

Research Article

Existence Results for Nonlinear Boundary Value Problems of Fractional Integro-differential Equations with Integral Boundary Conditions

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This paper deals with some existence results for a boundary value problem involving a nonlinear integro-differential equation of fractional order $q \in (1, 2]$ with integral boundary conditions. Our results are based on contraction mapping principle and Krasnosel'skiĭ's fixed point theorem.

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1. Introduction

In the last few decades, fractional-order models are found to be more adequate than integer-order models for some real world problems. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional differential equations in comparison with classical integer-order models. Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, and so forth, involves derivatives of fractional order. In consequence, the subject of fractional differential equations is gaining much importance and attention. For examples and details, see [1–22] and the references therein. However, the theory of boundary value problems for nonlinear fractional differential equations is still in the initial stages and many aspects of this theory need to be explored.

Integro-differential equations arise in many engineering and scientific disciplines, often as approximation to partial differential equations, which represent much of the continuum

phenomena. Many forms of these equations are possible. Some of the applications are unsteady aerodynamics and aero elastic phenomena, visco elasticity, visco elastic panel in super sonic gas flow, fluid dynamics, electrodynamics of complex medium, many models of population growth, polymer rheology, neural network modeling, sandwich system identification, materials with fading memory, mathematical modeling of the diffusion of discrete particles in a turbulent fluid, heat conduction in materials with memory, theory of lossless transmission lines, theory of population dynamics, compartmental systems, nuclear reactors, and mathematical modeling of a hereditary phenomena. For details, see [23–29] and the references therein.

Integral boundary conditions have various applications in applied fields such as blood flow problems, chemical engineering, thermoelasticity, underground water flow, population dynamics, and so forth. For a detailed description of the integral boundary conditions, we refer the reader to a recent paper [30]. For more details of nonlocal and integral boundary conditions, see [31–37] and references therein.

In this paper, we consider the following boundary value problem for a nonlinear fractional integrodifferential equation with integral boundary conditions

$$\begin{aligned} {}^c D^q x(t) &= f(t, x(t), (\chi x)(t)), \quad 0 < t < 1, \quad 1 < q \leq 2, \\ \alpha x(0) + \beta x'(0) &= \int_0^1 q_1(x(s)) ds, \quad \alpha x(1) + \beta x'(1) = \int_0^1 q_2(x(s)) ds, \end{aligned} \quad (1.1)$$

where ${}^c D$ is the Caputo fractional derivative, $f : [0, 1] \times X \times X \rightarrow X$, for $\gamma : [0, 1] \times [0, 1] \rightarrow [0, \infty)$,

$$(\chi x)(t) = \int_0^t \gamma(t, s)x(s) ds, \quad (1.2)$$

$q_1, q_2 : X \rightarrow X$ and $\alpha > 0$, $\beta \geq 0$ are real numbers. Here, $(X, \|\cdot\|)$ is a Banach space and $C = C([0, 1], X)$ denotes the Banach space of all continuous functions from $[0, 1] \rightarrow X$ endowed with a topology of uniform convergence with the norm denoted by $\|\cdot\|_C$.

2. Preliminaries

First of all, we recall some basic definitions [15, 18, 20].

Definition 2.1. For a function $f : [0, \infty) \rightarrow \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^c D^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s) ds, \quad n-1 < q < n, \quad n = [q] + 1, \quad (2.1)$$

where $[q]$ denotes the integer part of the real number q .

Definition 2.2. The Riemann-Liouville fractional integral of order q is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds, \quad q > 0, \quad (2.2)$$

provided the integral exists.

Definition 2.3. The Riemann-Liouville fractional derivative of order q for a function $f(t)$ is defined by

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt} \right)^n \int_0^t \frac{f(s)}{(t-s)^{q-n+1}} ds, \quad n = [q] + 1, \quad (2.3)$$

provided the right hand side is pointwise defined on $(0, \infty)$.

In passing, we remark that the definition of Riemann-Liouville fractional derivative, which did certainly play an important role in the development of theory of fractional derivatives and integrals, could hardly produce the physical interpretation of the initial conditions required for the initial value problems involving fractional differential equations. The same applies to the boundary value problems of fractional differential equations. It was Caputo definition of fractional derivative which solved this problem. In fact, the Caputo derivative becomes the conventional n th derivative of the function $f(t)$ as $q \rightarrow n$ and the initial conditions for fractional differential equations retain the same form as that of ordinary differential equations with integer derivatives. Another difference is that the Caputo derivative for a constant is zero while the Riemann-Liouville fractional derivative of a constant is nonzero. For more details, see [20].

Lemma 2.4 (see [22]). *For $q > 0$, the general solution of the fractional differential equation ${}^c D^q x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \quad (2.4)$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$ ($n = [q] + 1$).

In view of Lemma 2.4, it follows that

$$I^q {}^c D^q x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \quad (2.5)$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$ ($n = [q] + 1$).

Now, we state a known result due to Krasnosel'skiĭ [38] which is needed to prove the existence of at least one solution of (1.1).

Theorem 2.5. *Let M be a closed convex and nonempty subset of a Banach space X . Let A, B be the operators such that (i) $Ax + By \in M$ whenever $x, y \in M$, (ii) A is compact and continuous, (iii) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Lemma 2.6. For any $\zeta, \eta_1, \eta_2 \in C[0, 1]$, the unique solution of the boundary value problem

$$\begin{aligned} {}^c D^q x(t) &= \zeta(t), \quad 0 < t < 1, \quad 1 < q \leq 2, \\ \alpha x(0) + \beta x'(0) &= \int_0^1 \eta_1(s) ds, \quad \alpha x(1) + \beta x'(1) = \int_0^1 \eta_2(s) ds, \end{aligned} \quad (2.6)$$

is given by

$$x(t) = \int_0^1 G(t, s) \zeta(s) ds + \frac{1}{\alpha^2} \left[(\alpha(1-t) + \beta) \int_0^1 \eta_1(s) ds + (\beta + \alpha t) \int_0^1 \eta_2(s) ds \right], \quad (2.7)$$

where $G(t, s)$ is the Green's function given by

$$G(t, s) = \begin{cases} \frac{\alpha(t-s)^{q-1} + (\beta - \alpha t)(1-s)^{q-1}}{\alpha\Gamma(q)} + \frac{\beta(\beta - \alpha t)(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}, & s \leq t, \\ \frac{(\beta - \alpha t)(1-s)^{q-1}}{\alpha\Gamma(q)} + \frac{\beta(\beta - \alpha t)(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}, & t \leq s. \end{cases} \quad (2.8)$$

Proof. Using (2.5), for some constants $b_1, b_2 \in \mathbb{R}$, we have

$$x(t) = I^q \zeta(t) - b_1 - b_2 t = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \zeta(s) ds - b_1 - b_2 t. \quad (2.9)$$

In view of the relations ${}^c D^q I^q x(t) = x(t)$ and $I^q I^p x(t) = I^{q+p} x(t)$ for $q, p > 0$, $x \in L(0, 1)$, we obtain

$$x'(t) = \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} \zeta(s) ds - b_2. \quad (2.10)$$

Applying the boundary conditions for (2.6), we find that

$$\begin{aligned} b_1 &= \frac{1}{\alpha^2} \left[\beta \int_0^1 \eta_2(s) ds - (\beta + \alpha) \int_0^1 \eta_1(s) ds \right] - \frac{\beta}{\alpha\Gamma(q)} \int_0^1 (1-s)^{q-1} \zeta(s) ds \\ &\quad - \frac{\beta^2}{\alpha^2\Gamma(q-1)} \int_0^1 (1-s)^{q-2} \zeta(s) ds, \\ b_2 &= \frac{1}{\alpha} \left[\int_0^1 \eta_1(s) ds - \int_0^1 \eta_2(s) ds \right] + \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} \zeta(s) ds \\ &\quad + \frac{\beta}{\alpha\Gamma(q-1)} \int_0^1 (1-s)^{q-2} \zeta(s) ds. \end{aligned} \quad (2.11)$$

Thus, the unique solution of (2.6) is

$$\begin{aligned}
 x(t) &= \int_0^t \left[\frac{\alpha(t-s)^{q-1} + (\beta - \alpha t)(1-s)^{q-1}}{\alpha\Gamma(q)} + \frac{\beta(\beta - \alpha t)(1-s)^{q-2}}{\alpha^2\Gamma(q-1)} \right] \zeta(s) ds \\
 &\quad + \int_t^1 \left[\frac{(\beta - \alpha t)(1-s)^{q-1}}{\alpha\Gamma(q)} + \frac{\beta(\beta - \alpha t)(1-s)^{q-2}}{\alpha^2\Gamma(q-1)} \right] \zeta(s) ds \\
 &\quad + \frac{1}{\alpha^2} \left[(\alpha(1-t) + \beta) \int_0^1 \eta_1(s) ds + (\beta + \alpha t) \int_0^1 \eta_2(s) ds \right] \\
 &= \int_0^1 G(t,s) \zeta(s) ds + \frac{1}{\alpha^2} \left[(\alpha(1-t) + \beta) \int_0^1 \eta_1(s) ds + (\beta + \alpha t) \int_0^1 \eta_2(s) ds \right],
 \end{aligned} \tag{2.12}$$

where $G(t,s)$ is given by (2.8). This completes the proof. \square

3. Main Results

Theorem 3.1. Assume that $f : [0, 1] \times X \times X \rightarrow X$ is jointly continuous and maps bounded subsets of $[0, 1] \times X \times X$ into relatively compact subsets of X , $\gamma : [0, 1] \times [0, 1] \rightarrow [0, \infty)$ is continuous with $\gamma_0 = \max\{\gamma(t,s) : (t,s) \in [0, 1] \times [0, 1]\}$, and $q_1, q_2 : X \rightarrow X$ are continuous functions. Further, there exist positive constants $L_1, \bar{L}_1, L_2, L_3, M_2, M_3$ such that

$$(A_1) \quad \|f(t, x(t), (\chi x)(t)) - f(t, y(t), (\chi y)(t))\| \leq L_1 \|x - y\| + \bar{L}_1 \|\chi x - \chi y\|, \text{ for all } t \in [0, 1],$$

$$x, y \in X,$$

$$(A_2) \quad \|q_1(x) - q_1(y)\| \leq L_2 \|x - y\|, \|q_2(x) - q_2(y)\| \leq L_3 \|x - y\| \text{ with } \|q_1(x)\| \leq M_2, \|q_2(x)\| \leq M_3, \text{ for all } x, y \in X.$$

Then the boundary value problem (1.1) has a unique solution provided

$$(L_1 + \gamma_0 \bar{L}_1) \left[\frac{\beta + 2\alpha}{\alpha\Gamma(q+1)} + \frac{\beta^2 + \alpha\beta}{\alpha^2\Gamma(q)} \right] + \frac{\beta + \alpha}{\alpha^2} (L_2 + L_3) < 1 \tag{3.1}$$

with

$$L_1 + \gamma_0 \bar{L}_1 \leq \frac{1}{2} \left[\frac{\beta + 2\alpha}{\alpha\Gamma(q+1)} + \frac{\beta^2 + \alpha\beta}{\alpha^2\Gamma(q)} \right]^{-1}. \tag{3.2}$$

Proof. Define $F : C \rightarrow C$ by

$$\begin{aligned}
 (Fx)(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s), (\chi x)(s)) ds \\
 &\quad + \int_0^1 \left[\frac{(\beta - \alpha t)(1-s)^{q-1}}{\alpha\Gamma(q)} + \frac{\beta(\beta - \alpha t)(1-s)^{q-2}}{\alpha^2\Gamma(q-1)} \right] f(s, x(s), (\chi x)(s)) ds \\
 &\quad + \frac{1}{\alpha^2} \left[(\alpha(1-t) + \beta) \int_0^1 q_1(x(s)) ds + (\beta + \alpha t) \int_0^1 q_2(x(s)) ds \right], \quad t \in [0, 1].
 \end{aligned} \tag{3.3}$$

Setting $\sup_{t \in [0,1]} \|f(t, 0, 0)\| = M_1$ (by the assumption on f) and Choosing

$$r \geq 2 \left[M_1 \left(\frac{\beta + 2\alpha}{\alpha\Gamma(q+1)} + \frac{\beta^2 + \alpha\beta}{\alpha^2\Gamma(q)} \right) + \frac{\alpha + \beta}{\alpha^2} (M_2 + M_3) \right], \quad (3.4)$$

we show that $F B_r \subset B_r$, where $B_r = \{x \in C : \|x\| \leq r\}$. For $x \in B_r$, we have

$$\begin{aligned} \|(F x)(t)\| &\leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|f(s, x(s), (\chi x)(s))\| ds \\ &\quad + \int_0^1 |\beta - \alpha t| \left[\frac{(1-s)^{q-1}}{\alpha\Gamma(q)} + \frac{\beta(1-s)^{q-2}}{\alpha^2\Gamma(q-1)} \right] \|f(s, x(s), (\chi x)(s))\| ds \\ &\quad + \frac{\alpha + \beta}{\alpha^2} (M_2 + M_3) \\ &\leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [\|f(s, x(s), (\chi x)(s)) - f(s, 0, 0)\| + \|f(s, 0, 0)\|] ds \\ &\quad + \int_0^1 |\beta - \alpha t| \left[\frac{(1-s)^{q-1}}{\alpha\Gamma(q)} + \frac{\beta(1-s)^{q-2}}{\alpha^2\Gamma(q-1)} \right] \\ &\quad \times [\|f(s, x(s), (\chi x)(s)) - f(s, 0, 0)\| + \|f(s, 0, 0)\|] ds + \frac{\alpha + \beta}{\alpha^2} (M_2 + M_3) \\ &\leq ((L_1 + \gamma_0 \bar{L}_1)r + M_1) \left[\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds \right. \\ &\quad \left. + \int_0^1 |\beta - \alpha t| \left(\frac{(1-s)^{q-1}}{\alpha\Gamma(q)} + \frac{\beta(1-s)^{q-2}}{\alpha^2\Gamma(q-1)} \right) ds \right] \\ &\quad + \frac{\alpha + \beta}{\alpha^2} (M_2 + M_3) \\ &= ((L_1 + \gamma_0 \bar{L}_1)r + M_1) \left[\frac{t^q}{\Gamma(q+1)} + |\beta - \alpha t| \left(\frac{1}{\alpha\Gamma(q+1)} + \frac{\beta}{\alpha^2\Gamma(q)} \right) \right] \\ &\quad + \frac{\alpha + \beta}{\alpha^2} (M_2 + M_3) \\ &\leq (L_1 + \gamma_0 \bar{L}_1) \left[\frac{2\alpha + \beta}{\alpha\Gamma(q+1)} + \frac{\beta^2 + \alpha\beta}{\alpha^2\Gamma(q)} \right] r \\ &\quad + M_1 \left[\frac{2\alpha + \beta}{\alpha\Gamma(q+1)} + \frac{\beta^2 + \alpha\beta}{\alpha^2\Gamma(q)} \right] + \frac{\alpha + \beta}{\alpha^2} (M_2 + M_3) \leq r. \end{aligned} \quad (3.5)$$

Now, for $x, y \in C$ and for each $t \in [0, 1]$, we obtain

$$\begin{aligned}
& \|(Fx)(t) - (Fy)(t)\| \\
& \leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|f(s, x(s), (\chi x)(s)) - f(s, y(s), (\chi y)(s))\| ds \\
& \quad + \int_0^1 |\beta - \alpha t| \left[\frac{(1-s)^{q-1}}{\alpha \Gamma(q)} + \frac{\beta(1-s)^{q-2}}{\alpha^2 \Gamma(q-1)} \right] \|f(s, x(s), (\chi x)(s)) - f(s, y(s), (\chi y)(s))\| ds \\
& \quad + \frac{\alpha + \beta}{\alpha^2} \left[\int_0^1 \|q_1(x(s)) - q_1(y(s))\| ds + \int_0^1 \|q_2(x(s)) - q_2(y(s))\| ds \right] \\
& \leq (L_1 + \gamma_0 \bar{L}_1) \|x - y\|_C \left[\frac{t^q}{\Gamma(q+1)} + |\beta - \alpha t| \left(\frac{1}{\alpha \Gamma(q+1)} + \frac{\beta}{\alpha^2 \Gamma(q)} \right) \right] \\
& \quad + \frac{\alpha + \beta}{\alpha^2} (L_2 + L_3) \|x - y\|_C \\
& \leq \left\{ (L_1 + \gamma_0 \bar{L}_1) \left[\frac{2\alpha + \beta}{\alpha \Gamma(q+1)} + \frac{\beta^2 + \alpha\beta}{\alpha^2 \Gamma(q)} \right] + \frac{\alpha + \beta}{\alpha^2} (L_2 + L_3) \right\} \|x - y\|_C \\
& \leq \Lambda_{\alpha, \beta, q, \gamma_0, L_1, \bar{L}_1, L_2, L_3} \|x - y\|_C,
\end{aligned} \tag{3.6}$$

where

$$\Lambda_{\alpha, \beta, q, \gamma_0, L_1, \bar{L}_1, L_2, L_3} = (L_1 + \gamma_0 \bar{L}_1) \left[\frac{2\alpha + \beta}{\alpha \Gamma(q+1)} + \frac{\beta^2 + \alpha\beta}{\alpha^2 \Gamma(q)} \right] + \frac{\alpha + \beta}{\alpha^2} (L_2 + L_3), \tag{3.7}$$

which depends only on the parameters involved in the problem. As $\Lambda_{\alpha, \beta, q, L_1, \bar{L}_1, L_2, L_3} < 1$, therefore F is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle. \square

Theorem 3.2. Assume that (A_1) - (A_2) hold with $\|f(t, x(t), (\chi x)(t))\| \leq \mu(t)$, for all $(t, x, \chi x) \in [0, 1] \times X \times X$, where $\mu \in L^1([0, 1], \mathbb{R}^+)$ and

$$(L_1 + \gamma_0 \bar{L}_1) \left(\frac{\alpha + \beta}{\alpha \Gamma(q+1)} + \frac{\beta^2 + \alpha\beta}{\alpha^2 \Gamma(q)} \right) + \frac{\alpha + \beta}{\alpha^2} (L_2 + L_3) < 1. \tag{3.8}$$

Then the boundary value problem (1.1) has at least one solution on $[0, 1]$.

Proof. Let us fix

$$r \geq \|\mu\|_{L^1} \left[\frac{2\alpha + \beta}{\alpha \Gamma(q+1)} + \frac{\beta^2 + \alpha\beta}{\alpha^2 \Gamma(q)} \right] + \frac{\alpha + \beta}{\alpha^2} (M_2 + M_3), \tag{3.9}$$

and consider $B_r = \{x \in C : \|x\| \leq r\}$. We define the operators Φ and Ψ on B_r as

$$\begin{aligned} (\Phi x)(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s), (\chi x)(s)) ds, \\ (\Psi x)(t) &= \int_0^1 \left[\frac{(\beta - \alpha t)(1-s)^{q-1}}{\alpha \Gamma(q)} + \frac{\beta(\beta - \alpha t)(1-s)^{q-2}}{\alpha^2 \Gamma(q-1)} \right] f(s, x(s), (\chi x)(s)) ds \\ &\quad + \frac{1}{\alpha^2} \left[(\alpha(1-t) + \beta) \int_0^1 q_1(x(s)) ds + (\beta + \alpha t) \int_0^1 q_2(x(s)) ds \right]. \end{aligned} \quad (3.10)$$

For $x, y \in B_r$, we find that

$$\|\Phi x + \Psi y\| \leq \|\mu\|_{L^1} \left[\frac{2\alpha + \beta}{\alpha \Gamma(q+1)} + \frac{\beta^2 + \alpha\beta}{\alpha^2 \Gamma(q)} \right] + \frac{\alpha + \beta}{\alpha^2} (M_2 + M_3) \leq r. \quad (3.11)$$

Thus, $\Phi x + \Psi y \in B_r$. It follows from the assumption (A_1) , (A_2) that Ψ is a contraction mapping for

$$(L_1 + \gamma_0 \bar{L}_1) \left(\frac{\alpha + \beta}{\alpha \Gamma(q+1)} + \frac{\beta^2 + \alpha\beta}{\alpha^2 \Gamma(q)} \right) + \frac{\alpha + \beta}{\alpha^2} (L_2 + L_3) < 1. \quad (3.12)$$

Continuity of f implies that the operator Φ is continuous. Also, Φ is uniformly bounded on B_r as

$$\|\Phi x\| \leq \frac{\|\mu\|_{L^1}}{\Gamma(q+1)}. \quad (3.13)$$

Now we prove the compactness of the operator Φ . In view of (A_1) , we define $\sup_{(t,x,\chi x) \in \Omega} \|f(s, x(s), (\chi x)(s))\| = f_{\max}$, $\Omega = [0, 1] \times B_r \times B_r$, and consequently we have

$$\begin{aligned} &\|(\Phi x)(t_1) - (\Phi x)(t_2)\| \\ &= \left\| \frac{1}{\Gamma(q)} \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] f(s, x(s), (\chi x)(s)) ds \right. \\ &\quad \left. + \int_{t_1}^{t_2} (t_2 - s)^{q-1} f(s, x(s)) ds \right\| \\ &\leq \frac{f_{\max}}{\Gamma(q+1)} |2(t_2 - t_1)^q + t_1^q - t_2^q|, \end{aligned} \quad (3.14)$$

which is independent of x . So Φ is relatively compact on B_r . Hence, By Arzela Ascoli Theorem, Φ is compact on B_r . Thus all the assumptions of Theorem 2.5 are satisfied and

the conclusion of Theorem 2.5 implies that the boundary value problem (1.1) has at least one solution on $[0, 1]$. \square

Example 3.3. Consider the following boundary value problem:

$$\begin{aligned} {}^c D^q x(t) &= \frac{1}{(t+7)^2} \frac{|x|}{1+|x|} + \int_0^t \frac{e^{-(s-t)}}{49} x(s) ds, \quad t \in [0, 1], \quad 1 < q \leq 2, \\ x(0) + x'(0) &= \int_0^1 \frac{|x(s)|}{5+|x(s)|} ds, \quad x(1) + x'(1) = \int_0^1 \frac{|x(s)|}{7+|x(s)|} ds. \end{aligned} \quad (3.15)$$

Here, $f(t, x) = (1/(t+7)^2)(|x|/(1+|x|))$, $\gamma(t, s) = e^{-(s-t)}/49$, $q_1(x) = |x|/(5+|x|)$, $q_2(x) = |x|/(7+|x|)$, $\alpha = 1$, $\beta = 1$. As $\|f(t, x, \chi x) - f(t, y, \chi y)\| \leq (1/49)\|x - y\| + \|\chi x - \chi y\|$, $\|q_1(x) - q_1(y)\| \leq (1/5)\|x - y\|$, $\|q_2(x) - q_2(y)\| \leq (1/7)\|x - y\|$, therefore, (A_1) and (A_2) are satisfied with $L_1 = 1/49$, $\bar{L}_1 = 1$, $\gamma_0 = ((e-1)/49)L_2 = 1/5$, $L_3 = 1/7$. Further,

$$(L_1 + \gamma_0 \bar{L}_1) \left[\frac{\beta + 2\alpha}{\alpha \Gamma(q+1)} + \frac{\beta^2 + \alpha\beta}{\alpha^2 \Gamma(q)} \right] + \frac{\beta + \alpha}{\alpha^2} (L_2 + L_3) < 1 \iff \frac{e}{49} \left(\frac{3}{\Gamma(q+1)} + \frac{2}{\Gamma(q)} \right) < \frac{11}{35}. \quad (3.16)$$

Thus, by Theorem 3.1, the boundary value problem (3.15) has a unique solution on $[0, 1]$.

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