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Research Article

Hybrid Viscosity Iterative Method for Fixed Point, Variational Inequality and Equilibrium Problems

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We introduce an iterative scheme by the viscosity iterative method for finding a common element of the solution set of an equilibrium problem, the solution set of the variational inequality, and the fixed points set of infinitely many nonexpansive mappings in a Hilbert space. Then we prove our main result under some suitable conditions.

1. Introduction

Let H be a real Hilbert space with the inner product and the norm being denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty, closed, and convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The solution set of (1.1) is denoted by $\mathrm{EP}(F)$.

Let $A : C \to H$ be a mapping. The classical variational inequality, denoted by $\mathrm{VI}(A, C)$, is to find $x^* \in C$ such that

$$\langle Ax^*, v - x^* \rangle \geq 0, \quad \forall v \in C. \tag{1.2}$$

The variational inequality has been extensively studied in the literature (see, e.g., [1–3]). The mapping A is called α-inverse-strongly monotone if

$$\langle Au - Av, u - v\rangle \geq \alpha \|Au - Av\|^2, \quad \forall u, v \in C, \tag{1.3}$$

where α is a positive real number.

A mapping $T : C \to C$ is called strictly pseudocontractive if there exists k with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1.4}$$

It is easy to know that $I - T$ is $((1 - k)/2)$-inverse-strongly-monotone. If $k = 0$, then T is nonexpansive. We denote by $F(T)$ the fixed points set of T.

In 2003, for $x_0 \in C$, Takahashi and Toyoda [4] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n A x_n), \quad n \geq 0, \tag{1.5}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, A is an α-inverse-strongly monotone mapping, $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$, and P_C is the metric projection. They proved that if $F(S) \bigcap \mathrm{VI}(A, C) \neq \emptyset$, then $\{x_n\}$ converges weakly to some $z \in F(S) \bigcap \mathrm{VI}(A, C)$.

Recently, S. Takahashi and W. Takahashi [5] introduced an iterative scheme for finding a common element of the solution set of (1.1) and the fixed points set of a nonexpansive mapping in a Hilbert space. If F is bifunction which satisfies the following conditions:

- $(A1)$ $F(x, x) = 0$ for all $x \in C$;
- $(A2)$ F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
- $(A3)$ for each $x, y, z \in C$, $\lim_{t \to 0} F(tz + (1 - t)x, y) \leq F(x, y)$;
- $(A4)$ for each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous,

then they proved the following strong convergence theorem.

Theorem A (see [5]). *Let C be a closed and convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions $(A1)$–$(A4)$.*

Let $T : C \to H$ be a nonexpansive mapping such that $F(T) \bigcap EP(F) \neq \emptyset$ and let $f : H \to H$ be a contraction; that is, there is a constant $k \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq k\|x - y\|, \quad \forall x, y \in H, \tag{1.6}$$

and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in C$ and

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T u_n, \quad n \geq 1, \end{gathered} \tag{1.7}$$

where $\{\alpha_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0,\infty)$ *satisfy* $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$, $\liminf_{n\to\infty} r_n > 0$, *and* $\sum_{n=1}^{\infty}|r_{n+1}-r_n| < \infty$.

Then, $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $z \in F(T)\bigcap EP(F)$, *where* $z = P_{F(T)\bigcap EP(F)}f(z)$.

Let $\{T_n\}_{n=1}^{\infty}$ *be a sequence of nonexpansive mappings of* C *into itself and* $\{\lambda_n\}_{n=1}^{\infty}$ *a sequence of nonnegative numbers in* $[0,1]$. *For each* $n \geq 1$, *define a mapping* W_n *of* C *into itself as follows:*

$$\begin{aligned}
U_{n,n+1} &= I,\\
U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I,\\
U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I,\\
&\vdots\\
U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I,\\
U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I,\\
&\vdots\\
U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I,\\
W_n = U_{n,1} &= \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I.
\end{aligned} \tag{1.8}$$

Such a mapping W_n *is called the* W*-mapping generated by* $T_n, T_{n-1}, \ldots, T_1$ *and* $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$ *(see [6]).*

In this paper, we introduced a new iterative scheme generated by $x_1 \in C$ and find u_n such that

$$\begin{aligned}
F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle &\geq 0, \quad \forall y \in C,\\
y_n &= \beta_n f(x_n) + (1-\beta_n)x_n, \quad n \geq 1,\\
x_{n+1} &= \alpha_n y_n + (1-\alpha_n)W_n P_C(u_n - \delta_n A u_n),
\end{aligned} \tag{1.9}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, $\{r_n\}$ and $\{\delta_n\}$ are sequences in $(0,\infty)$, f is a fixed contractive mapping with contractive coefficient $k \in (0,1)$, A is an α-inverse-strongly monotone mapping of C to H, F is a bifunction which satisfies conditions $(A1)$–$(A4)$, and $\{W_n\}$ is generated by (1.8). Then we proved that the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* \in \bigcap_{n=1}^{\infty} F(T_n)\bigcap \mathrm{VI}(A,C)\bigcap \mathrm{EP}(F) = F$, where $x^* = P_F f(x^*)$.

2. Preliminaries

Let H be a real Hilbert space and let C be a closed and convex subset of H. P_C is the metric projection from H onto C, that is, for any $x \in H$, $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. It is easy to see that P_C is nonexpansive and

$$u \in \mathrm{VI}(A, C) \Longleftrightarrow u = P_C(u - \lambda A u), \quad \lambda > 0. \tag{2.1}$$

If A is an α-inverse-strongly monotone mapping of C to H, then it is obvious that A is $(1/\alpha)$-Lipschitz continuous. We also have that for all $x, y \in C$ and $\lambda > 0$,

$$\begin{aligned} \|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay\rangle + \lambda^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2. \end{aligned} \tag{2.2}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is nonexpansive.

Lemma 2.1 (see [7]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E, and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ for all $n \geq 1$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty}\|z_n - x_n\| = 0$.*

Lemma 2.2 (see [8]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \quad n \geq 1, \tag{2.3}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

$$\sum_{n=1}^{\infty} \alpha_n = \infty; \quad \limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0 \quad \text{or} \quad \sum_{n=1}^{\infty} |\delta_n| < \infty. \tag{2.4}$$

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.3 (see [9]). *Let C be a nonempty, closed, and convex subset of H and F a bifunction of $C \times C$ into $\mathbb{R}$ that satisfies conditions $(A1)$–$(A4)$. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{2.5}$$

Lemma 2.4 (see [9]). *Assume that $F : C \times C \to \mathbb{R}$ satisfies conditions $(A1)$–$(A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}. \tag{2.6}$$

Then, the following holds:

(i) T_r *is single-valued;*

(ii) T_r *is firmly nonexpansive, that is,*

$$\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y\rangle, \quad \forall x, y \in H; \tag{2.7}$$

(iii) $F(T_r) = EP(F)$;

(iv) $EP(F)$ *is closed and convex.*

Lemma 2.5 (Opial's theorem [10]). *Each Hilbert space H satisfies Opial's condition; that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\| \tag{2.8}$$

holds for each $y \in H$ with $x \ne y$.

Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C, where C is a nonempty, closed and convex subset of a real Hilbert space H. Given a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in $[0,1]$, one defines a sequence $\{W_n\}_{n=1}^{\infty}$ of self-mappings on C generated by (1.8). *Then one has the following results.*

Lemma 2.6 (see [6]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^{\infty} F(T_n) \ne \emptyset$ and $\{\lambda_n\}$ is a sequence in $(0,b]$ for some $b \in (0,1)$. Then, for every $x \in C$ and $k \ge 1$ the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Remark 2.7. It can be shown from Lemma 2.6 that if D is a nonempty and bounded subset of C, then for $\varepsilon > 0$ there exists $n_0 \ge k$ such that $\sup_{x\in D}\|U_{n,k}x - U_{n-1,k}x\| \le \varepsilon$ for all $n > n_0$.

Remark 2.8. Using Lemma 2.6, we can define a mapping $W : C \to C$ as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x \tag{2.9}$$

for all $x \in C$. Such a W is called the W-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$. Since W_n is nonexpansive, $W : C \to C$ is also nonexpansive. Indeed, observe that for each $x, y \in C$,

$$\|Wx - Wy\| = \lim_{n\to\infty}\|W_n x - W_n y\| \le \|x - y\|. \tag{2.10}$$

Let $\{x_n\}$ be a bounded sequence in C and $D = \{x_n : n \ge 0\}$. Then, it is clear from Remark 2.7 that for $\varepsilon > 0$ there exists $N_0 \ge 1$ such that for all $n > N_0$,

$$\|W_n x_n - W x_n\| = \|U_{n,1}x_n - U_1 x_n\| \le \sup_{x\in D}\|U_{n,1}x - U_1 x\| \le \varepsilon. \tag{2.11}$$

This implies that $\lim_{n\to\infty}\|W_n x_n - W x_n\| = 0$.

Lemma 2.9 (see [6]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^{\infty} F(T_n) \ne \emptyset$ and $\{\lambda_n\}$ is a sequence in $(0,b]$ for some $b \in (0,1)$. Then, $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.*

3. Strong Convergence Theorem

Theorem 3.1. *Let H be a Hilbert space. Let C be a nonempty, closed, and convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions $(A1)$–$(A4)$, A an α-inverse-strongly monotone mapping of C to H, f a contraction of C into itself, and $\{T_n\}_{n=1}^{\infty}$ a sequence of nonexpansive self-mappings on C such that $F \neq \emptyset$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\lambda_n\}$ are sequences in $(0,1)$, and $\{r_n\}$ and $\{\delta_n\}$ are sequences in $(0,\infty)$ which satisfies the following conditions:*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\lim_{n\to\infty} \beta_n = 0$; $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iii) $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(iv) $\delta_n \in [0,b]$, $b < 2\alpha$, $\lim_{n\to\infty} \delta_n = 0$;

(v) $\lambda_n \in [0,c]$, $c \in (0,1)$.

Then $\{x_n\}$ and $\{u_n\}$ generated by (1.9) *converge strongly to $x^* \in F$, where $x^* = P_F f(x^*)$.*

Proof. Let $p \in F$. It follows from Lemma 2.4 and (1.9) that $u_n = T_{r_n} x_n$, and hence,

$$\|u_n - p\| = \|T_{r_n} x_n - T_{r_n} p\| \leq \|x_n - p\|, \tag{3.1}$$

for all $n \in \mathbb{N}$. Let $z_n = P_C(u_n - \delta_n A u_n)$. Since $I - \delta_n A$ is nonexpansive and $p = P_C(p - \delta_n A p)$, we have

$$\|z_n - p\| \leq \|u_n - \delta_n A u_n - (p - \delta_n A p)\| \leq \|u_n - p\| \leq \|x_n - p\|, \tag{3.2}$$

$$\begin{aligned}
\|y_n - p\| &\leq \beta_n \|f(x_n) - p\| + (1-\beta_n)\|x_n - p\| \\
&\leq \beta_n \|f(x_n) - f(p)\| + \beta_n \|f(p) - p\| + (1-\beta_n)\|x_n - p\| \\
&\leq [1 - \beta_n(1-k)]\|x_n - p\| + \beta_n \|f(p) - p\|.
\end{aligned} \tag{3.3}$$

Thus,

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n y_n + (1-\alpha_n) W_n z_n - p\| \\
&\leq \alpha_n \|y_n - p\| + (1-\alpha_n)\|z_n - p\| \\
&\leq \alpha_n [1 - \beta_n(1-k)]\|x_n - p\| + \alpha_n \beta_n \|f(p) - p\| + (1-\alpha_n)\|x_n - p\| \\
&= [1 - \alpha_n \beta_n (1-k)]\|x_n - p\| + \alpha_n \beta_n (1-k) \frac{\|f(p) - p\|}{1-k} \\
&\leq \max\left\{ \|x_n - p\|, \frac{\|f(p) - p\|}{1-k} \right\}.
\end{aligned} \tag{3.4}$$

Hence $\{x_n\}$ is bounded. So $\{u_n\}$, $\{z_n\}$, $\{W_n x_n\}$, $\{W_n z_n\}$, and $\{f(x_n)\}$ are also bounded.

Next, we claim that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$. Indeed, assume that $x_{n+1} = \rho_n x_n + (1-\rho_n)t_n$, where $\rho_n = \alpha_n(1-\beta_n)$, $n \geq 0$. Then,

$$\begin{aligned}
t_{n+1}-t_n &= \frac{\alpha_{n+1}\beta_{n+1}f(x_{n+1})+(1-\alpha_{n+1})W_{n+1}z_{n+1}}{1-\rho_{n+1}} - \frac{\alpha_n\beta_n f(x_n)+(1-\alpha_n)W_n z_n}{1-\rho_n} \\
&= \frac{\alpha_{n+1}\beta_{n+1}f(x_{n+1})}{1-\rho_{n+1}} - \frac{\alpha_n\beta_n f(x_n)}{1-\rho_n} + \frac{1-\alpha_{n+1}}{1-\rho_{n+1}}(W_{n+1}z_{n+1}-W_{n+1}z_n) \\
&\quad + \frac{1-\alpha_{n+1}}{1-\rho_{n+1}}W_{n+1}z_n - \frac{1-\alpha_n}{1-\rho_n}W_n z_n \\
&\leq \frac{\alpha_{n+1}\beta_{n+1}f(x_{n+1})}{1-\rho_{n+1}} - \frac{\alpha_n\beta_n f(x_n)}{1-\rho_n} + \frac{1-\alpha_{n+1}}{1-\rho_{n+1}}(z_{n+1}-z_n) \\
&\quad + W_{n+1}z_n - \frac{\alpha_{n+1}\beta_{n+1}}{1-\rho_{n+1}}W_{n+1}z_n - W_n z_n + \frac{\alpha_n\beta_n}{1-\rho_n}W_n z_n,
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
\|z_{n+1}-z_n\| &\leq \|u_{n+1}-\delta_{n+1}Au_{n+1}-(u_n-\delta_n Au_n)\| \\
&\leq \|(I-\delta_{n+1}A)u_{n+1}-(I-\delta_{n+1}A)u_n\| + \|(I-\delta_{n+1}A)u_n-(I-\delta_n A)u_n\| \\
&\leq \|u_{n+1}-u_n\| + \|\delta_{n+1}-\delta_n\|\|Au_n\|.
\end{aligned} \tag{3.6}$$

Using (1.8) and the nonexpansivity of T_i, we deduce that

$$\begin{aligned}
\|W_{n+1}z_n - W_n z_n\| &= \|\lambda_1 T_1 U_{n+1,2}z_n - \lambda_1 T_1 U_{n,2}z_n\| \\
&\leq \lambda_1\|U_{n+1,2}z_n - U_{n,2}z_n\| \\
&\leq \lambda_1\|\lambda_2 T_2 U_{n+1,3}z_n - \lambda_2 T_2 U_{n,3}z_n\| \\
&\leq \lambda_1\lambda_2\|U_{n+1,3}z_n - U_{n,3}z_n\| \\
&\ \ \vdots \\
&\leq \left(\prod_{i=1}^{n}\lambda_i\right)\|U_{n+1,n+1}z_n - U_{n,n+1}z_n\| \\
&\leq M\prod_{i=1}^{n}\lambda_i,
\end{aligned} \tag{3.7}$$

for some constant $M \geq 0$. On the other hand, from $u_n = T_{r_n}x_n$ and $u_{n+1} = T_{r_{n+1}}x_{n+1}$, we obtain

$$F(u_n, y) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad \forall y \in C, \tag{3.8}$$

$$F(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y-u_{n+1}, u_{n+1}-x_{n+1}\rangle \geq 0, \quad \forall y \in C. \tag{3.9}$$

Setting $y = u_{n+1}$ in (3.8) and $y = u_n$ in (3.9), we get

$$F(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0,$$
$$F(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0. \tag{3.10}$$

From (A_2), we have

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\right\rangle \geq 0, \tag{3.11}$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\right\rangle \geq 0. \tag{3.12}$$

Without loss of generality, we may assume that there exists a real number r such that $r_n > r > 0$ for all $n \geq 0$. Then

$$\|u_{n+1} - u_n\|^2 \leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1})\right\rangle$$
$$\leq \|u_{n+1} - u_n\|\left(\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\|u_{n+1} - x_{n+1}\|\right), \tag{3.13}$$

and hence

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\|u_{n+1} - x_{n+1}\|$$
$$\leq \|x_{n+1} - \mathrm{x}_n\| + \frac{1}{r}|r_{n+1} - r_n|L, \tag{3.14}$$

where $L = \sup\{\|u_n - x_n\| : n \geq 0\}$. It follows from (3.5), (3.6), (3.7), and (3.14) that

$$\|t_{n+1} - t_n\| - \|x_{n+1} - x_n\| \leq \frac{\alpha_{n+1}\beta_{n+1}}{1 - \rho_{n+1}}\left[\|f(x_{n+1})\| + \|W_{n+1}z_n\|\right] + \frac{\alpha_n\beta_n}{1 - \rho_n}\left[\|f(x_n)\| + \|W_n z_n\|\right]$$
$$+ \frac{1 - \alpha_{n+1}}{1 - \rho_{n+1}}\left[\|x_{n+1} - x_n\| + \frac{L}{r}|r_{n+1} - r_n| + |\delta_{n+1} - \delta_n|\|Au_n\|\right]$$
$$+ M\prod_{i=1}^{n}\lambda_i - \|x_{n+1} - x_n\|$$

$$\leq \frac{\alpha_{n+1}\beta_{n+1}}{1-\rho_{n+1}}\left[\left\|f(x_{n+1})\right\| + \|W_{n+1}z_n\|\right] + \frac{\alpha_n\beta_n}{1-\rho_n}\left[\left\|f(x_n)\right\| + \|W_n z_n\|\right] + \frac{1-\alpha_{n+1}}{1-\rho_{n+1}}\left[\frac{L}{r}|r_{n+1}-r_n| + |\delta_{n+1}-\delta_n|\|Au_n\|\right] + M\prod_{i=1}^{n}\lambda_i. \tag{3.15}$$

Therefore, $\limsup_{n\to\infty}(\|t_{n+1}-t_n\| - \|x_{n+1}-x_n\|) \leq 0$.

Since $0 < \liminf_{n\to\infty}\alpha_n \leq \limsup_{n\to\infty}\alpha_n < 1$ and $\lim_{n\to\infty}\beta_n = 0$, hence,

$$0 < \liminf_{n\to\infty}\rho_n \leq \limsup_{n\to\infty}\rho_n < 1. \tag{3.16}$$

Lemma 2.1 yields that $\lim_{n\to\infty}\|t_n - x_n\| = 0$. Consequently, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\rho_n)\|t_{\rm n} - x_n\| = 0$.

For $p \in F$, we obtain

$$\begin{aligned}\left\|u_n - p\right\|^2 &= \left\|T_{r_n}x_n - T_{r_n}p\right\|^2\\ &\leq \langle T_{r_n}x_n - T_{r_n}p, x_n - p\rangle\\ &= \langle u_n - p, x_n - p\rangle\\ &= \frac{1}{2}\left(\left\|u_n - p\right\|^2 + \left\|x_n - p\right\|^2 - \|x_n - u_n\|^2\right),\end{aligned} \tag{3.17}$$

and hence

$$\left\|u_n - p\right\|^2 \leq \left\|x_n - p\right\|^2 - \|x_n - u_n\|^2. \tag{3.18}$$

This together with (3.2) yields that

$$\begin{aligned}\left\|x_{n+1} - p\right\|^2 &\leq \alpha_n\left\|y_n - p\right\|^2 + (1-\alpha_n)\left\|z_n - p\right\|^2\\ &\leq \alpha_n\left\|\beta_n(f(x_n) - p) + (1-\beta_n)(x_n - p)\right\|^2 + (1-\alpha_n)\left\|u_n - p\right\|^2\\ &\leq \alpha_n\beta_n\left\|f(x_n) - p\right\|^2 + \alpha_n(1-\beta_n)\left\|x_n - p\right\|^2\\ &\quad + (1-\alpha_n)\left(\left\|x_n - p\right\|^2 - \|u_n - x_n\|^2\right),\end{aligned} \tag{3.19}$$

and hence,

$$\begin{aligned}(1-\alpha_n)\|u_n - x_n\|^2 &\leq \alpha_n\beta_n\left\|f(x_n) - p\right\|^2 + (1-\alpha_n\beta_n)\left\|x_n - p\right\|^2 - \left\|x_{n+1} - p\right\|^2\\ &\leq \alpha_n\beta_n\left[\left\|f(x_n) - p\right\|^2 - \left\|x_n - p\right\|^2\right]\\ &\quad + \|x_{n+1} - x_n\|\left(\left\|x_n - p\right\| + \left\|x_{n+1} - p\right\|\right).\end{aligned} \tag{3.20}$$

So $\|u_n - x_n\| \to 0$ (note that $\lim_{n\to\infty}\beta_n = 0$ and $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$). Since

$$\begin{aligned}
\|W_n u_n - u_n\| &\le \|W_n u_n - W_n x_n\| + \|W_n x_n - x_n\| + \|x_n - u_n\| \\
&\le 2\|x_n - u_n\| + \|W_n x_n - x_n\|, \\
\|x_n - W_n x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - W_n x_n\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\beta_n\left\|f(x_n) - W_n x_n\right\| \\
&\quad + \alpha_n(1-\beta_n)\|x_n - W_n x_n\| + (1-\alpha_n)\|W_n z_n - W_n x_n\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\beta_n\left\|f(x_n) - W_n x_n\right\| \\
&\quad + \alpha_n(1-\beta_n)\|x_n - W_n x_n\| + (1-\alpha_n)\|P_C(u_n - \delta_n A u_n) - P_C x_n\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\beta_n\left\|f(x_n) - W_n x_n\right\| + \alpha_n(1-\beta_n)\|x_n - W_n x_n\| \\
&\quad + (1-\alpha_n)\|u_n - x_n\| + (1-\alpha_n)\delta_n\|A u_n\|,
\end{aligned} \tag{3.21}$$

we obtain $\lim_{n\to\infty}\|x_n - W_n x_n\| = 0$, and hence $\lim_{n\to\infty}\|u_n - W_n u_n\| = 0$. Thus, $\|u_n - W u_n\| \le \|u_n - W_n u_n\| + \|W_n u_n - W u_n\| \to 0$.

Let $Q = P_F$. Then Qf is a contraction of H into itself. In fact, there exists $k \in [0,1)$ such that $\|f(x) - f(y)\| \le k\|x - y\|$ for all $x, y \in H$. So

$$\left\|Qf(x) - Qf(y)\right\| \le \left\|f(x) - f(y)\right\| \le k\left\|x - y\right\| \tag{3.22}$$

for all $x, y \in H$. So Qf is a contraction by Banach contraction principle [11]. Since H is a complete space, there exists a unique element $x^* \in C \subset H$ such that $x^* = Qf(x^*)$.

Next we show that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle \le 0, \tag{3.23}$$

where $x^* = Qf(x^*)$. To show this inequality, we choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\limsup_{n\to\infty}\left\langle f(x^*) - x^*, u_n - x^*\right\rangle = \lim_{n\to\infty}\left\langle f(x^*) - x^*, u_{n_i} - x^*\right\rangle. \tag{3.24}$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence of $\{u_{n_i}\}$ which converges weakly to some $\omega \in C$, that is, $u_{n_i} \rightharpoonup \omega$. From $\|W u_n - u_n\| \to 0$, we obtain that $W u_{n_i} \rightharpoonup \omega$. Now we will show that $\omega \in F(W) \bigcap \mathrm{VI}(A, C) \bigcap \mathrm{EP}(F)$. First, we will show $\omega \in \mathrm{EP}(F)$. From $u_n = T_{r_n} x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\left\langle y - u_n, u_n - x_n\right\rangle \ge 0, \quad \forall y \in C. \tag{3.25}$$

By $(A2)$, we also have

$$\frac{1}{r_n}\left\langle y - u_n, u_n - x_n\right\rangle \ge F(y, u_n), \tag{3.26}$$

and hence

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq F(y, u_{n_i}). \tag{3.27}$$

Since $((u_{n_i} - x_{n_i})/r_{n_i}) \to 0$ and $u_{n_i} \rightharpoonup \omega$, it follows from $(A4)$ that $0 \geq F(y, \omega)$ for all $y \in C$. For any $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)\omega$. Since $y \in C$ and $\omega \in C$, then we have $y_t \in C$ and hence $F(y_t, \omega) \leq 0$. This together with $(A1)$ and $(A4)$ yields that

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, \omega) \leq tF(y_t, y), \tag{3.28}$$

and thus $0 \leq F(y_t, y)$. From $(A3)$, we have $0 \leq F(\omega, y)$ for all $y \in C$ and hence $\omega \in \mathrm{EP}(F)$. Now, we show that $\omega \in F(W)$. Indeed, we assume that $\omega \notin F(W)$; from Opial's condition, we have

$$\begin{aligned} \liminf_{i\to\infty} \|u_{n_i} - \omega\| &< \liminf_{i\to\infty} \|u_{n_i} - W\omega\| \\ &\leq \liminf_{i\to\infty} (\|u_{n_i} - Wu_{n_i}\| + \|Wu_{n_i} - W\omega\|) \\ &\leq \liminf_{i\to\infty} \|u_{n_i} - \omega\|. \end{aligned} \tag{3.29}$$

This is a contradiction. Thus, we obtain that $\omega \in F(W)$. Finally, by the same argument as in the proof of [3, Theorem 3.1], we can show that $\omega \in \mathrm{VI}(A, C)$. Hence $\omega \in F(W) \bigcap \mathrm{VI}(A, C) \bigcap \mathrm{EP}(F)$. Hence,

$$\begin{aligned} \limsup_{n\to\infty} \langle f(x^*) - x^*, x_n - x^* \rangle &= \limsup_{n\to\infty} \langle f(x^*) - x^*, u_n - x^* \rangle \\ &= \lim_{i\to\infty} \langle f(x^*) - x^*, u_{n_i} - x^* \rangle \\ &= \langle f(x^*) - x^*, \omega - x^* \rangle \leq 0. \end{aligned} \tag{3.30}$$

Now we show that $\lim_{n\to\infty} \|x_n - x^*\| = 0$.

From (1.9), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \langle \alpha_n \beta_n f(x_n) + \alpha_n (1-\beta_n) x_n + (1-\alpha_n) W_n z_n - x^*, x_{n+1} - x^* \rangle \\ &= \alpha_n \beta_n \langle f(x_n) - x^*, x_{n+1} - x^* \rangle + \alpha_n (1-\beta_n) \langle x_n - x^*, x_{n+1} - x^* \rangle \\ &\quad + (1-\alpha_n) \langle W_n z_n - x^*, x_{n+1} - x^* \rangle \\ &\leq \alpha_n \beta_n k \|x_n - x^*\| \|x_{n+1} - x^*\| + \alpha_n \beta_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle \\ &\quad + \alpha_n (1-\beta_n) \|x_n - x^*\| \|x_{n+1} - x^*\| + (1-\alpha_n) \|x_n - x^*\| \|x_{n+1} - x^*\| \\ &\leq [1 - \alpha_n \beta_n (1-k)] \frac{\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2}{2} \\ &\quad + \alpha_n \beta_n \langle f(x^*) - x^*, x_{n+1} - x^* \rangle, \end{aligned} \tag{3.31}$$

and hence,

$$\|x_{n+1} - x^*\|^2 \leq \left[1 - \alpha_n\beta_n(1-k)\right]\|x_n - x^*\|^2 + \alpha_n\beta_n(1-k)\frac{2}{1-k}\left\langle f(x^*) - x^*, x_{n+1} - x^*\right\rangle. \tag{3.32}$$

Using (3.23) and Lemma 2.2, we conclude that $\{x_n\}$ converges strongly to x^*. Consequently, $\{u_n\}$ converges strongly to x^*. This completes the proof. □

Using Theorem 3.1, we prove the following theorem.

Theorem 3.2. *Let H, C, F, f, and $\{T_n\}$ be given as in Theorem 3.1 and let S be an α-strictly pseudocontractive mapping such that $F \neq \emptyset$. Suppose that $\delta_n \in [0, b]$, $b < 1 - \alpha$ and $\lim_{n\to\infty}\delta_n = 0$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences and find u_n such that*

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \beta_n f(x_n) + (1-\beta_n)x_n, \quad n \geq 1, \\ x_{n+1} = \alpha_n y_n + (1-\alpha_n)W_n((1-\delta_n)u_n + \delta_n S u_n), \end{gathered} \tag{3.33}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$, and $\{\lambda_n\}$ are given as in Theorem 3.1. Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* \in F$, where $x^* = P_F f(x^*)$.

Proof. Put $A = I - S$. Then A is $((1-\alpha)/2)$-inverse-strongly-monotone. We have $F(S) = \mathrm{VI}(C, A)$ and put $P_C(u_n - \delta_n u_n) = (1-\delta_n)u_n + \delta_n S u_n$. So by Theorem 3.1 we obtain the desired result. □

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