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Research Article

A Survey on Semilinear Differential Equations and Inclusions Involving Riemann-Liouville Fractional Derivative

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We establish sufficient conditions for the existence of mild solutions for some densely defined semilinear functional differential equations and inclusions involving the Riemann-Liouville fractional derivative. Our approach is based on the C_0-semigroups theory combined with some suitable fixed point theorems.

1. Introduction

Differential equations and inclusions of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed we can find numerous applications in viscoelasticity, electrochemistry, electromagnetism, and so forth. For details, including some applications and recent results, see the monographs of Kilbas et al. [1], Kiryakova [2], Miller and Ross [3], Podlubny [4] and Samko et al. [5], and the papers of Agarwal et al. [6], Diethelm et al. [7, 8], El-Sayed [9–11], Gaul et al. [12], Glockle and Nonnenmacher [13], Lakshmikantham and Devi [14], Mainardi [15], Metzler et al. [16], Momani et al. [17, 18], Podlubny et al. [19], Yu and Gao [20] and the references therein. Some classes of evolution equations have been considered by El-Borai [21, 22], Jaradat et al. [23] studied the existence and uniqueness of mild solutions for a class of initial value problem for a semilinear integrodifferential equation involving the Caputo's fractional derivative.

In this survey paper, we give existence results for various classes of initial value problems for fractional semilinear functional differential equations and inclusions, both cases of finite and infinite delay are considered. More precisely the paper is organized as follows. In the second section we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper. In the third section we will be concerned with semilinear functional differential equations with finite as well infinite delay. In the forth section, we consider semilinear functional differential equation of neutral type for the both cases of finite and infinite delay. Section 5 is devoted to the study of functional differential inclusions, we examine the case when the right-hand side is convex valued as well as nonconvex valued. In Section 6, we will be concerned with perturbed functional differential equations and inclusions. In the last section, we give some existence results of extremal solutions in ordered Banach spaces.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $(E,\cdot)$ be a Banach space and J a compact real interval. $C(J,E)$ is the Banach space of all continuous functions from J into E with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\}. \tag{2.1}$$

For $\psi \in C([-r,0],E)$ the norm of ψ is defined by

$$\|\psi\|_C = \sup\{|\psi(\theta)|,\ \theta \in [-r,0]\}. \tag{2.2}$$

For $\phi \in C([-r,b],E)$ the norm of ϕ is defined by

$$\|\phi\|_{\mathfrak{D}} = \sup\{|\phi(\theta)|,\ \theta \in [-r,b]\}. \tag{2.3}$$

$B(E)$ denotes the Banach space of bounded linear operators from E into E, with norm

$$\|N\|_{B(E)} = \sup\{|N(y)| : |y| = 1\}. \tag{2.4}$$

$L^1(J,E)$ denotes the Banach space of measurable functions $y : J \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|dt. \tag{2.5}$$

Definition 2.1. A semigroup of class (C_0) is a one parameter family $T(t)_{t\geq 0} \subset B(E)$ satisfying the conditions

(i) $T(0) = I$;

(ii) $T(t)T(s) = T(t+s)$, for all $t,s \geq 0$;

(iii) the map $t \to T(t)(x)$ is strongly continuous, for each $x \in E$, that is,

$$\lim_{t\to 0} T(t)(x) = x, \quad \forall x \in E. \tag{2.6}$$

It is well known that the operator A generates a (C_0) semigroup if A satisfies

(i) $\overline{D(A)} = E$;

(ii) the Hille-Yosida condition, that is, there exists $M \geq 0$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$, $\sup\{(\lambda-\omega)^n |(\lambda I - A)^{-n}| : \lambda > \omega,\ n \in \mathbb{N}\} \leq M$, where $\rho(A)$ is the resolvent set of A and I is the identity operator in E.

For more details on strongly continuous operators, we refer the reader to the books of Goldstein [24], Fattorini [25], and the papers of Travis and Webb [26, 27], and for properties on semigroup theory we refer the interested reader to the books of Ahmed [28], Goldstein [24], and Pazy [29].

In all our paper we adopt the following definitions of fractional primitive and fractional derivative.

Definition 2.2 (see [4, 5]). The Riemann-Liouville fractional primitive of order α of a function $h : (0, b] \to \mathbb{R}$ of order $\alpha \in \mathbb{R}^+$ is defined by

$$I_0^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds, \tag{2.7}$$

provided the right side is pointwise defined on $(0, b]$, and where Γ is the gamma function.

For instance, I_0^α exists for all $\alpha > 0$, when $h \in C((0, b], \mathbb{R}) \cap L^1((0, b], \mathbb{R})$; note also that when $h \in C([0, b], \mathbb{R})$, then $I_0^\alpha h \in C([0, b], \mathbb{R})$ and moreover $I_0^\alpha h(0) = 0$.

Definition 2.3 (see [4, 5]). The Riemann-Liouville fractional derivative of order $\alpha \in (0, 1)$ of a continuous function $h : (0, b] \to \mathbb{R}$ is defined by

$$\begin{aligned} \frac{d^\alpha h(t)}{dt^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} h(s)ds \\ &= \frac{d}{dt} I_0^{1-\alpha} h(t). \end{aligned} \tag{2.8}$$

Let (X, d) be a metric space. We use the notations

$$\begin{aligned} \mathcal{P}_{\rm cl}(X) &= \{Y \in \mathcal{P}(X) : Y \text{ closed}\}, \\ \mathcal{P}_{\rm bd}(X) &= \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}, \\ \mathcal{P}_{\rm cv}(X) &= \{Y \in \mathcal{P}(X) : Y \text{ convex}\}, \\ \mathcal{P}_{\rm cp}(X) &= \{Y \in \mathcal{P}(X) : Y \text{ compact}\}. \end{aligned} \tag{2.9}$$

Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \bigcup\{\infty\}$ given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(A,b)\right\}, \tag{2.10}$$

where $d(A,b) = \inf_{a\in A} d(a,b)$, $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(\mathcal{P}_{\mathrm{bd,cl}}(X), H_d)$ is a metric space and $(\mathcal{P}_{\mathrm{cl}}(X), H_d)$ is a generalized metric space (see [30]).

A multivalued map $F : J \to \mathcal{P}_{\mathrm{cl}}(X)$ is said to be measurable if, for each $x \in X$, the function $Y : J \to \mathbb{R}$ defined by

$$Y(t) = d(x, F(t)) = \inf\{d(x,z) : z \in F(t)\} \tag{2.11}$$

is measurable.

Definition 2.4. A measurable multivalued function $F : J \to P_{\mathrm{bd,cl}}(X)$ is said to be integrably bounded if there exists a function $w \in L^1(J, R^+)$ such that $\|v\| \le w(t)$ a.e. $t \in J$ for all $v \in F(t)$.

A multivalued map $F : X \to \mathcal{P}(X)$ is convex (closed) valued if $F(x)$ is convex (closed) for all $x \in X$. F is bounded on bounded sets if $F(B) = \bigcup_{x\in B} F(x)$ is bounded in X for all $B \in \mathcal{P}_{\mathrm{bd}}(X)$, that is, $\sup_{x\in B}\{\sup\{|y| : y \in F(x)\}\} < \infty$.

F is called upper semicontinuous (u.s.c. for short) on X if for each $x_0 \in X$ the set $F(x_0)$ is nonempty, closed subset of X, and for each open set $\mathcal{U}$ of X containing $F(x_0)$, there exists an open neighborhood $\mathcal{V}$ of x_0 such that $F(\mathcal{V}) \subseteq \mathcal{U}$. F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in P_{\mathrm{bd}}(X)$. If the multivalued map F is completely continuous with nonempty compact valued, then F is u.s.c. if and only if F has closed graph, that is, $x_n \to x_*$, $y_n \to y_*$, $y_n \in F(x_*)$ imply $y_* \in F(x_*)$.

Definition 2.5. A multivalued map $\beta : J \times C([-r,0],E) \to P(E)$ is said to be Carathéodory if

(i) $t \mapsto \beta(t,x)$ is measurable for each $x \in C([-r,0],E)$,

(ii) $x \mapsto \beta(t,x)$ is u.s.c. for almost all $t \in J$.

Furthermore, a Carathéodory map β is said to be L^1-Carathéodory if

(iii) for each real number $\rho > 0$, there exists a function $h_\rho \in L^1(J, R_+)$ such that

$$\|\beta(t,x)\|_{P(E)} := \sup\{|v| : v \in \beta(t,x)\} \le h_\rho(t), \tag{2.12}$$

for a.e. $t \in J$, and for all $|x| \le \rho$.

Definition 2.6. A multivalued operator $F : X \to \mathcal{P}_{\mathrm{cl}}(X)$ is called

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(F(x), F(y)) \le \gamma d(x,y), \quad \text{for each } x, y \in X, \tag{2.13}$$

(b) contraction if and only if it is γ-Lipschitz with $\gamma < 1$,

(c) F has a fixed point if there exists $x \in X$ such that $x \in F(x)$.

The fixed point set of the multivalued operator F will be denoted by Fix F.

For more details on multivalued maps and the proof of the known results cited in this section we refer interested reader to the books of Deimling [31], Gorniewicz [32], and Hu and Papageorgiou [33].

Essential for the main results of this paper, we state a generalization of Gronwall's lemma for singular kernels [34, Lemma 7.1.1].

Lemma 2.7. *Let $v, w : [0, b] \to [0, \infty)$ be continuous functions. If $w(\cdot)$ is nondecreasing and there are constants $a > 0$ and $0 < \alpha < 1$ such that*

$$v(t) \leq w(t) + a\int_0^t \frac{v(s)}{(t-s)^\alpha} ds, \tag{2.14}$$

then there exists a constant $k = k(\alpha)$ such that

$$v(t) \leq \omega(t) + ka\int_0^t \frac{\omega(s)}{(t-s)^\alpha} ds, \tag{2.15}$$

for every $t \in [0, b]$.

In the sequel, the following fixed point theorems will be used. The following fixed point theorem for contraction multivalued maps is due to Covitz and Nadler [35].

Theorem 2.8. *Let (X, d) be a complete metric space. If $F : X \to \mathcal{P}_{cl}(X)$ is a contraction, then* Fix $F \neq \emptyset$.

The nonlinear alternative of Leray-Schauder applied to completely continuous operators [36].

Theorem 2.9. *Let X be a Banach space, and $C \subset X$ convex with $0 \in C$. Let $F : C \to C$ be a completely continuous operator. Then either*

(a) *F has a fixed point, or*

(b) *the set $\mathcal{E} = \{x \in C : x = \lambda F(x),\ 0 < \lambda < 1\}$ is unbounded.*

The following is the multivalued version of the previous theorem due to Martelli [37].

Theorem 2.10. *Let $T : X \to \mathcal{P}_{cp,cv}(X)$ be an upper semicontinuous and completely continuous multivalued map. If the set*

$$\mathcal{E} = \{u \in X : \lambda u \in Tu \text{ for some } \lambda > 1\} \tag{2.16}$$

is bounded, then T has a fixed point.

To state existence results for perturbed differential equations and inclusions we will use the following fixed point theorem of Krasnoselskii-Scheafer type of the sum of a completely continuous operator and a contraction one due to Burton and Kirk [38].

Theorem 2.11. *Let X be a Banach space, and $\mathcal{A}, \mathcal{B} : X \to X$ two operators satisfying*

(i) *$\mathcal{A}$ is a contraction;*

(ii) *$\mathcal{B}$ is completely continuous.*

Then either

(a) *the operator equation $y = \mathcal{A}(y) + \mathcal{B}(y)$ has a solution, or*

(b) *the set $\mathcal{E} = \{u \in X : u = \lambda\mathcal{A}(u/\lambda) + \lambda\mathcal{B}(u)\}$ is unbounded for $\lambda \in (0, 1)$.*

Recently Dhage states the multivalued version of the previous theorem.

Theorem 2.12 (see [39, 40]). *Let X be a Banach space, $\mathcal{A} : X \to P_{\mathrm{cl,cv,bd}}(X)$ and $\mathcal{B} : X \to P_{\mathrm{cp,cv}}(X)$, two multivalued operators satisfying*

(a) *$\mathcal{A}$ is a contraction;*

(b) *$\mathcal{B}$ is completely continuous.*

Then either

(i) *The operator inclusion $\lambda x \in \mathcal{A}x + \mathcal{B}x$ has a solution for $\lambda = 1$, or*

(ii) *the set $\mathcal{E} = \{u \in X \mid u \in \lambda\mathcal{A}u + \lambda\mathcal{B}u,\ 0 < \lambda < 1\}$ is unbounded.*

In the literature devoted to equations with finite delay, the phase space is much of time the space of all continuous functions on $[-r, 0]$, $r > 0$, endowed with the uniform norm topology. When the delay is infinite, the notion of the phase space plays an important role in the study of both qualitative and quantitative theory, a usual choice is a seminormed space $\mathcal{B}$ introduced by Hale and Kato [41] and satisfying the following axioms.

(A1) There exist a positive constant H and functions $K(\cdot)$, $M(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ with K continuous and M locally bounded, such that for any $a > 0$, if $x : (-\infty, a] \to E$, $x \in \mathcal{B}$, and $x(\cdot)$ is continuous on $[0, a]$, then for every $t \in [0, a]$ the following conditions hold:

  (i) x_t is in $\mathcal{B}$;

  (ii) $|x(t)| \leq H\|x_t\|_{\mathcal{B}}$;

  (iii) $\|x_t\|_{\mathcal{B}} \leq K(t)\sup\{|x(s)| : 0 \leq s \leq t\} + M(t)\|x_0\|_{\mathcal{B}}$, and H, K, and M are independent of $x(\cdot)$.

(A2) For the function $x(\cdot)$ in (A_1), x_t is a $\mathcal{B}$-valued continuous function on $[0, a]$.

(A3) The space $\mathcal{B}$ is complete.

Denote by

$$K_b = \sup\{K(t) : t \in J\}, \qquad M_b = \sup\{M(t) : t \in J\}. \tag{2.17}$$

Hereafter are some examples of phase spaces. For other details we refer, for instance, to the book by Hino et al. [42].

Example 2.13. The spaces BC, BUC, C^{∞}, and C^{0}.

BC is the space of bounded continuous functions defined from $(-\infty,0]$ to E;

BUC is the space of bounded uniformly continuous functions defined from $(-\infty,0]$ to E;

$$C^{\infty} := \left\{\phi \in \mathrm{BC} : \lim_{\theta\to-\infty}\phi(\theta) \text{ exist in } E\right\}; \tag{2.18}$$

$$C^{0} := \left\{\phi \in \mathrm{BC} : \lim_{\theta\to-\infty}\phi(\theta) = 0\right\}, \quad \text{endowed with the uniform norm} \\ \|\phi\| = \{|\phi(\theta)| : \theta \le 0\}. \tag{2.19}$$

We have that the spaces BUC, C^{∞}, and C^{0} satisfy conditions (A_1)–(A_3). BC satisfies (A_1), (A_3) but (A_2) is not satisfied.

Example 2.14. The spaces C_g, UC_g, C_g^{∞}, and C_g^{0}.

Let g be a positive continuous function on $(-\infty,0]$. We define

$$C_g := \left\{\phi \in C((-\infty,0],E) : \frac{\phi(\theta)}{g(\theta)} \text{ is bounded on } (-\infty,0]\right\};$$

$$C_g^{0} := \left\{\phi \in C_g : \lim_{\theta\to-\infty}\frac{\phi(\theta)}{g(\theta)} = 0\right\}, \quad \text{endowed with the uniform norm} \\ \|\phi\| = \left\{\frac{|\phi(\theta)|}{g(\theta)} : \theta \le 0\right\}. \tag{2.20}$$

We consider the following condition on the function g.

(g_1) For all $a > 0$, $\sup_{0\le t\le a}\sup\{(g(t+\theta)/g(\theta)) : -\infty < \theta \le -t\} < \infty$.

Then we have that the spaces C_g and C_g^{0} satisfy conditions (A_3). They satisfy conditions (A_1) and (A_3) if g_1 holds.

Example 2.15. The space C_{γ}.

For any real constant γ, we define the functional space C_{γ} bys

$$C_{\gamma} := \left\{\phi \in C((-\infty,0],E) : \lim_{\theta\to-\infty}e^{\gamma\theta}\phi(\theta) \text{ exist in } E\right\} \tag{2.21}$$

endowed with the following norm

$$\|\phi\| = \sup\{e^{\gamma\theta}|\phi(\theta)| : \theta \le 0\}. \tag{2.22}$$

Then in the space C_{γ} the axioms (A_1)–(A_3) are satisfied.

3. Semilinear Functional Differential Equations

3.1. Introduction

Functional differential and partial differential equations arise in many areas of applied mathematics and such equations have received much attention in recent years. A good guide to the literature for functional differential equations is the books by Hale [43] and Hale and Verduyn Lunel [44], Kolmanovskii and Myshkis [45], and Wu [46] and the references therein.

In a series of papers (see [47–50]), the authors considered some classes of initial value problems for functional differential equations involving the Riemann-Liouville and Caputo fractional derivatives of order $0 < \alpha \leq 1$. In [51, 52] some classes of semilinear functional differential equations involving the Riemann-Liouville have been considered. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann-Liouville and Caputo types see [53, 54].

In the following, we consider the semilinear functional differential equation of fractional order of the form

$$D^{\alpha}y(t) = Ay(t) + f(t, y_t), \quad t \in J := [0, b], \tag{3.1}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{3.2}$$

where D^{α} is the standard Riemann-Liouville fractional derivative, $f : J \times C([-r, 0], E) \to E$ is a continuous function, A is a closed linear operator (possibly unbounded), $\phi : [-r, 0] \to E$ a given continuous function with $\phi(0) = 0$, and $(E, |\cdot|)$ a real Banach space. For any function y defined on $[-r, b]$ and any $t \in J$ we denote by y_t the element of $C([-r, 0], E)$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0]. \tag{3.3}$$

Here $y_t(\cdot)$ represents the history of the state from time $t - r$, up to the present time t.

The reason for studying (3.1) is that it appears in mathematical models of viscoelasticity [55], and in other fields of science [54, 56]. Equation (3.1) is equivalent to solve an integral equation of convolution type. It is also of interest to explore the neighborhood of the diffusion ($\alpha = 1$). In this survey paper, we use the fractional derivative in the Riemann-Liouville sense. The problems considered in the survey are subject to zero data, which in this case, the Riemann-Liouville and Caputo fractional derivatives coincide. From a practical point of view, in some mathematical models it is more appropriate to consider traditional initial or boundary data. This is what we are considering in this survey.

In all our paper we suppose that the operator $A : D(A) \subset E \to E$ is the infinitesimal generator of a (C_0)-semigroup $\{T(t)\}_{t \geq 0}$. Denote by

$$M = \sup\left\{\|T(t)\|_{B(E)} : t \in J\right\}. \tag{3.4}$$

Before stating our main results in this section for problem (3.1) and (3.2) we give the definition of the mild solution.

Definition 3.1 (see [23]). One says that a continuous function $y : [-r,b] \to E$ is a mild solution of problem (3.1) and (3.2) if $y(t) = \phi(t),\ t \in [-r,0]$, and

$$y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} T(t-s) f(s,y_s)ds, \quad t \in J. \tag{3.5}$$

3.2. Existence Results for Finite Delay

By using the Banach's contraction principle, we get the following existence result for problem (3.1) and (3.2).

Theorem 3.2. *Let $f : J \times C([-r,0],E) \to E$ continuous. Assume the following.*

$(H1)$ *There exists a nonnegative constant k such that*

$$|f(t,u) - f(t,v)| \le k\|u-v\|_C, \quad \text{for } t \in J \text{ and every } u,v \in C([-r,0],E). \tag{3.6}$$

Then there exists a unique mild solution of problem (3.1) *and* (3.2) *on* $[-r,b]$.

Proof. Transform the IVP (3.1) and (3.2) into a fixed point problem. Consider the operator $\mathcal{F} : C([-r,b],E) \to C([-r,b],E)$ defined by

$$\mathcal{F}(y)(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} T(t-s) f\left(s,y_s\right)ds, & t \in [0,b]. \end{cases} \tag{3.7}$$

Let us define the iterates of operator $\mathcal{F}$ by

$$\mathcal{F}^1 = \mathcal{F}, \qquad \mathcal{F}^{n+1} = \mathcal{F} \circ \mathcal{F}^n. \tag{3.8}$$

It will be sufficient to prove that $\mathcal{F}^n$ is a contraction operator for n sufficiently large. For every $y,z \in C([-r,b],E)$ we have

$$\left|\mathcal{F}^n(y)(t) - \mathcal{F}^n(z)(t)\right| \le \frac{(kM)^n}{\Gamma(n\alpha+1)} t^{n\alpha}\|y-z\|_\infty. \tag{3.9}$$

Indeed,

$$
\begin{aligned}
\left|\mathcal{F}(y)(t)-\mathcal{F}(z)(t)\right| &\leq \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f(s,y_s)-f(s,z_s)\right|ds \\
&\leq \frac{kM}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\|y_s-z_s\right\|_C ds \\
&\leq \frac{kM}{\Gamma(\alpha)}\int_0^t \|y-z\|_\infty (t-s)^{\alpha-1} ds \\
&= \frac{kM}{\Gamma(\alpha+1)} t^\alpha \|y-z\|_\infty.
\end{aligned}
\tag{3.10}
$$

Therefore (3.9) is proved for $n=1$. Assuming by induction that (3.9) is valid for n, then

$$
\begin{aligned}
\left|\mathcal{F}^{n+1}(y)(t)-\mathcal{F}^{n+1}(z)(t)\right| &\leq \frac{kM}{\Gamma(\alpha)}\frac{(kM)^n}{\Gamma(n\alpha+1)}\|y-z\|_\infty \int_0^t (t-s)^{\alpha-1}s^{n\alpha}ds \\
&= \frac{(kM)^{n+1}}{\Gamma(\alpha)\Gamma(n\alpha+1)}\frac{\Gamma(\alpha)\Gamma(n\alpha+1)}{\Gamma((n+1)\alpha+1)} t^{\alpha+n\alpha}\|y-z\|_\infty \\
&= \frac{(kM)^{n+1}}{\Gamma((n+1)\alpha+1)} t^{(n+1)\alpha}\|y-z\|_\infty
\end{aligned}
\tag{3.11}
$$

and then (3.9) follows for $n+1$.

Now, taking n sufficiently large in (3.9) yield the contraction of operator $\mathcal{F}^n$.

Consequently $\mathcal{F}$ has a unique fixed point by the Banach's contraction principle, which gives rise to a unique mild solution to the problem (3.1) and (3.2). □

The following existence result is based upon Theorem 2.9.

Theorem 3.3. *Assume that the following hypotheses hold.*

($H2$) *The semigroup* $\{T(t)\}_{t\in J}$ *is compact for* $t>0$.

($H3$) *There exist functions* $p,q\in C(J,\mathbb{R}_+)$ *such that*

$$
|f(t,u)| \leq p(t)+q(t)\|u\|_C, \quad \text{for each } t\in J, \text{ and each } u\in C([-r,0],E).
\tag{3.12}
$$

Then the problem (3.1) *and* (3.2) *has at least one mild solution on* $[-r,b]$.

Proof. Transform the IVP (3.1) and (3.2) into a fixed point problem. Consider the operator $\mathcal{F}$ as defined in Theorem 3.2. To show that $\mathcal{F}$ is continuous, let us consider a sequence $\{y_n\}$ such that $y_n \to y$ in $C([-r,b],E)$. Then

$$
\begin{aligned}
\left|\mathcal{F}(y_n)(t)-\mathcal{F}(y)(t)\right| &\leq \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)\left[f(s,y_{n_s})-f(s,y_s)\right]ds\right| \\
&\leq \frac{Mb^\alpha}{\alpha\Gamma(\alpha)}\left\|f(\cdot,y_{n_\cdot})-f(\cdot,y_\cdot)\right\|_\infty.
\end{aligned}
\tag{3.13}
$$

Since f is a continuous function, then we have

$$\|\mathcal{F}(y_n)-\mathcal{F}(y)\|_\infty \leq \frac{Mb^\alpha}{\Gamma(\alpha+1)}\|f(\cdot,y_{n\cdot})-f(\cdot,y_\cdot)\|_\infty \longrightarrow 0 \quad \text{as } n\longmapsto\infty. \tag{3.14}$$

Thus $\mathcal{F}$ is continuous. Now for any $\rho>0$, and each $y\in B_\rho=\{y\in C([-r,b],E):\|y\|_\infty\leq\rho\}$ we have for each $t\in J$

$$\begin{aligned}
|\mathcal{F}(y)(t)| &= \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)f(s,y_s)ds\right| \\
&\leq \frac{M\|p\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds + M\frac{\|p\|_\infty+\rho\|q\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\
&\leq \frac{Mb^\alpha}{\Gamma(\alpha+1)}(\|p\|_\infty+\rho\|q\|_\infty) =: \delta<\infty.
\end{aligned} \tag{3.15}$$

Thus $\mathcal{F}$ maps bounded sets into bounded sets in $C([-r,b],E)$.

Now, let $\tau_1,\tau_2\in J$, $\tau_2>\tau_1$. Thus if $\epsilon>0$ and $\epsilon\leq\tau_1\leq\tau_2$ we have for any $y\in B_\rho$

$$\begin{aligned}
&\left|\mathcal{F}(y)(\tau_2)-\mathcal{F}(y)(\tau_1)\right| \\
&\quad\leq \frac{1}{\Gamma(\alpha)}\Bigg(\left|\int_0^{\tau_1-\epsilon}\left[(\tau_2-s)^{\alpha-1}T(\tau_2-s)-(\tau_1-s)^{\alpha-1}T(\tau_1-s)\right]f(s,y_s)ds\right| \\
&\qquad\qquad + \left|\int_{\tau_1-\epsilon}^{\tau_1}\left[(\tau_2-s)^{\alpha-1}T(\tau_2-s)-(\tau_1-s)^{\alpha-1}T(\tau_1-s)\right]f(s,y_s)ds\right| \\
&\qquad\qquad + \left|\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}T(\tau_2-s)f(s,y_s)ds\right|\Bigg) \\
&\quad\leq \frac{M(\|p\|_\infty+\rho\|q\|_\infty)}{\Gamma(\alpha)}\Bigg(\left|\int_0^{\tau_1-\epsilon}\left[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right]T(\tau_1-s)ds\right| \\
&\qquad\qquad + \left|\int_0^{\tau_1-\epsilon}(\tau_2-s)^{\alpha-1}T(\tau_1-\epsilon-s)(T(\tau_2-\tau_1-\epsilon)-T(\epsilon))ds\right| \\
&\qquad\qquad + \int_{\tau_1-\epsilon}^{\tau_1}\left[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right]ds + \int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}ds\Bigg) \\
&\quad\leq \frac{M(\|p\|_\infty+\rho\|q\|_\infty)}{\Gamma(\alpha)}\Bigg(\int_0^{\tau_1-\epsilon}\left[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right]ds \\
&\qquad\qquad + \|T(\tau_2-\tau_1-\epsilon)-T(\epsilon)\|_{B(E)}\int_0^{\tau_1-\epsilon}(\tau_2-s)^{\alpha-1}ds \\
&\qquad\qquad + \int_{\tau_1-\epsilon}^{\tau_1}\left((\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right)ds + \int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}ds\Bigg).
\end{aligned} \tag{3.16}$$

As $\tau_1 \to \tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero, since $T(t)$ is a strongly continuous operator and the compactness of $T(t)$ for $t > 0$ implies the continuity in the uniform operator topology [29]. By the Arzelá-Ascoli theorem it suffices to show that $\mathcal{F}$ maps B_ρ into a precompact set in E.

Let $0 < t < b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_\rho$ we define

$$\mathcal{F}_\epsilon(y)(t) = \frac{T(\epsilon)}{\Gamma(\alpha)}\int_0^{t-\epsilon}(t-s-\epsilon)^{\alpha-1}T(t-s-\epsilon)f(s,y_s)\,ds. \tag{3.17}$$

Since $T(t)$ is a compact operator for $t > 0$, the set

$$Y_\epsilon(t) = \{\mathcal{F}_\epsilon(y)(t) : y \in B_\rho\} \tag{3.18}$$

is precompact in E for every ϵ, $0 < \epsilon < t$. Moreover

$$\begin{aligned}\left|\mathcal{F}(y)(t) - \mathcal{F}_\epsilon(y)(t)\right| &\le \frac{M(\|p\|_\infty + \rho\|q\|_\infty)}{\Gamma(\alpha)}\Bigg(\int_0^{t-\epsilon}\left[(t-s)^{\alpha-1} - (t-s-\epsilon)^{\alpha-1}\right]ds \\ &\qquad + \int_{t-\epsilon}^{t}(t-s)^{\alpha-1}ds.\Bigg) \\ &\le \frac{M(\|p\|_\infty + \rho\|q\|_\infty)}{\Gamma(\alpha+1)}\left(t^\alpha - (t-\epsilon)^\alpha\right).\end{aligned} \tag{3.19}$$

Therefore, the set $Y(t) = \{\mathcal{F}(y)(t) : y \in B_\rho\}$ is precompact in E. Hence the operator $\mathcal{F}$ is completely continuous. Now, it remains to show that the set

$$\mathcal{E} = \{y \in C([-r,b],E) : y = \lambda\mathcal{F}(y) \text{ for some } 0 < \lambda < 1\} \tag{3.20}$$

is bounded.

Let $y \in \mathcal{E}$ be any element. Then, for each $t \in J$,

$$y(t) = \lambda\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}T(t-s)f(s,y_s)\,ds. \tag{3.21}$$

Then

$$|y(t)| \le \frac{Mb^\alpha\|p\|_\infty}{\Gamma(\alpha+1)} + \frac{M\|q\|_\infty}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|y_s\|\,ds. \tag{3.22}$$

We consider the function defined by

$$\mu(t) = \max\{|y(s)| : -r \le s \le t\}, \quad t \in J. \tag{3.23}$$

Let $t^* \in [-r,t]$ such that $\mu(t) = |y(t^*)|$, if $t^* \in [0,b]$ then by (3.22) we have, for $t \in J$, (note $t^* \leq t$)

$$\mu(t) \leq \frac{Mb^{\alpha}\|p\|_{\infty}}{\Gamma(\alpha+1)} + \frac{M\|q\|_{\infty}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mu(s)ds. \tag{3.24}$$

If $t^* \in [-r,0]$ then $\mu(t) = \|\phi\|_C$ and the previous inequality holds.

By Lemma 2.7 we have

$$\begin{aligned}\mu(t) &\leq \frac{Mb^{\alpha}\|p\|_{\infty}}{\Gamma(\alpha+1)} + k\frac{M\|q\|_{\infty}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\frac{Mb^{\alpha}\|p\|_{\infty}}{\Gamma(\alpha+1)}ds \\ &\leq \frac{Mb^{\alpha}\|p\|_{\infty}}{\Gamma(\alpha+1)}\left(1+k\frac{Mb^{\alpha}\|q\|_{\infty}}{\Gamma(\alpha+1)}\right) \\ &=: \Lambda.\end{aligned} \tag{3.25}$$

Hence

$$\|y\|_{\infty} \leq \max\left\{\|\phi\|_C, \Lambda\right\} \quad \forall y \in \mathcal{E}. \tag{3.26}$$

This shows that the set $\mathcal{E}$ is bounded. As a consequence of Theorem 2.9, we deduce that the operator $\mathcal{F}$ has a fixed point which is a mild solution of the problem (3.1) and (3.2). □

3.3. An Example

As an application of our results we consider the following partial functional differential equation of the form

$$\begin{gathered}\frac{\partial^{\alpha}}{\partial t^{\alpha}}z(t,x) = \frac{\partial^2}{\partial x^2}z(t,x) + Q(t,z(t-r,x)), \quad x \in [0,\pi],\ t \in [0,b],\ \alpha \in (0,1], \\ z(t,0) = z(t,\pi) = 0, \quad t \in [0,b], \\ z(t,x) = \phi(t,x), \quad t \in [-r,0],\ x \in [0,\pi],\end{gathered} \tag{3.27}$$

where $r > 0$, $\phi : [-r,0]\times[0,\pi] \to \mathbb{R}$ is continuous and $Q : [0,b]\times\mathbb{R} \to \mathbb{R}$ is a given function.

Let

$$\begin{gathered}y(t)(x) = z(t,x), \quad t \in J,\ x \in [0,\pi], \\ F(t,\phi)(x) = Q(t,\phi(\theta,x)), \quad \theta \in [-r,0],\ x \in [0,\pi], \\ \phi(\theta)(x) = \phi(\theta,x), \quad \theta \in [-r,0],\ x \in [0,\pi].\end{gathered} \tag{3.28}$$

Take $E = L^2[0,\pi]$ and define $A : D(A) \subset E \to E$ by $Aw = w''$ with domain

$$D(A) = \left\{w \in E, w, w' \text{ are absolutely continuous, } w'' \in E,\ w(0) = w(\pi) = 0\right\}. \tag{3.29}$$

Then

$$Aw = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, \quad w \in D(A), \tag{3.30}$$

where $(,)$ is the inner product in L^2 and $w_n(s) = \sqrt{(2/\pi)}\, \sin ns,\ n = 1, 2, \ldots$ is the orthogonal set of eigenvectors in A. It is well known (see [29]) that A is the infinitesimal generator of an analytic semigroup $T(t),\ t \in [0, b]$ in E and is given by

$$T(t)w = \sum_{n=1}^{\infty} \exp(-n^2 t)(w, w_n) w_n, \quad w \in E. \tag{3.31}$$

Since the analytic semigroup $T(t)$ is compact, there exists a constant $M \geq 1$ such that

$$\|T(t)\|_{B(E)} \leq M. \tag{3.32}$$

Also assume that there exist continuous functions $\sigma_1, \sigma_2 : [0, b] \to \mathbb{R}^+$ such that

$$|Q(t, w(t - r, x))| \leq \sigma_1(t)|w| + \sigma_2(t). \tag{3.33}$$

We can show that problem (3.1) and (3.2) is an abstract formulation of problem (3.27). Since all the conditions of Theorem 3.3 are satisfied, the problem (3.27) has a solution z on $[-r, b] \times [0, \pi]$.

3.4. *Existence Results for Infinite Delay*

In the following we will extend the previous results to the case when the delay is infinite. More precisely we consider the following problem

$$\begin{gathered} D^\alpha y(t) = Ay(t) + f(t, y_t), \quad t \in J := [0, b], \\ y_0 = \phi \in \mathcal{B}, \end{gathered} \tag{3.34}$$

where D^α is the standard Riemann-Liouville fractional derivative, $f : J \times \mathcal{B} \to E$ is a continuous function, $\mathcal{B}$ the phase space [41], $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$, $\phi : \mathcal{B} \to E$ a continuous function with $\phi(0) = 0$ and $(E, |\cdot|)$ a real Banach space. For any $t \in J$ the function $y_t \in \mathcal{B}$ is defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in (-\infty, 0]. \tag{3.35}$$

Consider the following space:

$$\mathcal{B}_b = \{y : (-\infty, b] \to E : y_{/J} \in C(J, E),\ y_0 \in \mathcal{B}\}, \tag{3.36}$$

where $y_{/J}$ is the restriction of y to J. Let $\|\cdot\|_b$ be the seminorm in $\mathcal{B}_b$ defined by

$$\|y\|_b = \|y_0\|_{\mathcal{B}} + \sup\{|y(s)| : 0 \le s \le b\}, \quad y \in \mathcal{B}_b. \tag{3.37}$$

Definition 3.4. One says that a function $y \in \mathcal{B}_b$ is a mild solution of problem (3.34) if $y_0 = \phi$ and

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) f(s, y_s) ds, \quad t \in J. \tag{3.38}$$

The first existence result is based on Banach's contraction principle.

Theorem 3.5. *Assume the following.*

$(H4)$ *There exists a nonnegative constant k such that*

$$|f(t,u) - f(t,v)| \le k\|u - v\|_{\mathcal{B}}, \quad \text{for } t \in J \text{ and every } u, v \in \mathcal{B}. \tag{3.39}$$

Then there exists a unique mild solution of problem (3.34) *on* $(-\infty, b]$.

Proof. Transform the IVP (3.34) into a fixed point problem. Consider the operator $\mathcal{N} : \mathcal{B}_b \to \mathcal{B}_b$ defined by

$$\mathcal{N}(y)(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} T(t-s) f(s, y_s) ds, & t \in J. \end{cases} \tag{3.40}$$

For $\phi \in \mathcal{B}$, we define the function

$$x(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ 0, & t \in J. \end{cases} \tag{3.41}$$

Then $x \in \mathcal{B}_b$. Set

$$y(t) = z(t) + x(t). \tag{3.42}$$

It is obvious that y satisfies (3.38) if and only if z satisfies $z_0 = 0$ and

$$z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) f(s, z_s + x_s) ds, \quad t \in J. \tag{3.43}$$

Let

$$\mathcal{B}_b^0 = \{z \in \mathcal{B}_b : z_0 = 0\}. \tag{3.44}$$

For any $z \in \mathcal{B}_b^0$, we have

$$\|z\|_b = \|z_0\|_B + \sup\{|z(s)| : 0 \le s \le b\} = \sup\{|z(s)| : 0 \le s \le b\}. \tag{3.45}$$

Thus $(\mathcal{B}_b^0, \|\cdot\|_b)$ is a Banach space. Let the operator $\mathcal{P} : \mathcal{B}_b^0 \to \mathcal{B}_b^0$ defined by

$$\mathcal{P}(z)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)f(s, z_s + x_s)ds. \tag{3.46}$$

It is obvious that $\mathcal{N}$ has a fixed point is equivalent to $\mathcal{P}$ has a fixed point, and so we turn to proving that $\mathcal{P}$ has a fixed point. As in Theorem 3.2, we show by induction that $\mathcal{P}^n$ satisfy for any $z, \overline{z} \in \mathcal{B}_b^0$, the following inequality:

$$\|\mathcal{P}^n(z) - \mathcal{P}^n(\overline{z})\|_b \le \frac{(kMK_b)^n}{\Gamma(n\alpha+1)} b^{n\alpha}\|z - \overline{z}\|_b, \tag{3.47}$$

which yields the contraction of $\mathcal{P}^n$ for sufficiently large values of n. Therefore, by the Banach's contraction principle $\mathcal{P}$ has a unique fixed point z^*. Then $y^*(t) = z^*(t) + x(t),\ t \in (-\infty, b]$ is a fixed point of the operator $\mathcal{N}$, which gives rise to a unique mild solution of the problem (3.34). □

Next we give an existence result based upon the nonlinear alternative of Leray-Schauder type.

Theorem 3.6. *Assume that the following hypotheses hold.*

(H5) *The semigroup* $\{T(t)\}_{t\in J}$ *is compact for* $t > 0$.

(H6) *There exist functions* $p, q \in C(J, \mathbb{R}_+)$ *such that*

$$|f(t,u)| \le p(t) + q(t)\|u\|_{\mathcal{B}}, \quad \text{for a.e. } t \in J, \text{ and each } u \in \mathcal{B}. \tag{3.48}$$

Then, the problem (3.34) *has at least one mild solution on* $(-\infty, b]$.

Proof. Transform the IVP (3.34) into a fixed point problem. Consider the operator $\mathcal{P}$ defined as in Theorem 3.5. We will show that the operator $\mathcal{P}$ is continuous and completely continuous. Let $\{z_n\}$ be a sequence such that $z_n \to z$ in $\mathcal{B}_b^0$. Then

$$\begin{aligned} |\mathcal{P}(z_n)(t) - \mathcal{P}(z)(t)| &\le \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)\left[f(s, z_{n_s} + x_s) - f(s, z_s + x_s)\right]ds\right| \\ &\le \frac{Mb^\alpha}{\alpha\Gamma(\alpha)}\|f(\cdot, z_{n\cdot} + x_\cdot) - f(\cdot, z_\cdot + x_\cdot)\|_\infty. \end{aligned} \tag{3.49}$$

Since f is a continuous function, then we have

$$\|\mathcal{P}(z_n) - \mathcal{P}(z)\|_b \le \frac{Mb^\alpha}{\Gamma(\alpha+1)}\|f(\cdot, z_{n\cdot} + x_\cdot) - f(\cdot, z_\cdot + x_\cdot)\|_\infty \longrightarrow 0 \quad \text{as } n \longmapsto \infty. \tag{3.50}$$

Thus $\mathcal{D}$ is continuous. To show that $\mathcal{D}$ maps bounded sets into bounded sets in $\mathcal{B}_b^0$ it is enough to show that for any $\rho > 0$ there exists a positive constant δ such that for each $z \in B_\rho = \{z \in \mathcal{B}_b^0 : \|z\|_b \leq \rho\}$ we have $\mathcal{D}(z) \in B_\delta$. Let $z \in B_\rho$, then

$$\begin{aligned}\|z_s + x_s\|_{\mathcal{B}} &\leq \|z_s\|_{\mathcal{B}} + \|x_s\|_{\mathcal{B}} \\ &\leq K_b\rho + M_b\|\phi\|_{\mathcal{B}} \\ &:= \rho^*.\end{aligned} \tag{3.51}$$

Then we have for each $t \in J$

$$\begin{aligned}|\mathcal{D}(z)(t)| &= \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)f(s, z_s + x_s)ds\right| \\ &\leq \frac{M}{\Gamma(\alpha)}\int_0^t \sup_{s\in[0,t]}|p(s)|(t-s)^{\alpha-1}ds + \frac{M\rho^*}{\Gamma(\alpha)}\int_0^t \sup_{s\in[0,t]}|q(s)|(t-s)^{\alpha-1}ds.\end{aligned} \tag{3.52}$$

Taking the supremum over t we have

$$\|\mathcal{D}(z)\|_b \leq \frac{Mb^\alpha(\|p\|_\infty + \rho^*\|q\|_\infty)}{\Gamma(\alpha+1)} =: \delta. \tag{3.53}$$

Now let $\tau_1, \tau_2 \in J$, $\tau_2 > \tau_1$. thus if $\epsilon > 0$ and $\epsilon \leq \tau_1 \leq \tau_2$ we have for each $z \in B_\rho$

$$\begin{aligned}\left|\mathcal{D}(z)(\tau_2) - \mathcal{D}(z)(\tau_1)\right| \leq{}& \frac{M(\|p\|_\infty + \rho^*\|q\|_\infty)}{\Gamma(\alpha)} \\ &\times \Bigg(\int_0^{\tau_1-\epsilon}\left[(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}\right]ds \\ &\quad + \left\|T(\tau_2-\tau_1-\epsilon) - T(\epsilon)\right\|_{B(E)}\int_0^{\tau_1-\epsilon}(\tau_2-s)^{\alpha-1}ds \\ &\quad + \int_{\tau_1-\epsilon}^{\tau_1}\left((\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}\right)ds + \int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}ds\Bigg).\end{aligned} \tag{3.54}$$

As $\tau_1 \to \tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero, since $T(t)$ is a strongly continuous operator and the compactness of $T(t)$ for $t > 0$ implies the continuity in the uniform operator topology (see [29]). By the Arzelá-Ascoli theorem it suffices to show that $\mathcal{D}$ maps B_ρ into a precompact set in E. Let $0 < t < b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $z \in B_\rho$ we define

$$\mathcal{D}_\epsilon(z)(t) = \frac{T(\epsilon)}{\Gamma(\alpha)}\int_0^{t-\epsilon}(t-s-\epsilon)^{\alpha-1}T(t-s-\epsilon)f(s, z_s + x_s)ds. \tag{3.55}$$

Since $T(t)$ is a compact operator for $t > 0$, the set

$$Z_\epsilon(t) = \{\mathcal{P}_\epsilon(z)(t) : z \in B_\rho\} \tag{3.56}$$

is precompact in E for every ϵ, $0 < \epsilon < t$. Moreover

$$\begin{aligned}\left|\mathcal{P}(z)(t) - \mathcal{P}_\epsilon(z)(t)\right| &\le \frac{M(\|p\|_\infty + \rho^*\|q\|_\infty)}{\Gamma(\alpha)}\left(\int_0^{t-\epsilon}\left[(t-s)^{\alpha-1} - (t-s-\epsilon)^{\alpha-1}\right]ds \right.\\ &\qquad \left. + \int_{t-\epsilon}^{t}(t-s)^{\alpha-1}ds.\right)\\ &\le \frac{M(\|p\|_\infty + \rho^*\|q\|_\infty)}{\Gamma(\alpha+1)}\left(t^\alpha - (t-\epsilon)^\alpha\right).\end{aligned} \tag{3.57}$$

Therefore, the set $Z(t) = \{\mathcal{P}(z)(t) : z \in B_\rho\}$ is precompact in E. Hence the operator $\mathcal{P}$ is completely continuous. Now, it remains to show that the set

$$\mathcal{E} = \left\{z \in \mathcal{B}_b^0 : z = \lambda\mathcal{P}(z) \text{ for some } 0 < \lambda < 1\right\} \tag{3.58}$$

is bounded. Let $z \in \mathcal{E}$ be any element. Then, for each $t \in J$,

$$z(t) = \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)f(s, z_s + x_s)ds. \tag{3.59}$$

Then

$$|z(t)| \le \frac{Mb^\alpha\|p\|_\infty}{\Gamma(\alpha+1)} + \frac{M\|q\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\|z_s + x_s\right\|_B ds, \tag{3.60}$$

but

$$\begin{aligned}\left\|z_t + x_t\right\|_B &\le K(t)\sup\{|z(s)| : 0 \le s \le t\} + M(t)\left\|z_0\right\|_B\\ &\quad + K(t)\sup\{|x(s)| : 0 \le s \le t\} + M(t)\left\|x_0\right\|_B\\ &\le K_b\sup\{|z(s)| : 0 \le s \le t\} + M_b\|\phi\|_B.\end{aligned} \tag{3.61}$$

Take the right-hand side of the above inequality as $v(t)$, then by (3.60) we have

$$|z(t)| \le \frac{Mb^\alpha\|p\|_\infty}{\Gamma(\alpha+1)} + \frac{M\|q\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds. \tag{3.62}$$

Using the above inequality and the definition of v we have

$$v(t) \le \frac{MK_b b^\alpha\|p\|_\infty}{\Gamma(\alpha+1)} + \frac{MK_b\|q\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds + M_b\|\phi\|_B. \tag{3.63}$$

By Lemma 2.7, there exists a constant $K = K(\alpha)$ such that we have

$$v(t) \leq \left[\frac{MK_b b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} + M_b\|\phi\|_B\right]\left[1 + \frac{MKK_b b^\alpha \|q\|_\infty}{\Gamma(\alpha+1)}\right] := \Lambda. \tag{3.64}$$

Then there exists a constant $d = d(\Lambda)$ such that $\|z\|_b \leq d$. This shows that the set $\mathcal{E}$ is bounded. As a consequence of the Leray-Schauder Theorem, we deduce that the operator $\mathcal{P}$ has a fixed point, then $\mathcal{N}$ has one which gives rise to a mild solution of the problem (3.34). □

3.5. An Example

To illustrate the previous results, we consider in this section the following model:

$$\begin{gathered}
\frac{\partial^\alpha}{\partial t^\alpha} v(t,\xi) = \frac{\partial^2 v}{\partial \xi^2}(t,\xi) + \int_{-\infty}^{0} P(\theta) r(t, v(t+\theta,\xi))d\theta, \quad t \in [0,T],\ \xi \in [0,\pi],\ \alpha \in (0,1], \\
v(t,0) = v(t,\pi) = 0, \quad \in [0,T], \\
v(\theta,\xi) = v_0(\theta,\xi), \quad -\infty < \theta \leq 0,\ \xi \in [0,\pi],
\end{gathered} \tag{3.65}$$

where $P : (-\infty, 0] \to R$, $r : [0,T] \times R \to R$, $v_0 : (-\infty, 0] \times [0,\pi] \to R$ are continuous functions.

Consider $E = L^2([0,\pi], R)$ and define A by $Aw = w''$ with domain

$$D(A) = \{w \in E : w, w' \text{ are absolutely continuous, } w'' \in E,\ w(0) = w(\pi) = 0\}. \tag{3.66}$$

Then A generates a C_0 semigroup $T(t)$ (see [29]).

For the phase space $\mathcal{B}$, we choose the well-known space $\mathrm{BUC}(R^-, E)$: the space of uniformly bounded continuous functions endowed with the following norm:

$$\|\varphi\| = \sup_{\theta \leq 0} |\varphi(\theta)| \quad \text{for } \varphi \in \mathcal{B}. \tag{3.67}$$

If we put for $\varphi \in \mathrm{BUC}(R^-, E)$ and $\xi \in [0,\pi]$

$$\begin{gathered}
y(t)(\xi) = v(t,\xi), \quad t \in [0,T],\ \xi \in [0,\pi], \\
\phi(\theta)(\xi) = v_0(\theta,\xi), \quad -\infty < \theta \leq 0,\ \xi \in [0,\pi], \\
f(t,\varphi)(\xi) = \int_{-\infty}^{0} P(\theta) r(t, \varphi(\theta)(\xi))d\theta, \quad -\infty < \theta \leq 0,\ \xi \in [0,\pi].
\end{gathered} \tag{3.68}$$

Then, problem (3.65) takes the abstract neutral evolution form (3.34).

4. Semilinear Functional Differential Equations of Neutral Type

4.1. Introduction

Neutral differential equations arise in many areas of applied mathematics, an extensive theory is developed, we refer the reader to the book by Hale and Verduyn Lunel [44] and Kolmanovskii and Myshkis [45]. The work for neutral functional differential equations with infinite delay was initiated by Hernández and Henríquez in [57, 58]. In the following, we will extend such results to arbitrary order functional differential equations of neutral type with finite as well as infinite delay. We based our main results upon the Banach's principle and the Leray-Schauder theorem.

4.2. Existence Results for the Finite Delay

First we will be concerned by the case when the delay is finite, more precisely we consider the following class of neutral functional differential equations

$$\begin{gathered} D^{\alpha}\left[y(t)-h\left(t,y_t\right)\right]=A\left[y(t)-h\left(t,y_t\right)\right]+f\left(t,y_t\right), \quad t\in J:=[0,b], \\ y(t)=\phi(t), \quad t\in[-r,0]. \end{gathered} \tag{4.1}$$

Definition 4.1. One says that a function $y\in C([-r,b],E)$ is a mild solution of problem (4.1) if $y(t)=\phi(t),\ t\in[-r,0]$ and

$$y(t)=h\left(t,y_t\right)-T(t)h(0,\phi)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}T(t-s)f\left(s,y_s\right)ds, \quad t\in J. \tag{4.2}$$

Our first existence result is based on the Banach's contraction principle.

Theorem 4.2. *Assume the following.*

$(H7)$ *There exists a nonnegative constant k such that*

$$|f(t,u)-f(t,v)|\leq k\|u-v\|_C, \quad \text{for } t\in J \text{ and every } u,v\in C([-r,0],E). \tag{4.3}$$

$(H8)$ *There exists a nonnegative constant l such that*

$$|h(t,u)-h(t,v)|\leq l\|u-v\|_C, \quad \text{for } t\in J \text{ and every } u,v\in C([-r,0],E). \tag{4.4}$$

Then there exists a unique mild solution of problem (4.1) *on* $[-r,b]$.

Proof. Transform the IVP (4.1) into a fixed point problem. Consider the operator $\mathcal{F} : C([-r,b],E) \to C([-r,b],E)$ defined by

$$\mathcal{F}(y)(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ h(t,y_t) - T(t)h(0,\phi), & \\ +\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}T(t-s)f(s,y_s)ds, & t \in J. \end{cases} \tag{4.5}$$

As in Theorem 3.2, we show by induction that $\mathcal{F}^n$ satisfy for any $y, z \in C([-r,b],E)$, the following inequality:

$$\|\mathcal{F}^n(y) - \mathcal{F}^n(z)\|_\infty \le \sum_{j=0}^{n} \frac{1}{\Gamma(j\alpha+1)} C_n^j (kMb^\alpha)^j l^{n-j} \|y-z\|_\infty, \tag{4.6}$$

which yields the contraction of $\mathcal{F}^n$ for sufficiently large values of n. Therefore, by the Banach's contraction principle $\mathcal{F}$ has a unique fixed point which gives rise to unique mild solution of problem (4.1). □

Next we give an existence result using the nonlinear alternative of Leray-Schauder.

Theorem 4.3. *Assume that the following hypotheses hold.*

$(H9)$ *The semigroup* $\{T(t)\}_{t\in J}$ *is compact for* $t > 0$.

$(H10)$ *There exist functions* $p, q \in C(J, \mathbb{R}_+)$ *such that*

$$|f(t,u)| \le p(t) + q(t)\|u\|_C, \quad \text{for a.e. } t \in J, \text{ and each } u \in C([-r,0],E). \tag{4.7}$$

$(H11)$ *The function* h *is continuous and completely continuous, and for every bounded set* $B \in C([-r,b],E)$, *the set* $\{t \mapsto h(t,y_t), y \in B\}$ *is equicontinuous in* E.

$(H12)$ *There exists constants:* $0 \le c_1 < 1$, *and* $c_2 \ge 0$ *such that*

$$|h(t,u)| \le c_1\|u\|_C + c_2, \quad \text{for } t \in J \text{ and } u \in C([-r,0],E). \tag{4.8}$$

Then the problem (4.1) *has at least one mild solution on* $[-r,b]$.

Proof. Consider the operator $\mathcal{F} : C([-r,b],E) \to C([-r,b],E)$ as in Theorem 4.2.

To show that the operator $\mathcal{F}$ is continuous and completely continuous it suffices to show, using $(H11)$, that the operator $\widetilde{\mathcal{F}} : C([-r,b],E) \to C([-r,b],E)$ defined by

$$\widetilde{\mathcal{F}}(y)(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}T(t-s)f(s,y_s)ds, & t \in J \end{cases} \tag{4.9}$$

is continuous and completely continuous. This can be done following the proof of Theorem 3.3.

Now, it remains to show that the set

$$\mathcal{E} = \{y \in C([-r,b],E) : y = \lambda \mathcal{F}(y) \text{ for some } 0 < \lambda < 1\} \tag{4.10}$$

is bounded. Let $y \in \mathcal{E}$ be any element. Then, for each $t \in J$,

$$\begin{aligned}|y(t)| &\leq \left|h(t,y_t)\right| + M|h(0,\phi)| + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f(s,y_s)\right|ds \\ &\leq c_1\|y_t\|_C + c_2 + M\|\phi\|_C + Mc_2 + \frac{Mb^\alpha\|p\|_\infty}{\Gamma(\alpha+1)} \\ &\quad + \frac{M\|q\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|y_s\|_C ds.\end{aligned} \tag{4.11}$$

We consider the function defined by

$$\mu(t) = \max\{|y(s)| : -r \leq s \leq t\}, \quad t \in J. \tag{4.12}$$

Let $t^* \in [-r,t]$ such that $\mu(t) = |y(t^*)|$, If $t^* \in [0,b]$ then we have, for $t \in J$, (note $t^* \leq t$)

$$(1-c_1)\mu(t) \leq M\|\phi\|_C + (1+M)c_2 + \frac{Mb^\alpha\|p\|_\infty}{\Gamma(\alpha+1)} + \frac{M\|q\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mu(s)ds. \tag{4.13}$$

If $t^* \in [-r,0]$ then $\mu(t) = \|\phi\|_C$ and the previous inequality holds.

By Lemma 2.7 there exists $K = K(\alpha)$ such that

$$\mu(t) \leq \frac{1}{1-c_1}\left(M\|\phi\|_C + (1+M)c_2 + \frac{Mb^\alpha\|p\|_\infty}{\Gamma(\alpha+1)}\right)\left(1 + \frac{MKb^\alpha\|q\|_\infty}{\Gamma(\alpha+1)}\right) := \Lambda. \tag{4.14}$$

This shows that the set $\mathcal{E}$ is bounded. As a consequence of the Leray-Schauder Theorem, we deduce that the operator $\mathcal{F}$ has a fixed point which gives rise to a mild solution of the problem (4.1). □

4.3. Existence Results for the Infinite Delay

In the following we will extend our previous results to the case of infinite delay, more precisely we consider the following problem:

$$\begin{gathered}D^\alpha\left[y(t) - h(t,y_t)\right] = A\left[y(t) - h(t,y_t)\right] + f(t,y_t), \quad t \in J := [0,b], \\ y_0 = \phi \in \mathcal{B}.\end{gathered} \tag{4.15}$$

Our first existence result is based on the Banach's contraction principle.

Theorem 4.4. *Assume that the following hypotheses hold.*

$(H13)$ *There exists a nonnegative constant k such that*

$$|f(t,u)-f(t,v)| \leq k\|u-v\|_{\mathcal{B}}, \quad \text{for } t \in J \text{ and every } u,v \in \mathcal{B}. \tag{4.16}$$

$(H14)$ *There exists a nonnegative constant l such that*

$$|h(t,u)-h(t,v)| \leq l\|u-v\|_{\mathcal{B}}, \quad \text{for } t \in J \text{ and every } u,v \in \mathcal{B}. \tag{4.17}$$

Then there exists a unique mild solution of problem (4.15) *on* $(-\infty,b]$.

Proof. Consider the operator $\mathcal{N}:\mathcal{B}_b \to \mathcal{B}_b$ defined by

$$\mathcal{N}(y)(t) = \begin{cases} \phi(t), & t \in (-\infty,0], \\ h(t,y_t) - T(t)h(0,\phi), & \\ +\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}T(t-s)f(s,y_s)ds, & t \in J. \end{cases} \tag{4.18}$$

In analogy to Theorem 3.2, we consider the operator $\mathcal{P}:\mathcal{B}_b^0 \to \mathcal{B}_b^0$ defined by

$$\mathcal{P}(z)(t) = h(t,z_t+x_t) - T(t)h(0,\phi) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)f(s,z_s+x_s)ds. \tag{4.19}$$

As in Theorem 3.2, we show by induction that $\mathcal{P}^n$ satisfy for any $z,\overline{z} \in \mathcal{B}_b^0$, the following inequality:

$$\|\mathcal{P}^n(z)-\mathcal{P}^n(\overline{z})\|_\infty \leq \sum_{j=0}^{n} \frac{K_b^n}{\Gamma(j\alpha+1)} C_n^j (kMb^\alpha)^j l^{n-j} \|z-\overline{z}\|_b, \tag{4.20}$$

which yields the contraction of $\mathcal{P}^n$ for sufficiently large values of n. Therefore, by the Banach's contraction principle $\mathcal{P}$ has a unique fixed point z^*. Then $y^*(t) = z^*(t) + x(t)$, $t \in (-\infty,b]$ is a fixed point of the operator $\mathcal{N}$, which gives rise to a unique mild solution of the problem (4.15). □

Next we give an existence result based upon the the nonlinear alternative of Leray-Schauder.

Theorem 4.5. *Assume that the following hypotheses hold.*

$(H15)$ *The semigroup $\{T(t)\}_{t\in J}$ is compact for $t>0$.*

$(H16)$ *There exist functions $p,q \in C(J,\mathbb{R}_+)$ such that*

$$|f(t,u)| \leq p(t) + q(t)\|u\|_{\mathcal{B}}, \quad \text{for a.e. } t \in J, \text{ and each } u \in \mathcal{B}. \tag{4.21}$$

(*H*17) *The function h is continuous and completely continuous, and for every bounded set* $B \in \mathcal{B}_b^0$*, the set* $\{t \mapsto h(t, y_t),\ y \in B\}$ *is equicontinuous in E.*

(*H*18) *There exists constants:* $0 \leq c_1 < 1/K_b$*, and* $c_2 \geq 0$ *such that*

$$|h(t,u)| \leq c_1 \|u\|_{\mathcal{B}} + c_2, \quad \text{for } t \in J,\ u \in \mathcal{B}. \tag{4.22}$$

Then the problem (4.15) *has at least one mild solution on* $(-\infty, b]$.

Proof. Let $\mathcal{P} : \mathcal{B}_b^0 \to \mathcal{B}_b^0$ defined as in Theorem 4.4. We can easily show that the operator $\mathcal{P}$ is continuous and completely continuous. Using (*H*17) it suffices to show that the operator $\mathcal{P} : \mathcal{B}_b^0 \to \mathcal{B}_b^0$ defined by

$$\mathcal{P}(z)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) f(s, z_s + x_s) ds \tag{4.23}$$

is continuous and completely continuous. Now, it remains to show that the set

$$\mathcal{E} = \left\{ z \in \mathcal{B}_b^0 : z = \lambda \mathcal{P}(z) \text{ for some } 0 < \lambda < 1 \right\} \tag{4.24}$$

is bounded.

Let $z \in \mathcal{E}$ be any element. Then, for each $t \in J$,

$$\begin{aligned} |z(t)| &\leq \left| h(t, z_t + x_t) \right| + M|h(0,\phi)| + \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f(s, z_s + x_s) \right| ds \\ &\leq c_1 \|z_t + x_t\|_B + c_2 + M\|\phi\|_B + Mc_2 + \frac{Mb^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} \\ &\quad + \frac{M\|q\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| z_s + x_s \right\|_B ds. \end{aligned} \tag{4.25}$$

Denote $v(t)$ as in Theorem 3.6. Then

$$\begin{gathered} \left\| z_t + x_t \right\|_B \leq v(t), \\ v(t) \leq c_1 K_b v(t) + K_b \left[\left(M + \frac{M_b}{K_b} \right) \|\phi\|_B + (1+M)c_2 + \frac{Mb^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} \right] \\ + \frac{K_b M \|q\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds. \end{gathered} \tag{4.26}$$

Then

$$v(t) \leq \frac{K_b}{1-c_1K_b}\left[\left(M+\frac{M_b}{K_b}\right)\|\phi\|_B+(1+M)c_2+\frac{Mb^{\alpha}\|p\|_{\infty}}{\Gamma(\alpha+1)}\right] + \frac{K_b}{1-c_1K_b}\frac{M\|q\|_{\infty}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds. \tag{4.27}$$

By Lemma 2.7 there exists a constant $K = K(\alpha)$ such that

$$v(t) \leq \frac{d_1K_b}{1-c_1K_b}\left(1+\frac{KK_b}{1-c_1K_b}\frac{Mb^{\alpha}\|q\|_{\infty}}{\Gamma(\alpha+1)}\right) := \Lambda \tag{4.28}$$

where

$$d_1 = \left(M+\frac{M_b}{K_b}\right)\|\phi\|_B+(1+M)c_2+\frac{Mb^{\alpha}\|p\|_{\infty}}{\Gamma(\alpha+1)}. \tag{4.29}$$

Then there exists a constant $d = d(\Lambda)$ such that $\|z\|_b \leq d$. This shows that the set $\mathcal{E}$ is bounded. As a consequence of the Leray-Schauder Theorem, we deduce that the operator $\mathcal{P}$ has a fixed point which gives rise to a mild solution of the problem (4.15). □

4.4. Example

To illustrate the previous results, we consider the following model arising in population dynamics:

$$\begin{gathered}
\frac{\partial^{\alpha}}{\partial t^{\alpha}}\left[v(t,\xi)-\int_{-\infty}^{0}K_1(\theta)g_1(t,v(t+\theta,\xi))d\theta\right] \\
=\frac{\partial^2}{\partial \xi^2}\left[v(t,\xi)-\int_{-\infty}^{0}K_1(\theta)g_1(t,v(t+\theta,\xi))d\theta\right] \\
+\int_{-\infty}^{0}K_2(\theta)g_2(t,v(t+\theta,\xi))d\theta \quad \text{for } t\in J,\ \xi\in[0,\pi],\ \alpha\in(0,1] \\
v(t,0)-\int_{-\infty}^{0}K_1(\theta)g_1(t,v(t+\theta,0))d\theta=0 \quad \text{for } t\in J=[0,b], \\
v(t,\pi)-\int_{-\infty}^{0}K_1(\theta)g_1(t,v(t+\theta,\pi))d\theta=0 \quad \text{for } t\in J, \\
v(\theta,\xi)=v_0(\theta,\xi) \quad \text{for } -\infty<\theta\leq 0,\ \xi\in[0,\pi],
\end{gathered} \tag{4.30}$$

where $K_1, K_2 : (-\infty, 0] \to R$ and $g_1, g_2 : J \times R \to R$ and $v_0 : (-\infty, 0] \times 0, \pi] \to R$ are continuous functions. Let $E = L^2([0,\pi]; R)$ and consider the operator

$$A : D(A) \subseteq E \longrightarrow E \tag{4.31}$$

defined by

$$\begin{gathered} D(A) = \{y \in E,\ y, y' \text{ are absolutely continuous with } y'' \in E \text{ and } y(0) = y(\pi) = 0\}, \\ Ay = y''. \end{gathered} \tag{4.32}$$

It is well known that A generates a C_0-semigroup (see [29]). For the phase space $\mathcal{B}$, we choose the well-known space $\mathrm{BUC}(R^-, E)$: the space of bounded uniformly continuous functions endowed with the following norm:

$$\|\varphi\| = \sup_{\theta \leq 0} |\varphi(\theta)| \quad \text{for } \varphi \in \mathcal{B}. \tag{4.33}$$

If we put for $\varphi \in \mathrm{BUC}(R^-, E)$ and $\xi \in [0,\pi]$

$$\begin{aligned} h(t,\varphi)(\xi) &= \int_{-\infty}^{0} K_1(\theta) g_1(t, \varphi(\theta)(\xi)) d\theta, \\ f(t,\varphi)(\xi) &= \int_{-\infty}^{0} K_2(\theta) g_2(t, \varphi(\theta)(\xi)) d\theta, \end{aligned} \tag{4.34}$$

then (4.30) take the abstract form (4.15). Under appropriate conditions on g_1, g_2, the problem (4.30) has by Theorem 4.5 a solution.

5. Semilinear Functional Differential Inclusions

Differential inclusions are generalization of differential equations, therefore all problems considered for differential equations, that is, existence of solutions, continuation of solutions, dependence on initial conditions and parameters, are present in the theory of differential inclusions. Since a differential inclusion usually has many solutions starting at a given point, new issues appear, such as investigation of topological properties of the set of solutions, and selection of solutions with given properties.

Functional differential inclusions with fractional order are first considered by El Sayed and Ibrahim [59]. Very recently Benchohra et al. [49], and Ouahab [60] have considered some classes of ordinary functional differential inclusions with delay, and in [6, 61] Agarwal et al. considered a class of boundary value problems for differential inclusion involving Caputo fractional derivative of order $\alpha \in (2,3]$. Chang and Nieto [62] considered a class of fractional differential inclusions of order $\alpha \in (1,2]$. Here we continue this study by considering partial functional differential inclusions involving the Riemann-Liouville derivative of order $\alpha \in (0,1]$. The both cases of convex valued and nonconvex valued of the right-hand side are considered, and where the delay is finite as well as infinite. Our approach is based on the C_0-semigroups theory combined with some suitable fixed point theorems.

In the following, we will be concerned with fractional semilinear functional differential inclusions with finite delay of the form

$$D^{\alpha}y(t) - Ay(t) \in F(t, y_t), \quad t \in J := [0, b],$$
$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{5.1}$$

where D^{α} is the standard Riemann-Liouville fractional derivative. $F : J \times C([-r, 0], E) \to \mathcal{P}(E)$ is a multivalued function. $\mathcal{P}(E)$ is the family of all nonempty subsets of E. $A : D(A) \subset E \to E$is a densely defined (possibly unbounded) operator generating a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ of bounded linear operators from E into E. $\phi : [-r, 0] \to E$ is a given continuous function such that $\phi(0) = 0$ and $(E, |\cdot|)$ is a real separable Banach spaces. For $\psi \in C([-r, 0], E)$ the norm of ψ is defined by

$$\|\psi\|_C = \sup\{|\psi(\theta)|, \theta \in [-r, 0]\}. \tag{5.2}$$

For $\phi \in C([-r, b], E)$ the norm of ϕ is defined by

$$\|\phi\|_{\mathfrak{D}} = \sup\{|\phi(\theta)|, \theta \in [-r, b]\}. \tag{5.3}$$

Recall that for each $y \in C([-r, b], E)$ the set

$$S_{F,y} = \left\{v \in L^1(J, E) : v(t) \in F(t, y_t) \text{ for a.e. } t \in J\right\} \tag{5.4}$$

is known as the set of selections of the multivalued F.

Definition 5.1. One says that a continuous function $y : [-r, b] \to E$ is a mild solution of problem (5.1) if there exists $f \in S_{F,y}$ such that $y(t) = \phi(t),\ t \in [-r, 0]$, and

$$y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)f(s)ds, \quad t \in J. \tag{5.5}$$

In the following, we give our first existence result for problem (5.1) with a convex valued right-hand side. Our approach is based upon Theorem 2.10.

Theorem 5.2. *Assume the following.*

($H19$) *$F : J \times C([-r, 0], E) \to \mathcal{P}_{\mathrm{cv,cp}}(E)$ is Carathéodory.*

($H20$) *The semigroup $\{T(t)\}_{t\in J}$ is compact for $t > 0$.*

($H21$) *There exist functions $p, q \in C(J, \mathbb{R}_+)$ such that*

$$\|F(t, u)\|_{\mathcal{P}(E)} \leq p(t) + q(t)\|u\|_C, \quad \text{for a.e. } t \in J, \text{ and each } u \in C([-r, 0], E). \tag{5.6}$$

Then the problem (5.1) *has at least one mild solution.*

Proof. Consider the multivalued operator

$$\mathcal{A} : C([-r,b],E) \longrightarrow \mathcal{P}(C([-r,b],E)) \tag{5.7}$$

defined by $\mathcal{A}(y) := \{h \in C([-r,b],E)\}$ such that

$$h(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0], \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} T(t-s) f(s) ds, & \text{if } t \in J, \end{cases} \tag{5.8}$$

where $f \in S_{F,y}$. It is obvious that fixed points of $\mathcal{A}$ are mild solutions of problem (5.1). We will show that $\mathcal{A}$ is a completely continuous multivalued operator, u.s.c. with convex values.

It is obvious that $\mathcal{A}$ is convex valued for each $y \in C([-r,b],E)$ since F has convex values.

To show that $\mathcal{A}$ maps bounded sets into bounded sets in $C([-r,b],E)$ it is enough to show that there exists a positive constant δ such that for each $h \in \mathcal{A}(y)$, $y \in B_\rho = \{y \in C(J,E) : \|y\|_\infty \le \rho\}$ one has $h \in B_\delta$. Indeed, if $h \in \mathcal{A}(y)$, then there exists $f \in S_{F,y}$ such that for each $t \in J$ we have

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) f(s) ds. \tag{5.9}$$

Using $(H21)$ we have for each $t \in J$,

$$|h(t)| \le \frac{Mb^\alpha}{\Gamma(\alpha+1)} \left(\|p\|_\infty + \rho\|q\|_\infty\right) =: \delta. \tag{5.10}$$

Then for each $h \in \mathcal{A}(B_\rho)$ we have $\|h\| \le \delta$.

Now let $h \in \mathcal{A}(y)$ for $y \in B_\rho$, and let $\tau_1, \tau_2 \in J$, $\tau_2 > \tau_1$. If $\epsilon > 0$ and $\epsilon \le \tau_1 \le \tau_2$ we have

$$\begin{aligned} \left|h(\tau_2) - h(\tau_1)\right| \le{}& \frac{1}{\Gamma(\alpha)} \left| \int_0^{\tau_1-\epsilon} \left[(\tau_2-s)^{\alpha-1} T(\tau_2-s) - (\tau_1-s)^{\alpha-1} T(\tau_1-s)\right] f(s) ds \right| \\ &+ \frac{1}{\Gamma(\alpha)} \left| \int_{\tau_1-\epsilon}^{\tau_1} \left[(\tau_2-s)^{\alpha-1} T(\tau_2-s) - (\tau_1-s)^{\alpha-1} T(\tau_1-s)\right] f(s) ds \right| \\ &+ \frac{1}{\Gamma(\alpha)} \left| \int_{\tau_1}^{\tau_2} (\tau_2-s)^{\alpha-1} T(\tau_2-s) f(s) ds \right|, \end{aligned} \tag{5.11}$$

where $f \in S_{F,y}$. Using the following semigroup identities

$$\begin{aligned} T(\tau_2-s) &= T(\tau_2-\tau_1+\epsilon) T(\tau_1-\epsilon-s), \\ T(\tau_1-s) &= T(\tau_1-\epsilon-s) T(\epsilon), \end{aligned} \tag{5.12}$$

we get

$$\begin{aligned}\left|h(\tau_2)-h(\tau_1)\right| \le{}& M\frac{\|p\|_\infty+\rho\|q\|_\infty}{\Gamma(\alpha)}\\ &\times\left(\int_0^{\tau_1-\epsilon}\left[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right]ds+M\int_{\tau_1-\epsilon}^{\tau_1}\left[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right]ds\right.\\ &\left.+M\|T(\tau_2-\tau_1+\epsilon)-T(\epsilon)\|_{B(E)}\int_0^{\tau_1}(\tau_2-s)^{\alpha-1}ds+M\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}ds\right).\end{aligned} \tag{5.13}$$

As $\tau_1 \to \tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero, since $T(t)$ is a strongly continuous operator and the compactness of $T(t)$ for $t>0$ implies the continuity in the uniform operator topology [29]. Let $0<t<b$ be fixed and let ϵ be a real number satisfying $0<\epsilon<t$. For $y\in B_\rho$ we define

$$h_\epsilon(t)=\frac{T(\epsilon)}{\Gamma(\alpha)}\int_0^{t-\epsilon}(t-s-\epsilon)^{\alpha-1}T(t-s-\epsilon)f(s)ds, \tag{5.14}$$

where $f\in S_{F,y}$. Since $T(t)$ is a compact operator, the set

$$H_\epsilon(t)=\left\{h_\epsilon(t): h_\epsilon\in\mathcal{A}(y)\right\} \tag{5.15}$$

is precompact in E for every $\epsilon,\ 0<\epsilon<t$. Moreover, for every $h\in\mathcal{A}(y)$ we have

$$\left|h(t)-h_\epsilon(t)\right|\le M\frac{\|p\|_\infty+\rho\|q\|_\infty}{\Gamma(\alpha)}\left(t^\alpha-(t-\epsilon)^\alpha\right). \tag{5.16}$$

Therefore, the set $H(t)=\{h(t): h\in\mathcal{A}(y)\}$ is totally bounded. Hence $H(t)=\{h(t): h\in\mathcal{A}(B_\rho)\}$ is precompact in E.

As a consequence of the Arzelá-Ascoli theorem we can conclude that the multivalued operator $\mathcal{A}$ is completely continuous.

Now we show that the operator $\mathcal{A}$ has closed graph. Let $y_n\to y_*$, $h_n\in\mathcal{A}(y_n)$, and $h_n\to h_*$. We will show that $h_*\in\mathcal{A}(y_*)$.

$h_n\in\mathcal{A}(y_n)$ means that there exists $f_n\in S_{F,y_n}$ such that

$$h_n(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}T(t-s)f_n(s)ds,\quad t\in J. \tag{5.17}$$

We must show that there exists $f_*\in S_{F,y_*}$ such that, for each $t\in J$

$$h_*(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}T(t-s)f_*(s)ds. \tag{5.18}$$

Since $F(\cdot,\cdot)$ has compact values, there exists a subsequence $f_{n_j}(\cdot)$ such that

$$\begin{gathered} f_{n_j}(\cdot) \longrightarrow f_*(\cdot) \quad \text{as } j \longmapsto \infty, \\ f_*(t) \in F\big(t, y_*(t)\big) \quad \text{a.e. } t \in J. \end{gathered} \tag{5.19}$$

Since $F(t,\cdot)$ is u.s.c., then for every $\epsilon > 0$ (particularly for $(\Gamma(\alpha+1)/Mb^\alpha)\epsilon$), there exist $n_0 = n_0(\epsilon) \geq 0$ such that for every $n \geq n_0$, we have

$$h_n(t) \in F\big(t, y_n(t)\big) \subset F\big(t, y_*(t)\big) + \epsilon B(0,1), \quad \text{a.e. } t \in J, \tag{5.20}$$

and hence,

$$\left|f_{n_j}(t) - f_*(t)\right| \leq \frac{\Gamma(\alpha+1)}{Mb^\alpha}\epsilon, \quad \text{a.e. } t \in J. \tag{5.21}$$

Then for each $t \in J$

$$\begin{aligned} \left|h_n(t) - h_*(t)\right| &\leq \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|f_n(s) - f_*(s)\right| ds \\ &\leq \frac{Mb^\alpha}{\Gamma(\alpha+1)} \left\|f_n - f_*\right\|_\infty. \end{aligned} \tag{5.22}$$

Hence,

$$\left\|h_n - h_*\right\|_\infty < \epsilon. \tag{5.23}$$

Now it remains to show that the set

$$\mathcal{E} = \{y \in C([-r,b],E) : \lambda y \in \mathcal{A} y \text{ for some } \lambda > 1\} \tag{5.24}$$

is bounded. Let $y \in \mathcal{E}$ be any element, then there exists $f \in S_{F,y}$ such that

$$y(t) = \lambda^{-1} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) f(s) ds. \tag{5.25}$$

Then by (H20) and (H21) for each $t \in J$ we have

$$|y(t)| \leq \frac{Mb^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} + \frac{M\|q\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\|y_s\right\|_C ds. \tag{5.26}$$

Consider the function defined by

$$\mu(t) = \max\{|y(s)| : -r \leq s \leq t\}, \quad t \in J. \tag{5.27}$$

Let $t^* \in [-r, t]$ such that $\mu(t) = |y(t^*)|$, If $t^* \in [0, b]$ then we have, for $t \in J$, (note $t^* \le t$)

$$\mu(t) \le \frac{Mb^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} + \frac{M\|q\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mu(s) ds. \tag{5.28}$$

If $t^* \in [-r, 0]$ then $\mu(t) = \|\phi\|_C$ and the previous inequality holds.

By Lemma 2.7 we have

$$\mu(t) \le \frac{Mb^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} \left(1 + k \frac{Mb^\alpha \|q\|_\infty}{\Gamma(\alpha+1)}\right) =: \Lambda. \tag{5.29}$$

Taking the supremum over $t \in J$ we get

$$\|y\|_C \le \Lambda. \tag{5.30}$$

Hence

$$\|y\|_{\mathfrak{D}} = \max\{\|\phi\|_C, \Lambda\}, \tag{5.31}$$

and so, the set $\mathcal{E}$ is bounded. Consequently the multivalued operator $\mathcal{A}$ has a fixed point which gives rise to a mild solution of problem (5.1) on $[-r, b]$. □

Now we will be concerned with existence results for problem (5.1) with nonconvex valued right-hand side. Our approach is based on the fixed point theorem for contraction multivalued maps due to Covitz and Nadler Jr. [35].

Theorem 5.3. *Assume that (H19) holds.*

($H22$) *There exists $l \in L^1(J, \mathbb{R}_+)$ such that*

$$H_d(F(t,x), F(t,y)) \le l(t)\|x-y\|_C \quad \text{a.e. } t \in J, \ \forall x, y \in C([-r,0], E), \tag{5.32}$$

with

$$d(0, F(t,0)) \le l(t) \quad \text{for a.e. } t \in J. \tag{5.33}$$

If

$$MI_0^\alpha l(b) < 1, \tag{5.34}$$

then the problem (5.1) *has at least one mild solution on* $[-r, b]$.

Proof. First we will prove that $\mathcal{A}(y) \in \mathcal{P}_{\text{cl}}(C([-r,b],E))$ for each $y \in C([-r,b],E)$. $(y_n)_{n\geq 0} \in \mathcal{A}(y)$ such that $y_n \to \tilde{y}$ in $C([-r,b],E)$. Then $\tilde{y} \in C([-r,b],E)$ and there exists $f_n \in S_{F,y}$ such that for each $t \in J$

$$y_n(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)f_n(s)ds. \tag{5.35}$$

Using the compactness property of the values of F and the second part of $(H22)$ we may pass to a subsequence if necessary to get that f_n converges weakly to $f \in L^1_\omega(J,E)$ (the space endowed with the weak topology). From Mazur's lemma (see [63]) there exists

$$f \in \overline{\text{conv}}\{f_n(t) : n \geq 1\}, \tag{5.36}$$

then there exists a subsequence $\{\overline{f}_n(t) : n \geq 1\}$ in $\overline{\text{conv}}\{f_n(t) : n \geq 1\}$, such that $\overline{f}_n$ converges strongly to f in $L^1(J,E) \Rightarrow f \in L^1(J,E)$. Then for each $t \in J$,

$$y_n(t) \longrightarrow \tilde{y}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)f(s)ds. \tag{5.37}$$

So, $\tilde{y} \in \mathcal{A}(y)$.

Now Let $y_1, y_2 \in C([-r,b],E)$ and $h_1 \in \mathcal{A}(y_1)$. Then there exists $f_1 \in S_{F,y_1}$ such that

$$h_1(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)f_1(s)ds, \quad t \in J. \tag{5.38}$$

Then from $(H22)$ there is $\omega \in S_{F,y_2}$ such that

$$\left|f_1(t) - \omega\right| \leq l(t)\left\|y_{1_t} - y_{2_t}\right\|_C, \quad t \in J. \tag{5.39}$$

Consider the multivalued operator $U : J \to \mathcal{P}(E)$ defined by

$$U(t) = \left\{\omega \in E : \left|f_1(t) - \omega\right| \leq l(t)\left\|y_{1_t} - y_{2_t}\right\|_C\right\}. \tag{5.40}$$

Since the multivalued operator $V(t) = U(t) \cap F(t, y_{2_t})$ is measurable (see [64, proposition III4]) there exists $f_2(t)$ a measurable selection for V. So, $f_2(t) \in F(t, y_{2_t})$ and

$$\left|f_1(t) - f_2(t)\right| \leq l(t)\left\|y_{1_t} - y_{2_t}\right\|_C, \quad t \in J. \tag{5.41}$$

Let us define for each $t \in J$

$$h_2(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)f_1(s)ds. \tag{5.42}$$

Then we have

$$\begin{aligned}\left|h_1(t)-h_2(t)\right| &\le \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f_1(s)-f_2(s)\right|ds\\ &\le \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} l(s)\left\|y_{1_s}-y_{2_s}\right\|_C ds\\ &\le M I_0^\alpha l(b)\left\|y_1-y_2\right\|_\infty.\end{aligned} \tag{5.43}$$

For $t \in [-r,0]$, the previous inequality is satisfied. Taking the supremum over t we get

$$\left\|h_1-h_2\right\|_{\mathfrak{D}} \le M I_0^\alpha l(b)\left\|y_1-y_2\right\|_{\mathfrak{D}}. \tag{5.44}$$

By analogous relation, obtained by interchanging the roles of y_1 and y_2, it follows that

$$H_d(\mathcal{A}(y_1),\mathcal{A}(y_2)) \le M I_0^\alpha l(b)\left\|y_1-y_2\right\|_{\mathfrak{D}}. \tag{5.45}$$

By (5.34) $\mathcal{A}$ is a contraction, and hence Theorem 2.8 implies that $\mathcal{A}$ has a fixed point which gives rise to a mild solution of problem (5.1). □

In the following, we will extend the previous results to the case when the delay is infinite. More precisely we consider the following problem:

$$\begin{gathered} D^\alpha y(t) - Ay(t) \in F(t,y_t), \quad t \in J := [0,b],\\ y_0 = \phi \in \mathcal{B}, \end{gathered} \tag{5.46}$$

where D^α is the standard Riemann-Liouville fractional derivative. $F : J \times \mathcal{B} \to \mathcal{P}(E)$ is a multivalued function. $\mathcal{B}$ is the phase space [41], $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\ge 0}$, $\phi : \mathcal{B} \to E$ a continuous function with $\phi(0) = 0$ and $(E, |\cdot|)$ a real Banach space. Consider the following space:

$$\mathcal{B}_b = \{y : (-\infty, b] \to E : y_{/J} \in C(J,E),\ y_0 \in \mathcal{B}\}, \tag{5.47}$$

where $y_{/J}$ is the restriction of y to J. Let $\|\cdot\|_b$ be the seminorm in $\mathcal{B}_b$ defined by

$$\|y\|_b = \left\|y_0\right\|_{\mathcal{B}} + \sup\{|y(s)| : 0 \le s \le b\}, \quad y \in \mathcal{B}_b. \tag{5.48}$$

Definition 5.4. One says that a function $y \in \mathcal{B}_b$ is a mild solution of problem (5.46) if $y_0 = \phi$ and there exists $f \in S_{F,y}$ such that

$$y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} T(t-s) f(s)ds, \quad t \in J. \tag{5.49}$$

In the following, we give an existence result for problem (5.46) with convex valued right-hand side. Our approach is based upon Theorem 2.10.

Theorem 5.5. *Assume the following.*

($H23$) $F : J \times \mathcal{B} \to \mathcal{P}_{\mathrm{cv,cp}}(\mathcal{B})$ *is Carathéodory.*

($H24$) *The semigroup* $\{T(t)\}_{t\in J}$ *is compact for* $t > 0$.

($H25$) *There exist functions* $p, q \in C(J, \mathbb{R}_+)$ *such that*

$$\|F(t,u)\|_{\mathcal{P}(E)} \leq p(t) + q(t)\|u\|_{\mathcal{B}}, \quad \text{for a.e. } t \in J, \text{ and each } u \in \mathcal{B}. \tag{5.50}$$

Then the problem (5.46) *has at least one mild solution.*

Proof. Consider the operator

$$\mathcal{N} : \mathcal{B}_b \longrightarrow \mathcal{P}(\mathcal{B}_b) \tag{5.51}$$

defined by

$$\mathcal{N}(y)(t) = \left\{ h \in \mathcal{B}_b : h(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} T(t-s) f(s) ds, & t \in J \end{cases} \right\}, \tag{5.52}$$

where $f \in S_{F,y}$.

For $\phi \in \mathcal{B}$, we define the function

$$x(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ 0, & t \in J. \end{cases} \tag{5.53}$$

Then $x \in \mathcal{B}_b$. Set

$$y(t) = z(t) + x(t). \tag{5.54}$$

It is obvious that y satisfies (5.49) if and only if z satisfies $z_0 = 0$ and

$$z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) f(s) ds, \quad t \in J. \tag{5.55}$$

Let

$$\mathcal{B}_b^0 = \{ z \in \mathcal{B}_b : z_0 = 0 \}. \tag{5.56}$$

For any $z \in \mathcal{B}_b^0$, we have

$$\|z\|_b = \|z_0\|_B + \sup\{|z(s)| : 0 \leq s \leq b\} = \sup\{|z(s)| : 0 \leq s \leq b\}. \tag{5.57}$$

Thus $(\mathcal{B}_b^0, \|\cdot\|_b)$ is a Banach space. Let the operator $\mathcal{A} : \mathcal{B}_b^0 \to (\mathcal{B}_b^0)$ defined by

$$\mathcal{A}(z)(t) = \left\{ h \in \mathcal{B}_b^0 : h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) f(s) ds,\ t \in J \right\}, \tag{5.58}$$

where $f \in L^1(J, E)$ and $f(t) \in F(t, z_t + x_t)$ for a.e. $t \in J$.

As in Theorem 5.2, we can show that the multivalued operator $\mathcal{A}$ is completely continuous, u.s.c. with convex values. It remains to show that the set

$$\mathcal{E} = \left\{ u \in \mathcal{B}_b^0 : \lambda u \in \mathcal{A} u \text{ for some } \lambda > 1 \right\} \tag{5.59}$$

is bounded.

Let $z \in \mathcal{E}$ be any element, then there exists a selection $f \in L^1(J, E)$ and $f(t) \in F(t, z_t + x_t)$ for a.e. $t \in J$ such that

$$\lambda z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) f(s) ds, \quad \text{for some } \lambda > 1. \tag{5.60}$$

Then for each $t \in J$ we have

$$\begin{aligned} |z(t)| &\leq \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| F(s, z_s + x_s) \right\|_{\mathcal{P}(E)} ds \\ &\leq \frac{M b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} + \frac{M \|q\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| z_s + x_s \right\|_{\mathcal{B}} ds. \end{aligned} \tag{5.61}$$

Following the proof of Theorem 3.6, we can show that the set $\mathcal{E}$ is bounded. Consequently, the multivalued operator $\mathcal{A}$ has a fixed point. Then $\mathcal{N}$ has one, witch gives rise to a mild solution of problem (5.46). □

Now we give an existence result for problem (5.46) with nonconvex valued right-hand side by using the fixed point Theorem 2.8.

Theorem 5.6. *Assume that (H23) holds. Then*

$(H26)$ *There exists $l \in L^1(J, \mathbb{R}_+)$ such that*

$$H_d(F(t,x), F(t,y)) \leq l(t) \|x - y\|_{\mathcal{B}} \quad \text{a.e. } t \in J,\ \forall x, y \in \mathcal{B}, \tag{5.62}$$

with

$$d(0, F(t,0)) \leq l(t) \quad \text{for a.e. } t \in J. \tag{5.63}$$

If

$$MK_bI_0^\alpha l(b) < 1, \tag{5.64}$$

then the problem (5.46) *has at least one mild solution on* $(-\infty, b]$.

Proof. As the previous theorem and following steps of the proof of Theorem 5.3. □

6. Perturbed Semilinear Differential Equations and Inclusions

In this section, we will be concerned with semilinear functional differential equations and inclusion of fractional order and where a perturbed term is considered. Our approach is based upon Burton-Kirk fixed point theorem (Theorem 2.11).

First, consider equations of the form

$$\begin{gathered} D^\alpha y(t) = Ay(t) + f(t, y_t) + g(t, y_t), \quad t \in J := [0, b], \\ y(t) = \phi(t), \quad t \in [-r, 0]. \end{gathered} \tag{6.1}$$

Definition 6.1. One says that a continuous function $y : [-r, b] \to E$ is a mild solution of problem (6.1) if $y(t) = \phi(t)$, $t \in [-r, 0]$, and

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) [f(s, y_s) + g(s, y_s)] ds, \quad t \in J. \tag{6.2}$$

Our first main result in this section reads as follows.

Theorem 6.2. *Assume that the following hypotheses hold.*

($H27$) *The semigroup* $\{T(t)\}_{t\in J}$ *is compact for* $t > 0$.

($H28$) *There exist functions* $p, q \in C(J, \mathbb{R}_+)$ *such that*

$$|f(t, u)| \le p(t) + q(t)\|u\|_C, \quad \text{for a.e. } t \in J, \text{ and each } u \in C([-r, 0], E). \tag{6.3}$$

($H29$) *There exists a nonnegative constant* k *such that*

$$|g(t, u) - g(t, v)| \le k\|u - v\|_C, \quad \text{for } t \in J \text{ and every } u, v \in C([-r, 0], E), \tag{6.4}$$

$$\frac{Mkb^\alpha}{\Gamma(\alpha+1)} < 1, \tag{6.5}$$

then the problem (6.1) *has at least one mild solution on* $[-r, b]$.

Proof. Transform the problem (6.1) into a fixed point problem. Consider the two operators

$$F, G : C([-r, b], E) \longrightarrow C([-r, b], E) \tag{6.6}$$

defined by

$$F(y)(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} T(t-s) f(s, y_s) ds, & t \in J, \end{cases}$$
$$G(y)(t) = \begin{cases} 0, & t \in [-r, 0], \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} T(t-s) g(s, y_s) ds, & t \in J. \end{cases} \tag{6.7}$$

Then the problem of finding the solution of IVP (6.1) is reduced to finding the solution of the operator equation $F(y)(t) + G(y)(t) = y(t),\ t \in [-r, b]$. We will show that the operators F and G satisfies all conditions of Theorem 2.11.

From Theorem 3.6, the operator F is completely continuous. We will show that the operator G is a contraction. Let $y, z \in C([-r, b], E)$, then for each $t \in [-r, b]$

$$\begin{aligned} |G(y)(t) - G(z)(t)| &\le \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| g(s, y_s) - g(s, z_s) \right| ds \\ &\le \frac{Mk}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y_s - z_s\|_C ds \\ &\le \frac{Mk}{\Gamma(\alpha)} \|y - z\|_\infty \int_0^t (t-s)^{\alpha-1} ds \\ &\le \frac{Mkb^\alpha}{\alpha\Gamma(\alpha)} \|y - z\|_\infty. \end{aligned} \tag{6.8}$$

Taking the supremum over t,

$$\|G(y) - G(z)\|_\infty \le \frac{Mkb^\alpha}{\Gamma(\alpha+1)} \|y - z\|_\infty, \tag{6.9}$$

which implies by (6.5) that G is a contraction. Now, it remains to show that the set

$$\mathcal{E} = \left\{ y \in C(J, E) : y = \lambda F(y) + \lambda G\left(\frac{y}{\lambda}\right) \text{ for some } 0 < \lambda < 1 \right\} \tag{6.10}$$

is bounded.

Let $y \in \mathcal{E}$ be any element. Then, for each $t \in J$,

$$\begin{aligned} y(t) = \lambda \frac{1}{\Gamma(\alpha)} &\int_0^t (t-s)^{\alpha-1} T(t-s) f(s, y_s) ds \\ &+ \lambda \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) g\left(s, \frac{y_s}{\lambda}\right) ds. \end{aligned} \tag{6.11}$$

Then

$$|y(t)| \leq \lambda M \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f(s, y_s) \right| ds$$
$$+ \lambda M \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| g\left(s, \frac{y_s}{\lambda}\right) - g(s,0) \right| ds$$
$$+ \lambda M \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g(s,0)| ds \tag{6.12}$$
$$\leq M \frac{b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} + M \frac{\|q\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y_s\|_C ds$$
$$+ M \frac{k}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y_s\|_C ds + M \frac{b^\alpha \sigma}{\Gamma(\alpha+1)},$$

where

$$\sigma = \sup_{s \in J} |g(s,0)|. \tag{6.13}$$

We consider the function defined by

$$\mu(t) = \max\{|y(s)| : -r \leq s \leq t\}, \quad t \in J. \tag{6.14}$$

Let $t^* \in [-r, t]$ such that $\mu(t) = |y(t^*)|$. If $t^* \in [0, b]$ then by the previous inequality we have, for $t \in J$, (note $t^* \leq t$)

$$\mu(t) \leq M \frac{b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} + M \frac{b^\alpha \sigma}{\Gamma(\alpha+1)} + M \frac{1}{\Gamma(\alpha)} \left(k + \|q\|_\infty\right) \int_0^t (t-s)^{\alpha-1} \mu(s) ds. \tag{6.15}$$

If $t^* \in [-r, 0]$ then $\mu(t) = \|\phi\|_C$ and the previous inequality holds.

By Lemma 2.7, there exists a constant $K = K(\alpha)$ such that we have

$$\mu(t) \leq M \frac{b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} + M \frac{b^\alpha \sigma}{\Gamma(\alpha+1)}$$
$$+ M \frac{K}{\Gamma(\alpha)} \left(k + \|q\|_\infty\right) \int_0^t (t-s)^{\alpha-1} \left(M \frac{b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} + M \frac{b^\alpha \sigma}{\Gamma(\alpha+1)} \right) ds$$
$$\leq \left[M \frac{b^\alpha}{\Gamma(\alpha+1)} \left(\sigma + \|p\|_\infty\right) \right] \left[1 + M \frac{K b^\alpha}{\Gamma(\alpha+1)} \left(k + \|q\|_\infty\right) \right] \tag{6.16}$$
$$=: \Lambda.$$

Hence,

$$\|y\|_\infty \leq \max\left\{ \|\phi\|_C, \Lambda \right\} \quad \forall y \in \mathcal{E}. \tag{6.17}$$

This shows that the set $\mathcal{E}$ is bounded. as a consequence of the Theorem 2.11, we deduce that the operator $F+G$ has a fixed point which gives rise to a mild solution of the problem (6.1).

Now we consider multivalued functional differential equations of the form

$$\begin{gathered} D^{\alpha} y(t) - Ay(t) \in F(t, y_t) + G(t, y_t), \quad t \in J := [0, b], \\ y(t) = \phi(t), \quad t \in [-r, 0]. \end{gathered} \tag{6.18}$$

□

Definition 6.3. One says that a continuous function $y : [-r, b] \to E$ is a mild solution of problem (6.18) if $y(t) = \phi(t)$, $t \in [-r, 0]$, and there exist $f \in S_{F,y}$ and $g \in S_{G,y}$ such that

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s)[f(s) + g(s)]ds, \quad t \in J. \tag{6.19}$$

Theorem 6.4. *Assume that the following hypotheses hold.*

($H30$) *The semigroup* $\{T(t)\}_{t \in J}$ *is compact for* $t > 0$.

($H31$) *The multifunction* $t \mapsto G(t, x)$ *is measurable, convex valued and integrably bounded for each* $x \in C([-r, 0], E)$.

($H32$) *There exists a function* $k \in C(J, R_+)$ *such that*

$$H_d(G(t, x), G(t, y)) \le k(t) \|x - y\| \quad \text{a.e. } t \in J, \ \forall x, y \in C([-r, 0], E), \tag{6.20}$$

with

$$\begin{gathered} d(0, G(t, 0)) \le k(t) \quad \text{for a.e. } t \in J, \\ M I_0^{\alpha} k(b) < 1. \end{gathered} \tag{6.21}$$

($H33$) $F : J \times C([-r, 0], E) \to \mathcal{P}_{\mathrm{cv,cp}}(E)$ *is Carathéodory.*

($H34$) *There exist functions* $p, q \in C(J, \mathbb{R}_+)$ *such that*

$$\|F(t, u)\|_{\mathcal{P}(E)} \le p(t) + q(t) \|u\|_C, \quad \text{for a.e. } t \in J, \text{ and each } u \in C([-r, 0], E). \tag{6.22}$$

Then IVP (6.18) *has at least one mild solution on* $[-r, b]$.

Proof. Consider the two multivalued operators

$$\mathcal{A} : C([-r, b], E) \longrightarrow \mathcal{P}(C([-r, b], E)) \tag{6.23}$$

defined by $\mathcal{A}(y) := \{h \in C([-r,b],E)\}$ such that

$$h(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0], \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} T(t-s) f(s) ds, & \text{if } t \in J, \end{cases} \tag{6.24}$$

$$\mathcal{B} : C([-r,b],E) \longrightarrow \mathcal{P}(C([-r,b],E))$$

defined by $\mathcal{B}(y) := \{h \in C([-r,b],E)\}$ such that

$$h(t) = \begin{cases} 0, & \text{if } t \in [-r,0], \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} T(t-s) g(s) ds, & \text{if } t \in J, \end{cases} \tag{6.25}$$

where $f \in S_{F,y}$ and $g \in S_{G,y}$. We will show that the operator $\mathcal{B}$ is closed, convex, and bounded valued and it is a contraction. Let $(y_n)_{n\geq 0} \in \mathcal{B}(y)$ such that $y_n \to \tilde{y}$ in $C(J,E)$. Using (H31), we can show that the values of Niemysky operator $S_{G,y}$ are closed in $L^1(J,E)$, and hence $\mathcal{B}(y)$ is closed for each $y \in C(J,E)$.

Now let $h_1, h_2 \in \mathcal{B}(y)$, then there exists $g_1, g_2 \in S_{G,y}$ such that, for each $t \in J$ we have

$$h_i(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) g_i(s) ds, \quad i = 1,2. \tag{6.26}$$

Let $0 \leq \delta \leq 1$. Then, for each $t \in J$, we have

$$(\delta h_1 + (1-\delta) h_2)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) \left[\delta g_1(s) + (1-\delta) g_2(s)\right] ds. \tag{6.27}$$

Since $G(t,y)$ has convex values, one has

$$\delta h_1 + (1-\delta) h_2 \in \mathcal{B}(y), \tag{6.28}$$

and hence $\mathcal{B}(y)$ is convex for each $y \in C(J,E)$.

Let $h \in \mathcal{B}(y)$ be any element. Then, there exists $g \in S_{G,y}$ such that

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) g(s) ds. \tag{6.29}$$

By (H31), we have for all $t \in J$

$$\begin{aligned} |h(t)| &\leq \frac{M}{\Gamma(\alpha)} \int_0^b (t-s)^{\alpha-1} \omega(s) ds \\ &= M I_0^\alpha \omega(b), \end{aligned} \tag{6.30}$$

where ω is from Definition 2.4. Then $\|h\|_\infty \leq MI_0^\alpha \omega(b)$ for all $h \in \mathcal{B}(y)$. Hence $\mathcal{B}(y)$ is a bounded subset of $C(J, E)$.

As in Theorem 5.3, we can easily show that the multivalued operator $\mathcal{B}$ is a contraction. Now, as in Theorem 5.2 we can show that the operator $\mathcal{A}$ satisfies all the conditions of Theorem 2.12.

It remains to show that the set

$$\mathcal{E} = \{y \in C(J, E) \mid y \in \lambda\mathcal{A}y + \lambda\mathcal{B}y,\ 0 < \lambda < 1\} \tag{6.31}$$

is bounded.

Let $y \in \mathcal{E}$ be any element. Then there exists $f \in S_{F,y}$ and $g \in S_{G,y}$ such that for each $t \in J$,

$$\begin{aligned} y(t) = {} & \lambda \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) f(s) ds \\ & + \lambda \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) g(s) ds. \end{aligned} \tag{6.32}$$

Then

$$\begin{aligned} |y(t)| \leq {} & M \frac{b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} + M \frac{\|q\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y_s\|_C ds \\ & + M \frac{\|k\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y_s\|_C ds + M \frac{b^\alpha \Delta}{\Gamma(\alpha+1)}, \end{aligned} \tag{6.33}$$

where

$$\Delta = \sup_{s \in J} |G(s, 0)|. \tag{6.34}$$

We consider the function defined by

$$\mu(t) = \max\{|y(s)| : -r \leq s \leq t\}, \quad t \in J. \tag{6.35}$$

Let $t^* \in [-r, t]$ such that $\mu(t) = |y(t^*)|$. If $t^* \in [0, b]$ then by the previous inequality we have, for $t \in J$, (note $t^* \leq t$)

$$\begin{aligned} \mu(t) \leq {} & M \frac{b^\alpha \|p\|_\infty}{\Gamma(\alpha+1)} + M \frac{b^\alpha \Delta}{\Gamma(\alpha+1)} \\ & + M \frac{1}{\Gamma(\alpha)} (\|k\|_\infty + \|q\|_\infty) \int_0^t (t-s)^{\alpha-1} \mu(s) ds. \end{aligned} \tag{6.36}$$

If $t^* \in [-r, 0]$ then $\mu(t) = \|\phi\|_C$ and the previous inequality holds.

By Lemma 2.7, there exists a constant $K = K(\alpha)$ such that we have

$$\mu(t) \leq \left[M\frac{b^\alpha}{\Gamma(\alpha+1)}(\|p\|_\infty + \Delta)\right]\left[1 + M\frac{Kb^\alpha}{\Gamma(\alpha+1)}(\|k\|_\infty + \|q\|_\infty)\right] =: \Lambda. \tag{6.37}$$

Hence

$$\|y\|_\infty \leq \max\left\{\|\phi\|_C, \Lambda\right\} \quad \forall y \in \mathcal{E}. \tag{6.38}$$

This shows that the set $\mathcal{E}$ is bounded. As a result, the conclusion (ii) of Theorem 2.12 does not hold. Hence, the conclusion (i) holds and consequently $\mathcal{A}(y) + \mathcal{B}(y)$ has a fixed point which is a mild solution of problem (6.18). □

7. Some Existence Results in Ordered Banach Spaces

In this section, we present some existence results in ordered Banach spaces using the method of upper and lower mild solutions. Before stating our main results let us introduce some preliminaries.

Definition 7.1. A nonempty closed subset C of a Banach space X is said to be a cone if

(i) $C + C \subset C$,

(ii) $\lambda C \subset C$ for $\lambda > 0$,

(iii) $\{-C\} \cap \{C\} = \{0\}$.

A cone C is called normal if the norm $\|\cdot\|$ is semimonotone on C, that is, there exists a constant $N > 0$ such that $\|x\| \leq N\|y\|$, whenever $x \leq y$. We equip the space $X = C(J, E)$ with the order relation $\leq$ induced by a regular cone C in E, that is for all $y, \overline{y} \in X : y \leq \overline{y}$ if and only if $\overline{y}(t) - y(t) \in C$ for all $t \in J$. In what follows will assume that the cone C is normal. Cones and their properties are detailed in [65, 66]. Let $a, b \in X$ be such that $a \leq b$. Then, by an order interval $[a, b]$ we mean a set of points in X given by

$$[a, b] = \{x \in X \mid a \leq x \leq b\}. \tag{7.1}$$

Definition 7.2. Let X be an ordered Banach space. A mapping $T : X \to X$ is called increasing if $T(x) \leq T(y)$ for any $x, y \in X$ with $x \leq y$. Similarly, T is called decreasing if $T(x) \geq T(y)$ whenever $x \leq y$.

Definition 7.3. A function $f(t, x)$ is called increasing in x for $t \in J$, if $f(t, x) \leq f(t, y)$ for each $t \in J$ for all $x, y \in X$ with $x \leq y$. Similarly $f(t, x)$ is called decreasing in x for $t \in J$, if $f(t, x) \geq f(t, y)$ for each $t \in J$ for all $x, y \in E$ with $x \leq y$.

Now suppose that E is an ordered Banach space and reconsider the initial value problem (3.1) and (3.2) with the same data.

Definition 7.4. One says that a continuous function $v : [-r,b] \to E$ is a lower mild solution of problem (3.1) and (3.2) if $v(t) = \phi(t),\ t \in [-r,0]$, and

$$v(t) \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)f(s,v_s)ds, \quad t \in J. \tag{7.2}$$

Similarly an upper mild solution w of IVP (3.1) and (3.2) is defined by reversing the order.

The following fixed point theorem is crucial for our existence result.

Theorem 7.5 (see [66]). *Let K be a normal cone in a partially ordered Banach space X. Let F be increasing on the interval $[a,b]$ and transform $[a,b]$ into itself, that is, $F(a) \ge a$ and $F(b) \le b$. Assume further that F is continuous and completely continuous. Then F has at least one fixed point $x \in (a,b)$.*

Our first main result reads as follows .

Theorem 7.6. *Assume that assumptions (H2)-(H3) hold. Assume moreover that*

$(H35)$ *The function $f(t,u)$ is increasing in u for each $t \in J$.*

$(H36)$ *$T(t)$ is order-preserving, that is, $T(t)(v) \ge 0$ whenever $v \ge 0$.*

$(H37)$ *The IVP (3.1) and (3.2) has a lower mild solution v and an upper mild solution w with $v \le w$.*

Then IVP (3.1) and (3.2) has at least one mild solution y on $[-r,b]$ with $v \le y \le w$.

Proof. It can be shown, as in the proof of Theorem 3.2, that F is continuous and completely continuous on $[v,w]$. We will show that F is increasing on $[v,w]$. Let $y,\overline{y} \in [a,b]$ be such that $y \le \overline{y}$. Then by (H35),(H36), we have for each $t \in J$

$$\begin{aligned} F(y)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)f(s,y_s)ds \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(t-s)f(s,\overline{y}_s)ds \\ &= F(\overline{y})(t). \end{aligned} \tag{7.3}$$

Therefore F is increasing on $[v,w]$. Finally, let $y \in [v,w]$ be any element. By (H37), we deduce that

$$v \le F(v) \le F(y) \le F(w) \le w, \tag{7.4}$$

which shows that $F(y) \in [v,w]$ for all $y \in [v,w]$. Thus, the functions F satisfies all conditions of Theorem 7.5, and hence IVP (3.1) and (3.2) has a mild solution on $[-r,b]$ belonging to the interval $[v,w]$. □

Now reconsider the perturbed initial value problem (6.1). To state our second main result in this section we use the following fixed point theorem due to Dhage and Henderson.

Theorem 7.7 (see [67]). *Let $[a,b]$ be an order interval in a Banach space and let $B_1, B_2 : [a,b] \to X$ be two functions satisfying*

(a) *B_1 is a contraction,*

(b) *B_2 is completely continuous,*

(c) *B_1 and B_2 are strictly monotone increasing,*

(d) $B_1(x) + B_2(x) \in [a,b]$, $\forall x \in [a,b]$.

Further if the cone K in X is normal, then the equation $x = B_1(x) + B_2(x)$ has at least fixed point x_* and a greatest fixed point $x^* \in [a,b]$. Moreover $x_* = \lim_{n\to\infty} x_n$ and $x^* = \lim_{n\to\infty} y_n$, where $\{x_n\}$ and $\{y_n\}$ are the sequences in $[a,b]$ defined by

$$x_{n+1} = B_1(x_n) + B_2(x_n), \qquad x_0 = a, \qquad y_{n+1} = B_1(y_n) + B_2(y_n), \qquad y_0 = b. \tag{7.5}$$

We need the following definitions in the sequel.

Definition 7.8. One says that a continuous function $v : [-r,b] \to E$ is a lower mild solution of problem (6.1) $y(t) \le \phi(t)$, $t \in [-r,0]$, and

$$y(t) \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s)\left[f(s,y_s) + g(s,y_s)\right] ds, \quad t \in J. \tag{7.6}$$

Similarly an upper mild solution w of IVP (6.1) is defined by reversing the order.

Theorem 7.9. *Assume that assumptions (H27)–(H29) hold. Suppose moreover that*

$(H38)$ *The functions $f(t,y)$ and $g(t,y)$ are increasing in y for each $t \in J$.*

$(H39)$ *$T(t)$ is order-preserving, that is, $T(t)(v) \ge 0$ whenever $v \ge 0$.*

$(H40)$ *The IVP* (6.1) *has a lower mild solution v and an upper mild solution w with $v \le w$.*

Then IVP (6.1) *has a minimal and a maximal mild solutions on $[-r,b]$.*

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