \tag{B.1}$$

For ε_R, given by (4.5), we obtain

$$\|\varepsilon_R\| \le \gamma, \qquad \left\|\varepsilon_R'\right\| \le \begin{cases} \dfrac{2\gamma b^2}{d}, & 2b\le 1, \\ \dfrac{\gamma b}{d}, & 2b>1. \end{cases} \tag{B.2}$$

Acknowledgments

Financial support by the Deutsche Forschungsgemeinschaft (Graduate College 695 ”Nonlinearities of optical materials”) is gratefully acknowledged. One of the authors (V. S. Serov) gratefully acknowledges the support by the Academy of Finland (Application no. 213476, Finnish Programme for Centres of Excellence in Research 2006–2011). The authors are grateful to anonymous referee whose valuable comments have very much helped to improve the quality of the presentation.

References

[1] W. Chen and D. L. Mills, “Optical response of a nonlinear dielectric film,” *Physical Review B*, vol. 35, no. 2, pp. 524–532, 1987.

[2] K. M. Leung, “Scattering of transverse-electric electromagnetic waves with a finite nonlinear film,” *Journal of the Optical Society of America B*, vol. 5, no. 2, pp. 571–574, 1988.

[3] W. Chen and D. L. Mills, "Optical behavior of a nonlinear thin film with oblique S-polarized incident wave," *Physical Review B*, vol. 38, no. 18, pp. 12814–12822, 1988.

[4] K. M. Leung, "Exact results for the scattering of electromagnetic waves with a nonlinear film," *Physical Review B*, vol. 39, pp. 3590–3598, 1989.

[5] Th. Peschel, "Investigation of optical tunneling through nonlinear films," *Journal of the Optical Society of America B*, vol. 5, no. 1, pp. 29–36, 1988.

[6] H. W. Schürmann and R. Schmoldt, "On the theory of reflectivity and transmissivity of a lossless nonlinear dielectric slab," *Zeitschrift für Physik B*, vol. 92, pp. 179–186, 1993.

[7] H. W. Schürmann, V. S. Serov, and Yu. V. Shestopalov, "Reflection and transmission of a plane TE-wave at a lossless nonlinear dielectric film," *Physica D*, vol. 158, no. 1–4, pp. 197–215, 2001.

[8] K. M. Leung, "Exact results for the scattering ofelectromagnetic waves with a nonlinear film," *Physical Review B*, vol. 39, pp. 3590–3598, 1989.

[9] H. W. Schürmann and R. Schmoldt, "Optical response of a nonlinear absorbing dielectric film," *Optics Letters*, vol. 21, no. 6, pp. 387–389, 1996.

[10] K. P. Yuen and K. W. Yu, "Optical response of a nonlinear composite film," *Journal of the Optical Society of America B*, vol. 14, no. 6, pp. 1387–1389, 1997.

[11] F. J. Gordillo-Vázquez and C. Pecharromán, "An effective-medium approach to the optical properties of heterogeneous materials with nonlinear properties," *Journal of Modern Optics*, vol. 50, no. 1, pp. 113–135, 2003.

[12] E. Zeidler, *Applied Functional Analysis: Applications to Mathematical Physics*, vol. 108 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1995.

[13] I. Stakgold, *Boundary Value Problems of Mathematical Physics*, vol. 1, Macmillan, New York, NY, USA, 1967.

[14] V. S. Serov, H. W. Schürmann, and E. Svetogorova, "Integral equation approach to reflection and transmission of a plane TE-wave at a (linear/nonlinear) dielectric film with spatially varying permittivity," *Journal of Physics. A*, vol. 37, no. 10, pp. 3489–3500, 2004.