Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 262691, 12 pages
doi:10.1155/2010/262691

Research Article

A New General Iterative Method for a Finite Family of Nonexpansive Mappings in Hilbert Spaces

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Received 10 February 2010; Revised 21 June 2010; Accepted 15 July 2010

Academic Editor: Massimo Furi

We introduce a new general iterative method by using the K-mapping for finding a common fixed point of a finite family of nonexpansive mappings in the framework of Hilbert spaces. A strong convergence theorem of the purposed iterative method is established under some certain control conditions. Our results improve and extend the results announced by many others.

1. Introduction

Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. A mapping T of C into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A point $x \in C$ is called a fixed point of T provided that $Tx = x$. We denote by $F(T)$ the set of fixed points of T (i.e., $F(T) = \{x \in H : Tx = x\}$). Recall that a self-mapping $f : C \rightarrow C$ is a contraction on C, if there exists a constant $\alpha \in (0, 1)$ such that $\|fx - fy\| \leq \alpha\|x - y\|$ for all $x, y \in C$. A bounded linear operator A on H is called *strongly positive* with coefficient $\overline{\gamma}$ if there is a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.1}$$

In 1953, Mann [1] introduced a well-known classical iteration to approximate a fixed point of a nonexpansive mapping. This iteration is defined as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T(x_n), \quad n \geq 0, \tag{1.2}$$

where the initial guess x_0 is taken in C arbitrarily, and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval $[0,1]$. But Mann's iteration process has only weak convergence, even in a Hilbert space setting. In general for example, Reich [2] showed that if E is a uniformly convex Banach space and has a Frehet differentiable norm and if the sequence $\{\alpha_n\}$ is such that $\Sigma_{n=1}^{\infty}\alpha_n(1-\alpha_n)=\infty$, then the sequence $\{x_n\}$ generated by process (1.2) converges weakly to a point in $F(T)$. Therefore, many authors try to modify Mann's iteration process to have strong convergence.

In 2005, Kim and Xu [3] introduced the following iteration process:

$$\begin{aligned} x_0 &= x \in C \text{ arbitrarily chosen}, \\ y_n &= \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} &= \alpha_n u + (1-\alpha_n)y_n. \end{aligned} \tag{1.3}$$

They proved in a uniformly smooth Banach space that the sequence $\{x_n\}$ defined by (1.3) converges strongly to a fixed point of T under some appropriate conditions on $\{\alpha_n\}$ and $\{\beta_n\}$.

In 2008, Yao et al. [4] alsomodified Mann's iterative scheme 1.2 to get a strong convergence theorem.

Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings with $F := \bigcap_{n=1}^{N} F(T_i) \neq \emptyset$. There are many authors introduced iterative method for finding an element of F which is an optimal point for the minimization problem. For $n > N$, T_n is understood as $T_{(n \bmod N)}$ with the mod function taking values in $\{1,2,\ldots,N\}$. Let u be a fixed element of H.

In 2003, Xu [5] proved that the sequence $\{x_n\}$ generated by

$$x_{n+1} = (1-\epsilon_n A)T_{n+1}x_n + \epsilon_{n+1}u \tag{1.4}$$

converges strongly to the solution of the quadratic minimization problem

$$\min_{x\in F}\frac{1}{2}\langle Ax, x\rangle - \langle x, u\rangle, \tag{1.5}$$

under suitable hypotheses on ϵ_n and under the additional hypothesis

$$F = F(T_1T_2\cdots T_N) = F(T_NT_1\cdots T_{N-1}) = \cdots = F(T_2T_3\cdots T_NT_1). \tag{1.6}$$

In 1999, Atsushiba and Takahashi [6] defined the mapping W_n as follows:

$$\begin{aligned} U_{n,0} &= I, \\ U_{n,1} &= \gamma_{n,1}T_1 + (1-\gamma_{n,1})I, \\ U_{n,2} &= \gamma_{n,2}T_2U_{n,1} + (1-\gamma_{n,2})I, \\ U_{n,3} &= \gamma_{n,3}T_3U_{n,2} + (1-\gamma_{n,3})I, \\ &\vdots \\ U_{n,N-1} &= \gamma_{n,N-1}T_N - 1U_{n,N-2} + (1-\gamma_{n,N-1})I, \\ W_n = U_{n,N} &= \gamma_{n,N}T_NU_{n,N-1} + (1-\gamma_{n,N})I, \end{aligned} \tag{1.7}$$

where $\{\gamma_{n,i}\}_i^N \subseteq [0,1]$. This mapping is called the W-mapping generated by $T_1, T_2, \ldots, T_N$ and $\gamma_{n,1}, \gamma_{n,2}, \ldots, \gamma_{n,N}$.

In 2000, Takahashi and Shimoji [7] proved that if X is strictly convex Banach space, then $F(W_n) = \bigcap_{i=1}^N F(T_i)$, where $0 < \lambda_{n,i} < 1, i = 1, 2, \ldots, N$.

In 2007,Shang et al. [8] introduced a composite iteration scheme as follows:

$$\begin{aligned} x_0 &= x \in C \text{ arbitrarily chosen,} \\ y_n &= \beta_n x_n + (1-\beta_n) W_n x_n, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n, \end{aligned} \tag{1.8}$$

where $f \in \prod_C$ is a contraction, and A is a linear bounded operator.

Note that the iterative scheme (1.8) is not well-defined, because $x_n (n \geq 1)$ may not lie in C, so $W_n x_n$ is not defined. However, if $C = H$, the iterative scheme (1.8) is well-defined and Theorem 2.1 [8] is obtained. In the case $C \neq H$, we have to modify the iterative scheme (1.8) in order to make it well-defined.

In 2009, Kangtunyakarn and Suantai [9] introduced a new mapping, called K-mapping, for finding a common fixed point of a finite family of nonexpansive mappings. For a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$ and sequence $\{\gamma_{n,i}\}_i^N$ in $[0,1]$, the mapping $K_n : C \to C$ is defined as follows:

$$\begin{aligned} U_{n,1} &= \gamma_{n,1} T_1 + (1-\gamma_{n,1}) I, \\ U_{n,2} &= \gamma_{n,2} T_2 U_{n,1} + (1-\gamma_{n,2}) U_{n,1}, \\ U_{n,3} &= \gamma_{n,3} T_3 U_{n,2} + (1-\gamma_{n,3}) U_{n,2}, \\ &\vdots \\ U_{n,N-1} &= \gamma_{n,N-1} T_N - 1 U_{n,N-2} + (1-\gamma_{n,N-1}) U_{n,N-2}, \\ K_n = U_{n,N} &= \gamma_{n,N} T_N U_{n,N-1} + (1-\gamma_{n,N}) U_{n,N-1}. \end{aligned} \tag{1.9}$$

The mapping K_n is called the *K-mapping* generated by $T_1, \ldots, T_N$ and $\gamma_{n,1}, \gamma_{n,2}, \ldots, \gamma_{n,N}$.

In this paper, motivated by Kim and Xu [3], Marino and Xu [10], Xu [5], Yao et al. [4], andShang et al. [8], we introduce a composite iterative scheme as follows:

$$\begin{aligned} x_0 &= x \in C \text{ arbitrarily chosen,} \\ y_n &= \beta_n x_n + (1-\beta_n) K_n x_n, \\ x_{n+1} &= P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n), \end{aligned} \tag{1.10}$$

where $f \in \prod_C$ is a contraction, and A is a bounded linear operator. We prove, under certain appropriate conditions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ that $\{x_n\}$ defined by (1.10) converges strongly to a common fixed point of the finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$, which solves a variational inequaility problem.

In order to prove our main results, we need the following lemmas.

Lemma 1.1. *For all $x, y \in H$, there holds the inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad x, y \in H. \tag{1.11}$$

Lemma 1.2 (see [11]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X, and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose that*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) z_n \tag{1.12}$$

for all integer $n \geq 0$, and

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{1.13}$$

Then $\lim_{n\to\infty}\|x_n - z_n\| = 0$.

Lemma 1.3 (see [5]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-\gamma_n)a_n + \delta_n$ $n \geq 0$, where $\{\gamma_n\} \subset (0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n\to\infty}\delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.4 (see [10]). *Let A be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma}$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 1.5 (see [10]). *Let H be a Hilbert space. Let A be a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \overline{\gamma}/\alpha$. Let $T : C \to C$ be a nonexpansive mapping with a fixed point $x_t \in C$ of the contraction $C \ni x \mapsto t\gamma f(x) + (1-tA)Tx$. Then x_t converges strongly as $t \to 0$ to a fixed point $\overline{x}$ of T, which solves the variational inequality*

$$\langle (A - \gamma f)\overline{x}, z - \overline{x}\rangle \geq 0, \quad z \in F(T). \tag{1.14}$$

Lemma 1.6 (see [1]). *Demiclosedness principle. Assume that T is nonexpansive self-mapping of closed convex subset C of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y, it follows that $(I-T)x = y$. Here, I is identity mapping of H.*

Lemma 1.7 (see [9]). *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, and let $\lambda_1, \ldots, \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, \ldots, N-1$ and $0 < \lambda_N \leq 1$. Let K be the K-mapping of C into itself generated by $T_1, \ldots, T_N$ and $\lambda_1, \ldots, \lambda_N$. Then $F(K) = \bigcap_{i=1}^{N} F(T_i)$.*

By using the same argument as in [9, Lemma 2.10], we obtain the following lemma.

Lemma 1.8. *Let C be a nonempty closed convex subset of Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpanxive mappings of C into itself and $\{\lambda_{n,i}\}_{i=1}^N$ sequences in $[0,1]$ such that $\lambda_{n,i} \to \lambda_i$, as $n \to \infty$, $(i = 1,2,\dots,N)$. Moreover, for every $n \in \mathbb{N}$, let K and K_n be the K-mappings generated by $T_1, T_2, \dots, T_N$ and $\lambda_1, \lambda_2, \dots, \lambda_N$, and $T_1, T_2, \dots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,N}$, respectively. Then, for every bounded sequence $x_n \in C$, one has $\lim_{n\to\infty}\|K_n x_n - K x_n\| = 0$.*

Let H be real Hilbert space with inner product $\langle \cdot,\cdot \rangle$, C a nonempty closed convex subset of H. Recall that the metric (nearest point) projection P_C from a real Hilbert space H to a closed convex subset C of H is defined as follows. Given that $x \in H$, $P_C x$ is the only point in C with the property $\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}$. Below Lemma 1.9 can be found in any standard functional analysis book.

Lemma 1.9. *Let C be a closed convex subset of a real Hilbert space H. Given that $x \in H$ and $y \in C$ then*

(i) *$y = P_C x$ if and only if the inequality $\langle x - y, y - z\rangle \geq 0$ for all $z \in C$,*

(ii) *P_C is nonexpansive,*

(iii) *$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2$ for all $x, y \in H$,*

(iv) *$\langle x - P_C x, P_C x - y\rangle \geq 0$ for all $x \in H$ and $y \in C$.*

2. Main Result

In this section, we prove strong convergence of the sequences $\{x_n\}$ defined by the iteration scheme (1.10).

Theorem 2.1. *Let H be a Hilbert space, C a closed convex nonempty subset of H. Let A be a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$, and let $f \in \prod_C$. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself, and let K_n be defined by (1.9). Assume that $0 < \gamma < \overline{\gamma}/\alpha$ and $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $x_0 \in C$, given that $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ are sequences in $(0,1)$, and suppose that the following conditions are satisfied:*

(C1) $\alpha_n \to 0$;

(C2) $\sum_{n=0}^\infty \alpha_n = \infty$;

(C3) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(C4) $\sum_{n=1}^\infty |\gamma_{n,i} - \gamma_{n-1,i}| < \infty$, *for all* $i = 1,2,\dots,N$ *and* $\{\gamma_{n,i}\}_{i=1}^N \subset [a,b]$, *where* $0 < a \leq b < 1$;

(C5) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;

(C6) $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$.

If $\{x_n\}_{n=1}^\infty$ is the composite process defined by (1.10), then $\{x_n\}_{n=1}^\infty$ converges strongly to $q \in F$, which also solves the following variational inequality:

$$\langle \gamma f(q) - Aq, p - q\rangle \leq 0, \quad p \in F. \tag{2.1}$$

Proof. First, we observe that $\{x_n\}_{n=0}^{\infty}$ is bounded. Indeed, take a point $u \in F$, and notice that

$$\|y_n - u\| \le \beta_n\|x_n - u\| + (1-\beta_n)\|K_n x_n - u\| \le \|x_n - u\|. \tag{2.2}$$

Since $\alpha_n \to 0$, we may assume that $\alpha_n \le \|A^{-1}\|$ for all n. By Lemma 1.4, we have $\|I - \alpha_n A\| \le 1 - \alpha_n\overline{\gamma}$ for all n.

It follows that

$$\begin{aligned}
\|x_{n+1} - u\| &= \left\|P_C(\alpha_n\gamma f(x_n) + (I-\alpha_n A)y_n) - P_C(u)\right\| \\
&\le \left\|\alpha_n(\gamma f(x_n) - Au) + (I-\alpha_n A)(y_n - u)\right\| \\
&\le \alpha_n\left\|\gamma f(x_n) - Au\right\| + (1-\alpha_n\overline{\gamma})\left\|y_n - u\right\| \\
&\le \alpha_n\left\|\gamma f(x_n) - \gamma f(u)\right\| + \alpha_n\left\|\gamma f(u) - Au\right\| + (1-\alpha_n\overline{\gamma})\left\|y_n - u\right\| \\
&\le \alpha\gamma\alpha_n\|x_n - u\| + \alpha_n\left\|\gamma f(u) - Au\right\| + (1-\alpha_n\overline{\gamma})\|x_n - u\| \\
&= (1 - (\overline{\gamma} - \gamma\alpha)\alpha_n)\|x_n - u\| + \alpha_n\left\|\gamma f(u) - Au\right\| \\
&= (1 - (\overline{\gamma} - \gamma\alpha)\alpha_n)\|x_n - u\| + (\overline{\gamma} - \gamma\alpha)\alpha_n\frac{\left\|\gamma f(u) - Au\right\|}{\overline{\gamma} - \gamma\alpha} \\
&\le \max\left\{\|x_n - u\|, \frac{\left\|\gamma f(u) - Au\right\|}{\overline{\gamma} - \gamma\alpha}\right\}.
\end{aligned} \tag{2.3}$$

By simple inductions, we have

$$\|x_n - u\| \le \max\left\{\|x_0 - u\|, \frac{\left\|\gamma f(u) - Au\right\|}{\overline{\gamma} - \gamma\alpha}\right\}, \quad n \ge 0. \tag{2.4}$$

Therefore $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{f(x_n)\}$. Since K_n is nonexpansive and $y_n = \beta_n x_n + (1-\beta_n)K_n x_n$, we also have

$$\begin{aligned}
\|y_{n+1} - y_n\| &\le \left\|(\beta_{n+1}x_{n+1} + (1-\beta_{n+1})K_{n+1}x_{n+1}) - (\beta_n x_n + (1-\beta_n)K_n x_n)\right\| \\
&= \left\|\beta_{n+1}x_{n+1} - \beta_{n+1}x_n + \beta_{n+1}x_n - \beta_n x_n + (1-\beta_{n+1})(K_{n+1}x_{n+1} - K_{n+1}x_n)\right. \\
&\qquad \left. + (1-\beta_{n+1})(K_{n+1}x_n - K_n x_n) + (1-\beta_{n+1})K_n x_n - (1-\beta_n)K_n x_n\right\| \\
&\le \beta_{n+1}\|x_{n+1} - x_n\| + \left|\beta_{n+1} - \beta_n\right|\|x_n\| + (1-\beta_{n+1})\|K_{n+1}x_{n+1} - K_{n+1}x_n\| \\
&\qquad + (1-\beta_{n+1})\|K_{n+1}x_n - K_n x_n\| + \left|\beta_n - \beta_{n+1}\right|\|K_n x_n\| \\
&\le \beta_{n+1}\|x_{n+1} - x_n\| + \left|\beta_{n+1} - \beta_n\right|\|x_n\| + (1-\beta_{n+1})\|x_{n+1} - x_n\| \\
&\qquad + (1-\beta_{n+1})\|K_{n+1}x_n - K_n x_n\| + \left|\beta_n - \beta_{n+1}\right|\|K_n x_n\| \\
&= \|x_{n+1} - x_n\| + \left|\beta_{n+1} - \beta_n\right|\|x_n\| + (1-\beta_{n+1})\|K_{n+1}x_n - K_n x_n\|\left|\beta_n - \beta_{n+1}\right|\|K_n x_n\|.
\end{aligned} \tag{2.5}$$

By using the inequalities (2.6) and (2.11) of [9, Lemma 2.11], we can conclude that

$$\|K_n x_{n-1} - K_{n-1}x_{n-1}\| \leq M\sum_{j=1}^{N}\left|\gamma_{n,j} - \gamma_{n-1,j}\right|, \tag{2.6}$$

where $M = \sup\{\sum_{j=2}^{N}(\|T_j U_{n,j-1}x_n\| + \|U_{n,j-1}x_n\|) + \|T_1 x_n\| + \|x_n\|\}$.

By (2.5) and (2.6), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \left\|\left(P_C\left(\alpha_n\gamma f(x_n) + (I-\alpha_n A)y_n\right)\right) - \left(P_C\left(\alpha_{n-1}\gamma f(x_{n-1}) + (I-\alpha_{n-1}A)y_{n-1}\right)\right)\right\| \\
&\leq \left\|(I-\alpha_n A)(y_n - y_{n-1}) - (\alpha_n - \alpha_{n-1})Ay_{n-1}\right. \\
&\quad \left. + \gamma\alpha_n\left(f(x_n) - f(x_{n-1})\right) + \gamma(\alpha_n - \alpha_{n-1})f(x_{n-1})\right\| \\
&\leq (1-\alpha_n\overline{\gamma})\left\|y_n - y_{n-1}\right\| + |\alpha_n - \alpha_{n-1}|\left\|Ay_{n-1}\right\| \\
&\quad + \gamma\alpha\alpha_n\|x_n - x_{n-1}\| + \gamma|\alpha_n - \alpha_{n-1}|\left\|f(x_{n-1})\right\| \\
&\leq (1-\alpha_n\overline{\gamma})\left[\|x_n - x_{n-1}\| + \left|\beta_n - \beta_{n-1}\right|\|x_{n-1}\|\right. \\
&\qquad \left. + \left|1-\beta_n\right|\|K_n x_{n-1} - K_{n-1}x_{n-1}\| + \left|\beta_{n-1} - \beta_n\right|\|K_{n-1}x_{n-1}\|\right] \\
&\quad + |\alpha_n - \alpha_{n-1}|\left\|Ay_{n-1}\right\| + \gamma\alpha\alpha_n\|x_n - x_{n-1}\| + \gamma|\alpha_n - \alpha_{n-1}|\left\|f(x_{n-1})\right\| \\
&\leq (1-\alpha_n\overline{\gamma})\|x_n - x_{n-1}\| + \left|\beta_n - \beta_{n-1}\right|\|x_{n-1}\| \\
&\quad + \left|1-\beta_n\right|\|K_n x_{n-1} - K_{n-1}x_{n-1}\| + \left|\beta_{n-1} - \beta_n\right|\|K_{n-1}x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|\|Ay_{n-1}\| + \gamma\alpha\alpha_n\|x_n - x_{n-1}\| + \gamma|\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\
&= \left(1 - (\overline{\gamma} - \gamma\alpha)\alpha_n\right)\|x_n - x_{n-1}\| + L\left|\beta_{n-1} - \beta_n\right| + M'|\alpha_n - \alpha_{n-1}| \\
&\quad + \left|1-\beta_n\right|M\sum_{j=1}^{N}\left|\gamma_{n,j} - \gamma_{n-1,j}\right|,
\end{aligned} \tag{2.7}$$

where $L = \sup\{\|x_{n-1}\| + \|K_{n-1}x_{n-1}\| : n \in \mathbb{N}\}$, $M' = \max\{\|Ay_{n-1}\| + \gamma\|f(x_{n-1})\|\}$. Since $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$, $\sum_{n=1}^{\infty}|\beta_{n-1} - \beta_n| < \infty$, and $\sum_{n=1}^{\infty}|\gamma_{n,j} - \gamma_{n-1,j}| < \infty$, for all $j = 1, 2, \ldots, N$, by Lemma 1.3, we obtain $\|x_{n+1} - x_n\| \to 0$. It follows that

$$\begin{aligned}
\left\|x_{n+1} - y_n\right\| &= \left\|P_C\left(\alpha_n\gamma f(x_n) + (I-\alpha_n A)y_n\right) - P_C(y_n)\right\| \\
&\leq \left\|\alpha_n\gamma f(x_n) + (I-\alpha_n A)y_n - y_n\right\| \\
&= \alpha_n\left\|\gamma f(x_n) + Ay_n\right\|.
\end{aligned} \tag{2.8}$$

Since $\alpha_n \to 0$ and $\{f(x_n)\}$, $\{Ay_n\}$ are bounded, we have $\|x_{n+1} - y_n\| \to 0$ as $n \to \infty$. Since

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|, \tag{2.9}$$

it implies that $\|x_n - y_n\| \to 0$ as $n \to \infty$.

On the other hand, we have

$$\|K_n x_n - x_n\| \leq \|x_n - y_n\| + \|y_n - K_n x_n\| = \|x_n - y_n\| + \beta_n \|x_n - K_n x_n\|, \tag{2.10}$$

which implies that $(1 - \beta_n)\|K_n x_n - x_n\| \leq \|x_n - y_n\|$.

From condition (C3) and $\|x_n - y_n\| \to 0$ as $n \to \infty$, we obtain

$$\|K_n x_n - x_n\| \to 0. \tag{2.11}$$

By (C4), we have $\lim_{n\to\infty}\gamma_{n,i} = \gamma_i \in [a,b]$ for all $i = 1,2,\ldots,N$. Let K be the K-mapping generated by $T_1,\ldots,T_N$ and $\gamma_1,\ldots,\gamma_N$. Next, we show that

$$\limsup_{n\to\infty}\langle \gamma f(q) - Aq, x_n - q\rangle \leq 0, \tag{2.12}$$

where $q = \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction $x \mapsto t\gamma f(x) + (I - tA)Kx$. Thus, x_t solves the fixed point equation $x_t = t\gamma f(x_t) + (I - tA)Kx_t$. By Lemma 1.5 and Lemma 1.7, we have $q \in F$ and $\langle \gamma f(q) - Aq,\ p - q\rangle \geq 0$ for all $p \in F$. It follows by (2.11) and Lemma 1.8 that $\|Kx_n - x_n\| \to 0$. Thus, we have $\|x_t - x_n\| = \|(I - tA)(Kx_t - x_n) + t(\gamma f(x_t) - Ax_n)\|$. It follows from Lemma 1.1 that for $0 < t < \|A\|^{-1}$,

$$\begin{aligned}
\|x_t - x_n\|^2 &= \left\|(I - tA)(Kx_t - x_n) + t(\gamma f(x_t) - Ax_n)\right\|^2 \\
&\leq (1 - \overline{\gamma}t)^2\|Kx_t - x_n\|^2 + 2t\langle \gamma f(x_t) - Ax_n, x_t - x_n\rangle \\
&\leq (1 - \overline{\gamma}t)^2\left(\|Kx_t - Kx_n\|^2 + 2\|Kx_t - K_n x_n\|\|Kx_n - x_n\| + \|Kx_n - x_n\|^2\right) \\
&\quad + 2t(\langle \gamma f(x_t) - Ax_t, x_t - x_n\rangle + \langle Ax_t - Ax_n, x_t - x_n\rangle) \\
&\leq \left(1 - 2\overline{\gamma}t + (\overline{\gamma}t)^2\right)\|x_t - x_n\|^2 + f_n(t) + 2t\langle \gamma f(x_t) - Ax_t, x_t - x_n\rangle \\
&\quad + 2t\langle Ax_t - Ax_n, x_t - x_n\rangle,
\end{aligned} \tag{2.13}$$

where

$$f_n(t) = (2\|x_t - x_n\| + \|x_n - Kx_n\|)\|x_n - Kx_n\| \longrightarrow 0, \quad \text{as } n \to 0. \tag{2.14}$$

It follows that

$$
\begin{aligned}
\langle Ax_t - \gamma f(x_t), x_t - x_n \rangle &\le \left(\frac{-2\overline{\gamma}t + (\overline{\gamma}t)^2}{2t} \right) \|x_t - x_n\|^2 + \frac{1}{2t} f_n(t) + \langle Ax_t - Ax_n, x_t - x_n \rangle \\
&\le \left(\frac{-2 + \overline{\gamma}t}{2} \right) \overline{\gamma} \|x_t - x_n\|^2 + \frac{1}{2t} f_n(t) + \langle Ax_t - Ax_n, x_t - x_n \rangle \\
&\le \left(-1 + \frac{\overline{\gamma}t}{2} \right) \langle Ax_t - Ax_n, x_t - x_n \rangle + \frac{1}{2t} f_n(t) + \langle Ax_t - Ax_n, x_t - x_n \rangle \\
&\le \frac{\overline{\gamma}t}{2} \langle Ax_t - Ax_n, x_t - x_n \rangle + \frac{1}{2t} f_n(t).
\end{aligned}
\tag{2.15}
$$

Letting $n \to \infty$ in (2.15) and (2.14), we get

$$
\limsup_{n \to \infty} \langle Ax_t - \gamma f(x_t), x_t - x_n \rangle \le \frac{t}{2} M_0, \tag{2.16}
$$

where $M_0 > 0$ is a constant such that $M_0 \ge \overline{\gamma} \langle Ax_t - Ax_n, x_t - x_n \rangle$ for all $t \in (0,1)$ and $n \ge 1$. Taking $t \to 0$ in (2.16), we have

$$
\limsup_{t \to 0} \limsup_{n \to \infty} \langle Ax_t - \gamma f(x_t), x_t - x_n \rangle \le 0. \tag{2.17}
$$

On the other hand, one has

$$
\begin{aligned}
\langle \gamma f(q) - Aq, x_n - q \rangle &= \langle \gamma f(q) - Aq, x_n - q \rangle - \langle \gamma f(q) - Aq, x_n - x_t \rangle \\
&\quad + \langle \gamma f(q) - Aq, x_n - x_t \rangle - \langle \gamma f(q) - Ax_t, x_n - x_t \rangle \\
&\quad + \langle \gamma f(q) - Ax_t, x_n - x_t \rangle - \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle \\
&\quad + \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle. \\
&= \langle \gamma f(q) - Aq, x_t - q \rangle + \langle Ax_t - Aq, x_n - x_t \rangle \\
&\quad + \langle \gamma f(q) - \gamma f(x_t), x_n - x_t \rangle + \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle \\
&\le \|\gamma f(q) - Aq\| \|x_t - q\| + (\|A\| \|x_t - q\| + \gamma\alpha \|x_t - q\|) \|x_n - x_t\| \\
&\quad + \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle \\
&= \|\gamma f(q) - Aq\| \|x_t - q\| + (\|A\| + \gamma\alpha) \|x_t - q\| \|x_n - x_t\| \\
&\quad + \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle.
\end{aligned}
\tag{2.18}
$$

It follows that

$$
\begin{aligned}
\limsup_{n \to \infty} \langle \gamma f(q) - Aq, x_n - q \rangle \le \|\gamma f(q) - Aq\| \|x_t - q\| + (\|A\| + \gamma\alpha) \|x_t - q\| \limsup_{n \to \infty} \|x_n - x_t\| \\
+ \limsup_{n \to \infty} \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle.
\end{aligned}
\tag{2.19}
$$

Therefore, from (2.17) and $\lim_{t\to 0}\|x_t - q\| = 0$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle \gamma f(q) - Aq, x_n - q\rangle &\le \limsup_{t\to 0}\left(\limsup_{n\to\infty}\langle \gamma f(q) - Aq, x_n - q\rangle\right)\\ &\le \limsup_{t\to 0}\limsup_{n\to\infty}\langle \gamma f(x_t) - Ax_t, x_n - x_t\rangle \le 0.\end{aligned} \tag{2.20}$$

Hence (2.12) holds. Finally, we prove that $x_n \to q$. By using (2.2) and together with the Schwarz inequality, we have

$$\begin{aligned}\|x_{n+1} - q\|^2 &= \|P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n) - P_C(q)\|^2\\
&\le \|\alpha_n(\gamma f(x_n) - Aq) + (I - \alpha_n A)(y_n - q)\|^2\\
&= \|(I - \alpha_n A)(y_n - q)\|^2 + \alpha_n^2\|\gamma f(x_n) - Aq\|^2\\
&\quad + 2\alpha_n\langle (I - \alpha_n A)(y_n - q), \gamma f(x_n) - Aq\rangle\\
&\le (1 - \alpha_n\overline{\gamma})^2\|y_n - q\|^2 + \alpha_n^2\|\gamma f(x_n) - Aq\|^2\\
&\quad + 2\alpha_n\langle y_n - q, \gamma f(x_n) - Aq\rangle - 2\alpha_n^2\langle A(y_n - q), \gamma f(x_n) - Aq\rangle\\
&\le (1 - \alpha_n\overline{\gamma})^2\|x_n - q\|^2 + \alpha_n^2\|\gamma f(x_n) - Aq\|^2\\
&\quad + 2\alpha_n\langle y_n - q, \gamma f(x_n) - \gamma f(q)\rangle + 2\alpha_n\langle y_n - q, \gamma f(q) - Aq\rangle\\
&\quad - 2\alpha_n^2\langle A(y_n - q), \gamma f(x_n) - Aq\rangle\\
&\le (1 - \alpha_n\overline{\gamma})^2\|x_n - q\|^2 + \alpha_n^2\|\gamma f(x_n) - Aq\|^2\\
&\quad + 2\alpha_n\|y_n - q\|\|\gamma f(x_n) - \gamma f(q)\| + 2\alpha_n\langle y_n - q, \gamma f(q) - Aq\rangle\\
&\quad - 2\alpha_n^2\langle A(y_n - q), \gamma f(x_n) - Aq\rangle\\
&\le (1 - \alpha_n\overline{\gamma})^2\|x_n - q\|^2 + \alpha_n^2\|\gamma f(x_n) - Aq\|^2\\
&\quad + 2\gamma\alpha\alpha_n\|y_n - q\|\|x_n - q\| + 2\alpha_n\langle y_n - q, \gamma f(q) - Aq\rangle\\
&\quad - 2\alpha_n^2\langle A(y_n - q), \gamma f(x_n) - Aq\rangle\\
&\le (1 - \alpha_n\overline{\gamma})^2\|x_n - q\|^2 + \alpha_n^2\|\gamma f(x_n) - Aq\|^2\\
&\quad + 2\gamma\alpha\alpha_n\|x_n - q\|^2 + 2\alpha_n\langle y_n - q, \gamma f(q) - Aq\rangle\\
&\quad - 2\alpha_n^2\langle A(y_n - q), \gamma f(x_n) - Aq\rangle\\
&\le \left((1 - \alpha_n\overline{\gamma})^2 + 2\gamma\alpha\alpha_n\right)\|x_n - q\|^2 + 2\alpha_n\langle y_n - q, \gamma f(x_n) - Aq\rangle\\
&\quad + \alpha_n^2\|\gamma f(x_n) - Aq\|^2 + 2\alpha_n^2\|A(y_n - q)\|\|\gamma f(x_n) - Aq\|\\
&= (1 - 2(\overline{\gamma} - \gamma\alpha)\alpha_n)\|x_n - q\|^2\\
&\quad + \alpha_n\Big(2\langle y_n - q, \gamma f(q) - Aq\rangle\\
&\qquad + \alpha_n\left(\|\gamma f(x_n) - Aq\|^2 + 2\|A(y_n - q)\|\|\gamma f(x_n) - Aq\| + \overline{\gamma}^2\|x_n - q\|^2\right)\Big).\end{aligned} \tag{2.21}$$

Since $\{x_n\}$, $\{f(x_n)\}$, and $\{y_n\}$ are bounded, we can take a constant $\eta > 0$ such that

$$\eta \geq \|\gamma f(x_n) - Aq\|^2 + 2\|A(y_n - q)\|\|\gamma f(x_n) - Aq\| + \overline{\gamma}^2\|x_n - q\|^2 \tag{2.22}$$

for all $n \geq 0$. It then follows that

$$\|x_{n+1} - q\|^2 \leq (1 - 2(\overline{\gamma} - \gamma\alpha)\alpha_n)\|x_n - q\|^2 + \alpha_n\beta_n, \tag{2.23}$$

where $\beta_n = 2\langle y_n - q, \gamma f(q) - Aq\rangle + \eta\alpha_n$. By $\limsup_{n\to\infty}\langle(\gamma f - A)q, y_n - q\rangle \leq 0$, we get $\limsup_{n\to\infty}\beta_n \leq 0$. By applying Lemma 1.3 to (2.23), we can conclude that $x_n \to q$. This completes the proof. □

If $A = I$ and $\gamma = 1$ in Theorem 2.1, we obtain the following result.

Corollary 2.2. *Let H be a Hilbert space, C a closed convex nonempty subset of H, and let $f \in \prod_C$. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself, and let K_n be defined by (1.9). Assume that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $x_0 \in C$, given that $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ are sequences in $(0,1)$, and suppose that the following conditions are satisfied:*

(C1) $\alpha_n \to 0$;

(C2) $\sum_{n=0}^\infty \alpha_n = \infty$;

(C3) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(C4) $\sum_{n=1}^\infty |\gamma_{n,i} - \gamma_{n-1,i}| < \infty$, *for all* $i = 1, 2, \ldots, N$ *and* $\{\gamma_{n,i}\}_{i=1}^N \subset [a,b]$, *where* $0 < a \leq b < 1$;

(C5) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;

(C6) $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$.

If $\{x_n\}_{n=1}^\infty$ is the composite process defined by

$$\begin{aligned} y_n &= \beta_n x_n + (1 - \beta_n)K_n x_n, \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)y_n, \end{aligned} \tag{2.24}$$

then $\{x_n\}_{n=1}^\infty$ converges strongly to $q \in F$, which also solves the following variational inequality:

$$\langle(f - I)q, p - q\rangle \leq 0, \quad p \in F. \tag{2.25}$$

If $N = 1$, $A = I$, $\gamma = 1$, and $f \equiv u \in C$ is a constant in Theorem 2.1, we get the results of Kim and Xu [3].

Corollary 2.3. *Let H be a Hilbert space, C a closed convex nonempty subset of H, and let $f \in \prod_C$. Let T be a nonexpansive mapping of C into itself. $F(T) \neq \emptyset$. Let $x_0 \in C$, given that $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ are sequences in $(0,1)$, and suppose that the following conditions are satisfied:*

(C1) $\alpha_n \to 0$;

(C2) $\sum_{n=0}^\infty \alpha_n = \infty$;

(C3) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(C4) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(C5) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

If $\{x_n\}_{n=1}^{\infty}$ *is the composite process defined by*

$$\begin{aligned} y_n &= \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} &= \alpha_n u + (I-\alpha_n)y_n, \end{aligned} \tag{2.26}$$

then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $q \in F$*, which also solves the following variational inequality:*

$$\langle u-q, p-q\rangle \leq 0, \quad p \in F. \tag{2.27}$$

Acknowledgments

The authors would like to thank the referees for valuable suggestions on the paper and thank the Center of Excellence in Mathematics, the Thailand Research Fund, and the Graduate School of Chiang Mai University for financial support.

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