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*Research Article*

# Approximate Fixed Points for Nonexpansive and Quasi-Nonexpansive Mappings in Hyperspaces

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This paper provides a few convergence results of the Ishikawa iteration sequence with errors for nonexpansive and quasi-nonexpansive mappings in hyperspaces. The results presented in this paper improve and generalize some results in the literature.



## 1. Introduction and Preliminaries

Browder [1] and Kirk [2] established that a nonexpansive mapping $T$ which maps a closed bounded convex subset $C$ of a uniformly convex Banach space into itself has a fixed point in $C$. Since then, many researchers have studied, under various conditions, the convergence of the Mann and Ishikawa iteration methods dealing with nonexpansive and quasi-nonexpansive mappings (see [3–11] and the references therein). Rhoades [9] pointed out that the Picard iteration schemes for nonexpansive mappings need not converge. Senter and Dotson [10] obtained conditions under which the Mann iteration schemes generated by nonexpansive and quasi-nonexpansiv mappings in uniformly convex Banach spaces, converge to fixed points of these mappings, respectively. Ishikawa [7] established that the Mann iteration methods can be used to approximate fixed points of nonexpansive mappings in Banach spaces. Deng [3] obtained similar results for Ishikawa iteration processes in normed linear spaces and Banach spaces.

Our aim is to prove several convergence theorems of the Ishikawa iteration sequence with errors for nonexpansive and quasi-nonexpansive mappings in hyperspaces. Our results presented in this paper extend substantially the results due to Deng [3], Ishikawa [7], and Senter and Dotson [10].

Assume that $X$ is a nonempty subset of a normed linear space $(E, \|\cdot\|)$ and $CC(X)$ denotes the family of all nonempty convex compact subsets of $X$, and $H$ is the Hausdorff metric induced by the norm $\|\cdot\|$. For $x \in E$, $X \subset E$, $A, B \in CC(X)$, $\Im \subseteq CC(X)$, $T : (\Im, H) \to (CC(X), H)$, and $t \in R = (-\infty, \infty)$, let

$$\begin{gathered} d(x,, A) = \inf\{\|x-a\| : a \in A\}, \qquad D(A, \Im) = \inf\{H(A, C) : C \in \Im\}, \\ \Im_X = \{\{x\} : x \in X\}, \qquad A + B = \{a + b : a \in A,\ b \in B\}, \qquad tA = \{ta : a \in A\}, \\ co(\Im) = \left\{ \sum_{i=1}^{n} t_i A_i : t_i \ge 0, \sum_{i=1}^{n} t_i = 1,\ A_i \in \Im,\ n \ge 1 \right\}, \qquad F(T) = \{A \in \Im : TA = A\}. \end{gathered} \tag{1.1}$$

It is easy to see that $tA + (1-t)A = A$ and $tA + (1-t)B \in CC(E)$ for all $t \in [0,1]$ and $A, B \in CC(E)$. Hence $CC(E)$ is convex. Hu and Huang [12] proved that if $(E, \|\cdot\|)$ is a Banach space, then $(CC(X), H)$ is a complete metric space. Now we introduce the following concepts in hyperspaces.

*Definition 1.1.* Let $\Im$ be a nonempty subset of $CC(E)$ and let $T : (\Im, H) \to (CC(E), H)$ be a mapping. Assume that $\{t_n\}_{n\ge0}$, $\{t'_n\}_{n\ge0}$, $\{s_n\}_{n\ge0}$, and $\{s'_n\}_{n\ge0}$ are arbitrary real sequences in $[0,1]$ satisfying $t_n + t'_n \le 1$ and $s_n + s'_n \le 1$ for $n \ge 1$ and $\{P_n\}_{n\ge0}$ and $\{Q_n\}_{n\ge0}$ are any bounded sequences of the elements in $CC(E)$.

(i) For $A_0 \in \Im$, the sequence $\{A_n\}_{n\ge0}$ defined by

$$\begin{gathered} B_n = \left(1 - s_n - s'_n\right) A_n + s_n T A_n + s'_n P_n, \\ A_{n+1} = \left(1 - t_n - t'_n\right) A_n + t_n T B_n + t'_n Q_n, \quad n \ge 0 \end{gathered} \tag{1.2}$$

is called the Ishikawa iteration sequence with errors provided that $\{A_n, B_n : n \ge 0\} \subseteq \Im$.

(ii) If $s'_n = t'_n = 0$ for all $n \ge 0$ in (1.2), the sequence $\{A_n\}_{n\ge0}$ defined by

$$B_n = (1 - s_n) A_n + s_n T A_n, \quad A_{n+1} = (1 - t_n) A_n + t_n T B_n, \quad n \ge 0, \tag{1.3}$$

is called the Ishikawa iteration sequence provided that $\{A_n, B_n : n \ge 0\} \subseteq \Im$.

(iii) If $s_n = s'_n = 0$ for all $n \ge 0$ in (1.2), the sequence $\{A_n\}_{n\ge0}$ defined by

$$A_{n+1} = \left(1 - t_n - t'_n\right) A_n + t_n T A_n + t'_n Q_n, \quad n \ge 0, \tag{1.4}$$

is called the Mann iteration sequence with errors provided that $\{A_n : n \ge 0\} \subseteq \Im$.

(iv) If $s'_n = t'_n = s_n = 0$ for all $n \ge 0$ in (1.2), the sequence $\{A_n\}_{n\ge0}$ defined by

$$A_{n+1} = (1 - t_n) A_n + t_n T A_n, \quad n \ge 0, \tag{1.5}$$

is called the Mann iteration sequence provided that $\{A_n : n \ge 0\} \subseteq \Im$.

*Definition 1.2.* Let $\Im$ be a nonempty subset of $CC(E)$. A mapping $T : (\Im, H) \to (CC(E), H)$ is said to be

(i) nonexpansive if $H(TA, TB) \leq H(A, B)$ for all $A, B \in \Im$;

(ii) quasi-nonexpansive if $F(T) \neq \emptyset$ and $H(TA, P) \leq H(A, P)$ for all $A \in \Im$ and $P \in F(T)$.

*Definition 1.3.* Let $\Im$ be a nonempty subset of $CC(E)$. A mapping $T : (\Im, H) \to (CC(E), H)$ with $F(T) \neq \emptyset$ is said to be satisfy the following.

(i) Condition A if there is a continuous function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for $t \in (0, \infty)$, such that $H(A, TA) \geq f(D(A, F(T)))$ for all $A \in \Im$.

(ii) Condition B if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for $t \in (0, \infty)$, such that $H(A, TA) \geq f(D(A, F(T)))$ for all $A \in \Im$.

*Remark 1.4.* In case $\Im = \Im_X$, where X is a nonempty subset of $E$, and $T : \Im_X \to \Im_E \subseteq CC(E)$ is a mapping, then Definitions 1.1, 1.2, and 1.3(ii) reduce to the corresponding concepts in [1–11, 13]. It is well known that every nonexpansive mapping with nonempty fixed point set is quasi-nonexpansive, but the converse is not true; see [8]. Examples 3.1 and 3.4 in this paper reveal that the class of nonexpansive mappings with nonempty fixed point set is a proper subclass of quasi-nonexpansive mappings with both Condition A and Condition B.

The following lemmas play important roles in this paper.

**Lemma 1.5** (see [12]). *Let $(E, \|\cdot\|)$ be a Banach space and $\Im$ a compact subset of $(CC(E), H)$. Then $(\overline{co(\Im)}, H)$ is compact, where $\overline{co(\Im)}$ stands for the closure of $co(\Im)$.*

**Lemma 1.6** (see [4]). *Suppose that $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$, and $\{c_n\}_{n\geq 0}$ are three sequences of nonnegative numbers such that $a_{n+1} \leq (1 + b_n)a_n + c_n$ for all $n \geq 0$. If $\sum_{n=0}^{\infty} b_n$ and $\sum_{n=0}^{\infty} c_n$ converge, then $\lim_{n\to\infty} a_n$ exists.*

**Lemma 1.7** (see [14]). *Let $(X, d)$ be a metric space. Let $A$ and $B$ be compact subsets of $X$. Then for any $a \in A$, there exists $b \in B$ such that $d(a, b) \leq H(A, B)$, where $H$ is the Hausdorff metric induced by $d$.*

**Lemma 1.8.** *Let $(E, \|\cdot\|)$ be a normed linear space. Then*

$$H((1-t-s)A+tB+sC, (1-t-s)L + tM + sN) \leq (1-t-s)H(A, L) + tH(B, M) + sH(C, N) \tag{1.6}$$

*for all $A, B, C, L, M, N \in CC(E)$ and $t, s \in [0, 1]$ with $s + t \leq 1$.*

*Proof.* Set

$$r = (1-t-s)H(A, L) + tH(B, M) + sH(C, N). \tag{1.7}$$

For any $a \in A$, $b \in B$, $c \in C$, by Lemma 1.7 we infer that there exist $l \in L$, $m \in M$, $n \in N$ such that

$$\|a-l\| \leq H(A, L), \qquad \|b-m\| \leq H(B, M), \qquad \|c-n\| \leq H(C, N), \tag{1.8}$$

which imply that

$$\|(1-t-s)a+tb+sc-(1-t-s)l-tm-sn\| \le (1-t-s)\|a-l\|+t\|b-m\|+s\|c-n\| \le r. \tag{1.9}$$

That is,

$$\sup\{d((1-t-s)a+tb+sc,(1-t-s)L+tM+sN) : a \in A,\ b \in B,\ c \in C\} \le r. \tag{1.10}$$

Similarly we have

$$\sup\{d((1-t-s)l+tm+sn,(1-t-s)A+tB+sC) : l \in L,\ m \in M,\ n \in N\} \le r. \tag{1.11}$$

Thus (1.6) follows from (1.10) and (1.11). This completes the proof. □

**Lemma 1.9.** *Let $(E,\|\cdot\|)$ be a normed linear space and $\Im$ a nonempty closed subset of $(CC(E),H)$. If $T:(\Im,H)\to(CC(E),H)$ is quasi-nonexpansive, then $F(T)$ is closed.*

*Proof.* Let $\{P_n\}_{n\ge0}$ be in $F(T)$ with $\lim_{n\to\infty}H(P_n,P)=0$. Since $T$ is quasi-nonexpansive, it follows that

$$H(P,TP) \le H(P_n,P)+H(P_n,TP) \le 2H(P_n,P) \longrightarrow 0 \tag{1.12}$$

as $n\to\infty$. Hence $P \in F(T)$. That is, $F(T)$ is closed. This completes the proof. □

## 2. Main Results

Our results are as follows.

**Theorem 2.1.** *Let $(E,\|\cdot\|)$ be a normed linear space and let $\Im$ be a nonempty subset of $CC(E)$. Assume that $T:(\Im,H)\to(CC(E),H)$ is nonexpansive and $A_0\in\Im$. Suppose that there exists a constant $t$ satisfying*

$$0<t_n+t'_n\le t<1,\quad n\ge0, \tag{2.1}$$

$$\sum_{n=0}^{\infty}t_n=\infty,\quad \sum_{n=0}^{\infty}s_n<\infty,\quad \sum_{n=0}^{\infty}s'_n<\infty,\quad \sum_{n=0}^{\infty}t'_n<\infty,\quad \sum_{n=0}^{\infty}t'_n(t_n+t'_n)^{-1}<\infty. \tag{2.2}$$

*If the Ishikawa iteration sequence with errors $\{A_n\}_{n\ge0}$ is bounded, then $\lim_{n\to\infty}H(A_n,TA_n)=0$.*

*Proof.* Since $T$ is nonexpansive, $\{A_n\}_{n\ge0}$, $\{P_n\}_{n\ge0}$, and $\{Q_n\}_{n\ge0}$ are bounded, it follows that

$$a:=\sup\{H(A,B): A\in\{A_n,B_n,P_n,Q_n : n\ge0\}, B\in\{A_n,B_n,TA_n,TB_n : n\ge0\}\}<\infty. \tag{2.3}$$

Let $n$ and $k$ be arbitrary nonnegative integers. In view of (1.2), (2.3), Lemma 1.8, and the nonexpansiveness of $T$, we conclude that

$$H(B_n, A_n) \leq s_n H(A_n, TA_n) + as'_n, \tag{2.4}$$

$$H(TB_n, A_n) \leq H(TB_n, TA_n) + H(TA_n, A_n) \leq (1+s_n)H(A_n, TA_n) + as'_n, \tag{2.5}$$

$$H(A_{n+1}, A_n) \leq t_n H(TB_n, A_n) + at'_n \leq t_n(1+s_n)H(A_n, TA_n) + a\left(t_n s'_n + t'_n\right), \tag{2.6}$$

$$\begin{aligned} H(A_{n+1}, TA_k) &\leq \left(1 - t_n - t'_n\right)H(A_n, TA_k) + t_n H(TB_n, TA_k) + at'_n \\ &\leq \left(1 - t_n - t'_n\right)H(A_n, TA_k) + t_n H(B_n, A_k) + at'_n, \end{aligned} \tag{2.7}$$

which yields that

$$H(A_n, TA_k) \geq \left(1 - t_n - t'_n\right)^{-1}\left(H(A_{n+1}, TA_k) - t_n H(B_n, A_k) - at'_n\right). \tag{2.8}$$

Using (1.2), (2.3)–(2.6), Lemma 1.8, and the nonexpansiveness of $T$, we have

$$\begin{aligned} H(B_n, A_{n+k+1}) &\leq H(B_n, A_{n+1}) + \sum_{i=1}^{k} H(A_{n+i}, A_{n+i+1}) \\ &\leq \left(1 - s_n - s'_n\right)H(A_n, A_{n+1}) + s_n H(TA_n, A_{n+1}) + as'_n \\ &\quad + \sum_{i=1}^{k}\left[(1+s_{n+i})t_{n+i}H(A_{n+i}, TA_{n+i}) + a\left(t_{n+i}s'_{n+i} + t'_{n+i}\right)\right] \\ &\leq \left(1 - s_n - s'_n\right)\left[t_n(1+s_n)H(TA_n, A_n) + a\left(t_n s'_n + t'_n\right)\right] \\ &\quad + s_n\left[\left(1 - t_n - t'_n\right)H(TA_n, A_n) + t_n H(TB_n, TA_n) + at'_n\right] + as'_n \\ &\quad + \sum_{i=1}^{k}(1+s_{n+i})t_{n+i}H(A_{n+i}, TA_{n+i}) + a\sum_{i=1}^{k}\left(t_{n+i}s'_{n+i} + t'_{n+i}\right) \\ &\leq \left(t_n + s_n - s_n t_n - s_n^2 t_n - s'_n t_n - s_n t'_n - s'_n t_n s_n\right)H(A_n, TA_n) \\ &\quad + a\left(1 - s_n - s'_n\right)\left(t_n s'_n + t'_n\right) + s_n t_n\left(s_n H(A_n, TA_n) + as'_n\right) \\ &\quad + as_n t'_n + as'_n + \sum_{i=1}^{k}(1+s_{n+i})t_{n+i}H(A_{n+i}, TA_{n+i}) + a\sum_{i=1}^{k}\left(t_{n+i}s'_{n+i} + t'_{n+i}\right) \\ &\leq \sum_{i=0}^{k}(t_{n+i} + s_{n+i})H(A_{n+i}, TA_{n+i}) + a\left[s'_n + \sum_{i=0}^{k}\left(t_{n+i}s'_{n+i} + t'_{n+i}\right)\right], \end{aligned} \tag{2.9}$$

$$
\begin{aligned}
H(TA_{n+1},A_{n+1}) &\le (1-t_n-t'_n)H(A_n,TA_{n+1})+t_nH(TB_n,TA_{n+1})+t'_nH(Q_n,TA_{n+1})\\
&\le (1-t_n-t'_n)(H(A_{n+1},TA_{n+1})+H(A_n,A_{n+1}))+t_nH(B_n,A_{n+1})+at'_n\\
&\le (1-t_n-t'_n)H(A_{n+1},TA_{n+1})\\
&\quad+(1-t_n-t'_n)\left[(1+s_n)t_nH(A_n,TA_n)+a\left(t_ns'_n+t'_n\right)\right]\\
&\quad+t_n\left[(t_n+s_n)H(A_n,TA_n)+a\left(s'_n+t_ns'_n+t'_n\right)\right]+at'_n\\
&\le (1-t_n-t'_n)H(A_{n+1},TA_{n+1})+t_n(1+2s_n)H(A_n,TA_n)+2a\left(t_ns'_n+t'_n\right)
\end{aligned}
\tag{2.10}
$$

which implies that

$$
\begin{aligned}
H(A_{n+1},TA_{n+1}) &\le \left(t_n+t'_n\right)^{-1}\left[t_n(1+2s_n)H(A_n,TA_n)+2a\left(t_ns'_n+t'_n\right)\right]\\
&\le (1+2s_n)H(A_n,TA_n)+2a\left(s'_n+t_n\left(t'_n+t'_n\right)^{-1}\right).
\end{aligned}
\tag{2.11}
$$

Lemma 1.6, (2.2), and (2.11) yield that there exists a nonnegative constant $r$ satisfying

$$
\lim_{n\to\infty}H(A_n,TA_n)=r,
\tag{2.12}
$$

which implies that for any $\varepsilon>0$ there exists a positive integer $N$ such that

$$
r-\varepsilon\le H(A_n,TA_n)\le r+\varepsilon\quad \text{for } n\ge N.
\tag{2.13}
$$

Now we prove by induction that the following inequality holds for all $n\ge 1$:

$$
\begin{aligned}
H\left(A_p,TA_{p+n}\right) &\ge (r+\varepsilon)\left(1+\sum_{i=0}^{n-1}t_{p+i}\right)-2\varepsilon\prod_{i=0}^{n-1}\left(1-t_{p+i}-t'_{p+i}\right)^{-1}\\
&\quad-(r+\varepsilon)\sum_{i=0}^{n-1}\left[t_{p+i}\left(\sum_{j=i}^{n-1}s_{p+j}\right)\prod_{k=0}^{i}\left(1-t_{p+k}-t'_{p+k}\right)^{-1}\right]\\
&\quad-a\sum_{i=0}^{n-1}\left\{\left[t_{p+i}\left(s'_{p+i}+\sum_{j=i}^{n-1}\left(t_{p+j}s'_{p+j}+t'_{p+j}\right)\right)+t'_{p+i}\right]\right.\\
&\qquad\left.\times\prod_{k=0}^{i}\left(1-t_{p+k}-t'_{p+k}\right)^{-1}\right\},\quad p\ge N.
\end{aligned}
\tag{2.14}
$$

According to (1.2), (2.8), (2.9), and (2.13), we derive that

$$\begin{aligned}
&H(A_p, TA_{p+1})\\
&\quad\geq \left(1-t_p-t'_p\right)^{-1}\left(H(A_{p+1}, TA_{p+1}) - t_pH(B_p, A_{p+1}) - at'_p\right)\\
&\quad\geq \left(1-t_p-t'_p\right)^{-1}\left(r-\varepsilon-(r+\varepsilon)t_p(t_p+s_p) - at_p\left(s'_p+t_ps'_p+t'_p\right)-at'_p\right)\\
&\quad= \left(1-t_p-t'_p\right)^{-1}\left[r-\varepsilon-(r+\varepsilon)\left(1-2(1-t_p)+(1-t_p)^2+t_ps_p\right)-a\left(t_p\left(s'_p+t_ps'_p+t'_p\right)+t'_p\right)\right]\\
&\quad\geq (r+\varepsilon)(1+t_p) - 2\varepsilon\left(1-t_p-t'_p\right)^{-1} - (r+\varepsilon)t_ps_p\left(1-t_p-t'_p\right)^{-1}\\
&\qquad - a\left[t_p\left(s'_p+t_ps'_p+t'_p\right)+t'_p\right]\left(1-t_p-t'_p\right)^{-1}, \quad p\geq N.
\end{aligned} \tag{2.15}$$

Hence (2.14) holds for $n=1$. Suppose that (2.14) holds for $n=m\geq 1$. That is,

$$\begin{aligned}
H(A_p, TA_{p+m}) \geq{}& (r+\varepsilon)\left(1+\sum_{i=0}^{m-1}t_{p+i}\right) - 2\varepsilon\prod_{i=0}^{m-1}\left(1-t_{p+i}-t'_{p+i}\right)^{-1}\\
&- (r+\varepsilon)\sum_{i=0}^{m-1}\left[t_{p+i}\left(\sum_{j=i}^{m-1}s_{p+j}\right)\prod_{k=0}^{i}\left(1-t_{p+k}-t'_{p+k}\right)^{-1}\right]\\
&- a\sum_{i=0}^{m-1}\left\{\left[t_{p+i}\left(s'_{p+i}+\sum_{j=i}^{m-1}\left(t_{p+j}s'_{p+j}+t'_{p+j}\right)\right)+t'_{p+i}\right]\right.\\
&\left.\times\prod_{k=0}^{i}\left(1-t_{p+k}-t'_{p+k}\right)^{-1}\right\}, \quad p\geq N.
\end{aligned} \tag{2.16}$$

In view of (1.2), (2.8), (2.9), and (2.16), we infer that

$$\begin{aligned}
&H(A_p, TA_{p+m+1})\\
&\quad\geq \left(1-t_p-t'_p\right)^{-1}(H(A_{p+1}, TA_{p+m+1}) - t_pH(B_p, A_{p+m+1}) - at'_n)\\
&\quad\geq \left(1-t_p-t'_p\right)^{-1}\left\{(r+\varepsilon)\left(1+\sum_{i=0}^{m-1}t_{p+1+i}\right) - 2\varepsilon\prod_{i=0}^{m-1}\left(1-t_{p+1+i}-t'_{p+1+i}\right)^{-1}\right.\\
&\qquad - (r+\varepsilon)\sum_{i=0}^{m-1}\left[t_{p+1+i}\left(\sum_{j=i}^{m-1}s_{p+1+j}\right)\prod_{k=0}^{i}\left(1-t_{p+1+k}-t'_{p+1+k}\right)^{-1}\right]\\
&\qquad - a\sum_{i=0}^{m-1}\left[\left(t_{p+1+i}\left(s'_{p+1+i}+\sum_{j=i}^{m-1}\left(t_{p+1+j}s'_{p+1+j}+t'_{p+1+j}\right)\right)+\ t'_{p+1+i}\right)\right.\\
&\qquad\left.\times\prod_{k=0}^{i}\left(1-t_{p+1+k}-t'_{p+1+k}\right)^{-1}\right] - t_p\sum_{i=0}^{m-1}(t_{p+i}+s_{p+i})(r+\varepsilon)\\
&\qquad\left. -at_p\left(s'_p+\sum_{i=0}^{m}\left(t_{p+i}s'_{p+i}+t'_{p+i}\right)\right)-at'_p\right\}
\end{aligned}$$

$$
\begin{aligned}
&= -2\varepsilon\prod_{i=0}^{m}\left(1-t_{p+i}-t'_{p+i}\right)^{-1} + \left(1-t_p-t'_p\right)^{-1}(r+\varepsilon)\\
&\quad\times\left[1+\sum_{i=0}^{m}t_{p+1+i}-1+2(1-t_p)-(1-t_p)^2-t_p\sum_{i=1}^{m}t_{p+i}-t_p\sum_{i=0}^{m}s_{p+i}\right]\\
&\quad-(r+\varepsilon)\sum_{i=0}^{m-1}\left[t_{p+1+i}\left(\sum_{j=i}^{m-1}s_{p+1+j}\right)\prod_{k=0}^{i+1}\left(1-t_{p+k}-t'_{p+k}\right)^{-1}\right]\\
&\quad-a\left\{\left[t_p\left(s'_p+\sum_{i=0}^{m}\left(t_{p+i}s'_{p+i}+t'_{p+i}\right)\right)+t'_p\right]\left(1-t_p-t'_p\right)^{-1}\right.\\
&\qquad+\sum_{i=0}^{m-1}\left[\left(t_{p+1+i}\left(s'_{p+1+i}+\sum_{j=i}^{m-1}\left(t_{p+1+j}s'_{p+1+j}+t'_{p+1+j}\right)\right)+t'_{p+1+i}\right)\right.\\
&\qquad\left.\left.\times\prod_{k=0}^{i+1}\left(1-t_{p+k}-t'_{p+k}\right)^{-1}\right]\right\}\\
&= -2\varepsilon\prod_{i=0}^{m}\left(1-t_{p+i}-t'_{p+i}\right)^{-1}+(r+\varepsilon)(1-t_p)\left(1-t_p-t'_p\right)^{-1}\left(1+\sum_{i=0}^{m}t_{p+i}\right)\\
&\quad-(r+\varepsilon)\left\{t_p\left(1-t_p-t'_p\right)^{-1}\sum_{i=0}^{m}s_{p+i}+\sum_{i=0}^{m-1}\left[t_{p+1+i}\left(\sum_{j=i}^{m-1}s_{p+1+j}\right)\prod_{k=0}^{i+1}\left(1-t_{p+k}-t'_{p+k}\right)^{-1}\right]\right\}\\
&\quad-a\sum_{i=0}^{m}\left\{\left[t_{p+i}\left(s'_{p+i}+\sum_{j=i}^{m}\left(t_{p+j}s'_{p+j}+t'_{p+j}\right)\right)+t'_{p+i}\right]\prod_{k=0}^{i}\left(1-t_{p+k}-t'_{p+k}\right)^{-1}\right\}\\
&\geq (r+\varepsilon)\left(1+\sum_{i=0}^{m}t_{p+i}\right)-2\varepsilon\prod_{i=0}^{m}\left(1-t_{p+i}-t'_{p+i}\right)^{-1}\\
&\quad+(r+\varepsilon)\sum_{i=0}^{m}\left[t_{p+i}\left(\sum_{j=i}^{m}s_{p+j}\right)\prod_{k=0}^{i}\left(1-t_{p+k}-t'_{p+k}\right)^{-1}\right]\\
&\quad-a\sum_{i=0}^{m}\left\{\left[t_{p+i}\left(s'_{p+i}+\sum_{j=i}^{m}\left(t_{p+j}s'_{p+j}+t'_{p+j}\right)\right)+\ t'_{p+i}\right]\prod_{k=0}^{i}\left(1-t_{p+k}-t'_{p+k}\right)^{-1}\right\},\quad p\geq N.
\end{aligned}
\tag{2.17}
$$

That is, (2.14) holds for $n=m+1$. Hence (2.14) holds for all $n\geq 1$.

We now assert that $r=0$. If not, then $r>0$. Let $m$ be an arbitrary positive integer and

$$
\varepsilon=\min\left\{r, 2^{-1}rt(2r+a)^{-1}(1-t)^{m}, r(1-t)^{m}\left(2+at^{-1}\right)^{-1}\right\}. \tag{2.18}
$$

According to (2.1), (2.2), and (2.12), we know that there exists a positive integer $N = N(\varepsilon)$ satisfying (2.13) and

$$\max\left\{\sum_{k=n}^{n+p} s_k, s'_i + \sum_{k=n}^{n+p}\left(t_k s'_k + t'_k\right)\right\} < \varepsilon \quad \text{for } n,\ i \geq N,\ p \geq 1. \tag{2.19}$$

It follows from (2.1), (2.2), (2.13), (2.14), and (2.19) that

$$\begin{aligned}
H(A_N, TA_{N+m}) &\geq (r+\varepsilon)\left(1+\sum_{i=0}^{m-1} t_{N+i}\right) - 2\varepsilon\prod_{i=0}^{m-1}\left(1-t_{N+i}-t'_{N+i}\right)^{-1} \\
&\quad - (r+\varepsilon)\sum_{i=0}^{m-1}\left[t_{N+i}\left(\sum_{j=i}^{m-1} s_{N+j}\right)\prod_{k=0}^{i}\left(1-t_{N+k}-t'_{N+k}\right)^{-1}\right] \\
&\quad - a\sum_{i=0}^{m-1}\left\{\left[t_{N+i}\left(s'_{N+i}+\sum_{j=i}^{m-1}\left(t_{N+j}s'_{N+j}+t'_{N+j}\right)\right)+t'_{N+i}\right]\right. \\
&\qquad\qquad \left.\times\prod_{k=0}^{i}\left(1-t_{N+k}-t'_{N+k}\right)^{-1}\right\} \\
&\geq (r+\varepsilon)\left(1+\sum_{i=0}^{m-1} t_{N+i}\right) - 2\varepsilon(1-t)^{-m} \\
&\quad - (r+\varepsilon)\varepsilon\sum_{i=0}^{m-1} t_{N+i}(1-t)^{-i-1} - a\sum_{i=0}^{m-1}\left[\left(t_{N+i}\varepsilon + t'_{N+i}\right)(1-t)^{-i-1}\right] \\
&\geq (r+\varepsilon)\left(1+\sum_{i=0}^{m-1} t_{N+i}\right) - 2\varepsilon(1-t)^{-m} - (r+\varepsilon)\varepsilon\sum_{i=0}^{m-1}\left[t_{N+i}\sum_{j=0}^{i}(1-t)^{-j-1}\right] \\
&\quad - a\varepsilon\sum_{i=0}^{m-1} t_{N+i}\sum_{j=0}^{i}(1-t)^{-j-1} - a\sum_{i=0}^{m-1}\left[(1-t)^{-i-1}\sum_{j=0}^{i} t'_{N+j}\right] \\
&\geq (r+\varepsilon)\left(1+\sum_{i=0}^{m-1} t_{N+i}\right) - 2\varepsilon(1-t)^{-m} - (r+\varepsilon)\varepsilon t^{-1}(1-t)^{-m}\sum_{i=0}^{m-1} t_{N+i} \\
&\quad - a\varepsilon t^{-1}(1-t)^{-m}\sum_{i=0}^{m-1} t_{N+i} - a\varepsilon\sum_{i=0}^{m-1}(1-t)^{-i-1} \\
&\geq \left(r+\varepsilon-(r+\varepsilon+a)\varepsilon t^{-1}(1-t)^{-m}\right)\sum_{i=0}^{m-1} t_{N+i} \\
&\quad + \left(r+\varepsilon-2\varepsilon(1-t)^{-m} - a\varepsilon t^{-1}(1-t)^{-m}\right) \\
&\geq \left(r-(2r+a)\varepsilon t^{-1}(1-t)^{-m}\right)\sum_{i=0}^{m-1} t_{N+i} + \left(r-\left(2+at^{-1}\right)\varepsilon(1-t)^{-m}\right) \\
&\geq 2^{-1} r\sum_{i=0}^{m-1} t_{N+i} \longrightarrow \infty
\end{aligned} \tag{2.20}$$

as $m \to \infty$. Thus (2.3) and (2.20) yield that $a = \infty$, which is absurd. Hence $r = 0$. This completes the proof. □

**Theorem 2.2.** *Let* $(E, \|\cdot\|)$ *be a Banach space and* $\Im$ *a nonempty closed subset of* $CC(E)$. *Assume that* $T : (\Im, H) \to (CC(E), H)$ *is nonexpansive and there exists a compact subset* $\Omega$ *of* $CC(E)$ *such that* $T(\Im) \cup \{P_n, Q_n : n \geq 0\} \subseteq \Omega$. *If* (2.1) *and* (2.2) *hold, then* $T$ *has a fixed point in* $\Im$. *Moreover, given* $A_0 \in \Im$, *the Ishikawa iteration sequence with errors* $\{A_n\}_{n\geq 0}$ *converges to a fixed point of* $T$.

*Proof.* Setting $\Im_0 = \overline{co(\{A_0\} \cup \Omega)}$, by Lemma 1.5 and the compactness $\Omega$ we conclude that $\Im_0$ is compact. It is evident that $\{A_n\}_{n\geq 0} \subseteq \Im_0$, which yields that $\{A_n\}_{n\geq 0}$ is bounded. Since $\Im$ is closed and $\{A_n\}_{n\geq 0} \subseteq \Im$, we conclude that there exist a subsequence $\{A_{n_i}\}_{i\geq 0}$ of $\{A_n\}_{n\geq 0}$ and $A \in \Im$ such that

$$\lim_{i\to\infty} H(A_{n_i}, A) = 0. \tag{2.21}$$

It follows from (2.21), Theorem 2.1, and the nonexpansiveness of $T$ that

$$\begin{aligned} H(A, TA) &\leq H(A, A_{n_i}) + H(A_{n_i}, TA_{n_i}) + H(TA_{n_i}, TA) \\ &\leq 2H(A, A_{n_i}) + H(A_{n_i}, TA_{n_i}) \longrightarrow 0 \end{aligned} \tag{2.22}$$

as $i \to \infty$. That is, $A = TA$. Put

$$b = \sup\{H(P_n, A), H(Q_n, A) : n \geq 0\}. \tag{2.23}$$

In view of (1.2), Lemma 1.8 and the nonexpansiveness of $T$, we derive that

$$\begin{aligned} H(A_{n+1}, A) &\leq \left(1 - t_n - t_n'\right) H(A_n, A) + t_n H(TB_n, A) + bt_n' \\ &\leq \left(1 - t_n - t_n'\right) H(A_n, A) \\ &\quad + t_n\left(\left(1 - s_n - s_n'\right) H(A_n, A) + s_n H(TA_n, A) + bs_n'\right) + bt_n' \end{aligned} \tag{2.24}$$

for $n \geq 0$. It follows from Lemma 1.6, (2.2), (2.23), and (2.24) that $\lim_{i\to\infty} H(A_n, A)$ exists. Using (2.21) we get that $\lim_{i\to\infty} H(A_n, A) = 0$. This completes the proof. □

**Theorem 2.3.** *Let* $(E, \|\cdot\|)$ *be a Banach space and* $\Im$ *a nonempty closed subset of* $CC(E)$. *Suppose that* $T : (\Im, H) \to (CC(E), H)$ *is a qusi-nonexpansive mapping and satisfies Condition A. Assume that* (2.1) *and* (2.2) *hold and* $A_0$ *is in* $\Im$. *If* $F(T)$ *is bounded, then the Ishikawa iteration sequence with errors* $\{A_n\}_{n\geq 0}$ *converges to a fixed point of* $T$ *in* $\Im$.

*Proof.* Let $b = \sup\{H(P_n, A), H(Q_n, A) : n \geq 0 \text{ and } A \in F(T)\}$. Then $b < \infty$. As in the proof of Theorem 2.2, we get that (2.24) holds and $\lim_{i\to\infty} H(A_n, A)$ exists, where $A \in F(T)$. Consequently, $\{A_n\}_{n\geq 0}$ is bounded and

$$D(A_{n+1}, F(T)) \leq D(A_n, F(T)) + b\left(s_n' + t_n'\right) \quad \forall n \geq 0. \tag{2.25}$$

It follows from Lemma 1.6, (2.2), and (2.25) that $\lim_{n\to\infty} D(A_n, F(T)) = s \geq 0$. In view of Theorem 2.1 and Condition A, we have

$$\lim_{n\to\infty} H(A_n, TA_n) = 0, \quad f(D(A_n, F(T))) \leq H(A_n, TA_n) \quad \forall n \geq 0. \tag{2.26}$$

Using the continuity of $f$, we know that $f(s) = 0$. That is, $s = 0$ and

$$\lim_{n\to\infty} D(A_n, F(T)) = 0. \tag{2.27}$$

Clearly (2.27) ensures that for any $i \geq 0$ there exist $N_i \geq 1$ and $P_i \in F(T)$ such that $H(A_{N_i}, P_i) < 2^{-i}$, which implies from (2.24) that

$$H(A_n, P_i) < 2^{-i} + b\sum_{k=N_i}^{n-1} \left(s'_k + t'_k\right) \quad \text{for } n \geq N_i. \tag{2.28}$$

We require $N_{i+1} > N_i$ for all $i \geq 0$. Notice that for any $j > i \geq 0$

$$\begin{aligned} H(P_i, P_j) &\leq \sum_{k=i}^{j-1} \left(H(P_k, A_{N_{k+1}}) + H(A_{N_{k+1}}, P_{k+1})\right) \\ &\leq \sum_{k=i}^{j-1} \left(2^{-k} + b\sum_{m=N_k}^{N_{k+1}-1} \left(s'_m + t'_m\right) + 2^{-k-1}\right) \\ &= 3\left(2^{-i} - 2^{-j}\right) + b\sum_{l=N_i}^{N_j-1} \left(s'_l + t'_l\right). \end{aligned} \tag{2.29}$$

Thus (2.2) and (2.29) yield that $\{P_i\}_{i\geq 0}$ is a Cauchy sequence in $F(T)$. It follows from Lemma 1.9 that there exists $P \in F(T)$ satisfying $\lim_{i\to\infty} P_i = P$. For any $\varepsilon > 0$ there exists $i_0 > 0$ such that

$$\max\left\{2^{-i_0}, H(P_{i_0}, P), b\sum_{k=N_{i_0}}^{n-1} \left(s'_k + t'_k\right)\right\} < 3^{-1}\varepsilon \quad \text{for } n > N_{i_0}. \tag{2.30}$$

Using (2.28) and (2.30) we have

$$\begin{aligned} H(A_n, P) &\leq H(A_n, P_{i_0}) + H(P_{i_0}, P) \\ &\leq 2^{-i_0} + b\sum_{k=N_{i_0}}^{n-1} \left(s'_k + t'_k\right) + H(P_{i_0}, P) \\ &< \varepsilon \end{aligned} \tag{2.31}$$

for $n > N_{i_0}$. That is, $\{A_n\}_{n\geq 0}$ converges to $P \in F(T)$. This completes the proof. $\square$

A proof similar to that of Theorem 2.3 gives the following result and is thus omitted.

**Theorem 2.4.** *Let $(E, \|\cdot\|)$ be a Banach space and let $\Im$ be a nonempty closed subset of $CC(E)$. Suppose that $T : (\Im, H) \to (CC(E), H)$ is a qusi-nonexpansive mapping and satisfies Condition B. Assume that $A_0$ is in $\Im$ and there exists a constant $t$ satisfying*

$$0 < t_n \leq t < 1, \quad n \geq 0, \quad \sum_{n=0}^{\infty} t_n = \infty, \quad \sum_{n=0}^{\infty} s_n < \infty. \tag{2.32}$$

*Then the Ishikawa iteration sequence $\{A_n\}_{n\geq 0}$ converges to a fixed point of $T$ in $\Im$.*

Let $X$ be a nonempty subset of $(E, \|\cdot\|)$. It is easy to see that $(\Im_X, H)$ is isometric to $(X, \|\cdot\|)$. Thus Theorems 2.1–2.4 yield the following results.

**Corollary 2.5.** *Let $X$ be a nonempty subset of a normed linear space $(E, \|\cdot\|)$. Assume that $T : (X, \|\cdot\|) \to (E, \|\cdot\|)$ is nonexpansive and $A_0 \in X$. Suppose that* (2.1) *and* (2.2) *hold. If the Ishikawa iteration sequence with errors $\{A_n\}_{n\geq 0}$ is bounded, then $\lim_{n\to\infty}\|A_n - TA_n\| = 0$.*

*Remark 2.6.* Corollary 2.5 extends Theorem 1 in [3] and Lemma 2 in [7] from the Ishikawa iteration scheme and Mann iteration scheme into the Ishikawa iteration scheme with errors, respectively.

**Corollary 2.7.** *Let $X$ be a nonempty closed subset of a Banach space $(E, \|\cdot\|)$. Assume that $T : (X, \|\cdot\|) \to (E, \|\cdot\|)$ is nonexpansive and there exists a compact subset $Y$ of $E$ with $T(X) \cup \{P_n, Q_n : n \geq 0\} \subseteq Y$. Suppose that* (2.1) *and* (2.2) *hold. Then $T$ has a fixed point in $X$. Moreover for any $A_0 \in X$, the Ishikawa iteration sequence with errors $\{A_n\}_{n\geq 0}$ converges to a fixed point of $T$.*

*Remark 2.8.* Theorem 3 in [3] and Theorem 1 in [7] and [8] are special cases of Corollary 2.7.

**Corollary 2.9.** *Let $X$ be a nonempty closed subset of a Banach space $(E, \|\cdot\|)$ and let $T : (X, \|\cdot\|) \to (E, \|\cdot\|)$ be quasi-nonexpansive. Assume that* (2.1) *and* (2.2) *hold and $T$ satisfies Condition A. If $F(T)$ is bounded, then for any $A_0 \in X$, the Ishikawa iteration sequence with errors $\{A_n\}_{n\geq 0}$ converges to a fixed point of $T$ in $X$.*

**Corollary 2.10.** *Let $X$ be a nonempty closed subset of a Banach space $(E, \|\cdot\|)$ and let $T : (X, \|\cdot\|) \to (E, \|\cdot\|)$ be quasi-nonexpansive. Assume that* (2.32) *holds and $A_0$ is in $X$. If $T$ satisfies Condition B, then the Ishikawa iteration sequence $\{A_n\}_{n\geq 0}$ converges to a fixed point of $T$ in $X$.*

*Remark 2.11.* Corollary 2.10 extends, improves, and unifies Theorem 4 in [3], Theorem 2 in [7] and [8] in the following ways:

(i) the Mann iteration method in [7, 8], and Ishikawa iteration method in [3] are replaced by the more general Ishikawa iteration method with errors;

(ii) the nonexpansive mappings in [3, 7, 8] are replaced by the more general quasi-nonexpansive mappings.

## 3. Examples and Problems

Now we construct a few nontrivial examples to illustrate the results in Section 2. The following example reveals that Corollary 2.10 extends properly Theorem 4 in [3], Theorem 2 in [7] and [8].

*Example 3.1.* Let $E = R$ with the usual norm $|\cdot|$ and let $X = [0,1]$. Define $T : X \to E$ and $f : [0,\infty) \to [0,\infty)$ by

$$Tx = \begin{cases} \dfrac{3}{4}x, & \text{for } x \in \left[0, \dfrac{1}{2}\right], \\ \dfrac{1}{2}x, & \text{for } x \in \left(\dfrac{1}{2}, 1\right], \end{cases} \tag{3.1}$$

and $f(t) = (1/4)t$ for $t \geq 0$. Set $t_n = (2+\sqrt{n})^{-1}$ and $s_n = (1+n^2)^{-1}$ for $n \geq 0$ and $A_0 \in X$. Then $F(T) = \{0\}$ and

$$|Tx - 0| \leq \frac{3}{4}|x-0|, \quad |x - Tx| \geq \frac{1}{4}|x| = f(d(x, F(T))) \quad \text{for} \quad x \in X. \tag{3.2}$$

Thus the assumptions of Corollary 2.10 are satisfied. However, Theorem 4 in [3], Theorem 2 in [7] and [8] are not applicable since

$$\left| T\frac{1}{2} - T\frac{17}{32} \right| = \frac{7}{64} > \frac{1}{32} = \left| \frac{1}{2} - \frac{17}{32} \right|, \tag{3.3}$$

that is, $T$ is not nonexpansive.

The examples below show that Theorems 2.1–2.4 extend substantially Corollaries 2.5–2.10, respectively.

*Example 3.2.* Let $E = R^2$ with the usual norm $|\cdot|$ and let $X = [0,1]^2$. For any $(a,b) \in X$, $\Delta(0,0)(a,0)(0,b)$ stands for the triangle with vertices $(0,0), (a,0)$, and $(0,b)$. Let $\Im = \{\Delta(0,0)(a,0)(0,b) : (a,b) \in X\}$ and $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ be in $\Im$. Define $T : \Im \to CC(E)$ by

$$T\Delta(0,0)(a,0)(0,b) = \Delta(0,0)\left(2^{-1}(a+b), 0\right)\left(0, 4^{-1}\left|b^2 - a^2\right|\right) \quad \text{for } (a,b) \in X. \tag{3.4}$$

Put $t_n = (2+\sqrt{n})^{-1}$, $t'_n = (2+n^{7/4})^{-1}$, $s_n = s'_n = (3+n^{3/2})^{-1}$ for $n \geq 0$ and $A_0 \in X$. It follows that $\Im$ is a compact subset of $CC(E)$, $F(T) = \{\Delta(0,0)(0,0)(0,0)\}$ and

$$\begin{aligned}
&H(T\Delta(0,0)(a,0)(0,b), T\Delta(0,0)(c,0)(0,d)) \\
&\quad = H\Big(\Delta(0,0)\Big(2^{-1}(a+b),0\Big)\Big(0,4^{-1}\Big|b^2-a^2\Big|\Big), \Delta(0,0)\Big(2^{-1}(c+d),0\Big)\Big(0,4^{-1}\Big|d^2-c^2\Big|\Big)\Big) \\
&\quad = \max\Big\{2^{-1}|a+b-c-d|, 4^{-1}\Big|\Big|b^2-a^2\Big|-\Big|d^2-c^2\Big|\Big|\Big\} \\
&\quad \leq \max\{|a-c|,|b-d|\} \\
&\quad = H(\Delta(0,0)(a,0)(0,b), \Delta(0,0)(c,0)(0,d))
\end{aligned} \tag{3.5}$$

for $(a,b),(c,d) \in X$. That is, the conditions of Theorems 2.1 and 2.2 are fulfilled. Hence we can invoke our Theorems 2.1 and 2.2 show that the Ishikawa iteration sequence with errors $\{A_n\}_{n\geq 0}$ converges to $\Delta(0,0)(0,0)(0,0)$ and $\lim_{n\to\infty} H(A_n, TA_n) = 0$.

*Example 3.3.* Let $E, X, \Im$, $\{P_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$, $\{s_n\}_{n\geq 0}$, $\{s'_n\}_{n\geq 0}$, $\{t_n\}_{n\geq 0}$, $\{t'_n\}_{n\geq 0}$, and $A_0$ be as in Example 3.2. Define $T : \Im \to CC(E)$ and $f : [0,\infty) \to [0,\infty)$ by

$$\begin{gathered}
T\Delta(0,0)(a,0)(0,b) = \Delta(0,0)\Big(a^3\Big(1+a^2\Big)^{-1},0\Big)\Big(0,b^3\Big(1+b^2\Big)^{-1}\Big) \quad \text{for } (a,b) \in X, \\
f(t) = t\Big(1+t^2\Big)^{-1} \quad \text{for } t \in [0,\infty).
\end{gathered} \tag{3.6}$$

Obviously, $F(T) = \{\Delta(0,0)(0,0)(0,0)\}$,

$$\begin{aligned}
&H(T\Delta(0,0)(a,0)(0,b), \Delta(0,0)(0,0)(0,0)) \\
&\quad = H\Big(\Delta(0,0)\Big(a^3\Big(1+a^2\Big)^{-1},0\Big)\Big(0,b^3\Big(1+b^2\Big)^{-1}\Big), \Delta(0,0)(0,0)(0,0)\Big) \\
&\quad = \max\Big\{a^3\Big(1+a^2\Big)^{-1}, b^3\Big(1+b^2\Big)^{-1}\Big\} \\
&\quad \leq \max\{a,b\} \\
&\quad = H(\Delta(0,0)(a,0)(0,b), \Delta(0,0)(0,0)(0,0)), \\
&H(\Delta(0,0)(a,0)(0,b), T\Delta(0,0)(a,0)(b,0)) \\
&\quad = H\Big(\Delta(0,0)(a,0)(b,0), \Delta(0,0)\Big(a^3\Big(1+a^2\Big)^{-1},0\Big)\Big(0,b^3\Big(1+b^2\Big)^{-1}\Big)\Big) \\
&\quad = \max\{f(a), f(b)\} \\
&\quad \geq f(\max\{a,b\}) \\
&\quad = f(D(\Delta(0,0)(a,0)(0,b), F(T)))
\end{aligned} \tag{3.7}$$

for $(a,b) \in X$. Therefore the conditions of Theorem 2.3 are fulfilled.

*Example 3.4.* Let $E, X, \mathfrak{I}$, and $A_0$ be as in Example 3.2. Define $T : \mathfrak{I} \to CC(E)$, $f : [0,\infty) \to [0,\infty)$ and $h : [0,1] \to [1,2]$ by

$$T\Delta(0,0)(a,0)(0,b) = \Delta(0,0)\left(2^{-1}ah(a),0\right)\left(0,2^{-1}b^2\right) \quad \text{for } (a,b) \in X,$$
$$f(t) = 8^{-1}t \quad \text{for } t \geq 0,$$
$$h(x) = \begin{cases} \dfrac{7}{4}, & \text{for } x \in \left[0,\dfrac{1}{2}\right], \\ 1, & \text{for } x \in \left(\dfrac{1}{2},1\right]. \end{cases} \tag{3.8}$$

It follows that $F(T) = \{\Delta(0,0)(0,0)(0,0)\}$,

$$\begin{aligned} &H(T\Delta(0,0)(a,0)(0,b),\Delta(0,0)(0,0)(0,0)) \\ &\quad = \max\left\{2^{-1}ah(a),2^{-1}b^2\right\} \\ &\quad \leq \max\{a,b\} \\ &\quad = H(\Delta(0,0)(a,0)(0,b),\Delta(0,0)(0,0)(0,0)), \\ &H(\Delta(0,0)(a,0)(0,b),T\Delta(0,0)(a,0)(0,b)) \\ &\quad = H\left(\Delta(0,0)(a,0)(b,0),\Delta(0,0)\left(2^{-1}ah(a),0\right)\left(0,2^{-1}b^2\right)\right) \\ &\quad = \max\left\{a\left(1-2^{-1}h(a)\right),b\left(1-2^{-1}b^2\right)\right\} \\ &\quad \geq 8^{-1}\max\{a,b\} \\ &\quad = f(D(\Delta(0,0)(a,0)(0,b),F(T))) \end{aligned} \tag{3.9}$$

for $(a,b) \in X$. Obviously, the assumptions of Theorem 2.4 are fulfilled. On the other hand, $T$ is not nonexpansive since

$$\begin{aligned} &H\left(T\Delta(0,0)\left(\frac{1}{2},0\right)\left(0,\frac{1}{2}\right),T\Delta(0,0)\left(\frac{9}{16},0\right)\left(0,\frac{1}{2}\right)\right) \\ &\quad = \frac{1}{2}\left|\frac{1}{2}h\left(\frac{1}{2}\right)-\frac{9}{16}h\left(\frac{9}{16}\right)\right| \\ &\quad = \frac{5}{32} > \frac{1}{16} \\ &\quad = H\left(\Delta(0,0)\left(\frac{1}{2},0\right)\left(0,\frac{1}{2}\right),\Delta(0,0)\left(\frac{9}{16},0\right)\left(0,\frac{1}{2}\right)\right). \end{aligned} \tag{3.10}$$

We conclude with the following problems.

*Problem 3.5.* Can Condition A in Theorem 2.3 be replaced by Condition B?

*Problem 3.6.* Can the boundedness of $F(T)$ in Theorem 2.3 be removed?

*Problem 3.7.* Can Theorem 2.4 be extended to the Ishikawa iteration method with errors?

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