## Research Article

# A General Iterative Method for Equilibrium Problems and Fixed Point Problems in Hilbert Spaces 

Meijuan Shang, Yongfu Su, and Xiaolong Qin

Received 14 May 2007; Revised 15 August 2007; Accepted 18 September 2007
Recommended by Hichem Ben-El-Mechaiekh

We introduce a general iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. Our results improve and extend the corresponding ones announced by S. Takahashi and W. Takahashi in 2007, Marino and Xu in 2006, Combettes and Hirstoaga in 2005, and many others.

Copyright © 2007 Meijuan Shang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

## 1. Introduction

Let $H$ be a real Hilbert space and let $C$ be nonempty closed convex subset of $H$. Recall that a mapping $S$ of $C$ into itself is called nonexpansive if $\|S x-S y\| \leq\|x-y\|$ for all $x, y \in C$. We denote by $F(S)$ the set of fixed points of $S$. Let $B$ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $B: C \times C \rightarrow \mathbb{R}$ is to find $x \in C$ such that

$$
\begin{equation*}
B(x, y) \geq 0 \quad \forall y \in C . \tag{1.1}
\end{equation*}
$$

The set of solutions of (1.1) is denoted by $E P(B)$. Give a mapping $T: C \rightarrow H$, let $B(x, y)=$ $\langle T x, y-x\rangle$ for all $x, y \in C$. Then $z \in E P(B)$ if and only if $\langle T z, y-z\rangle \geq 0$ for all $y \in$ $C$, that is, $z$ is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.1). Some methods have been proposed to solve the equilibrium problem; see, for instance, [1, 2]. Recently, Combettes and Hirstoaga [1] introduced an iterative scheme of finding the best approximation to the initial data when $E P(B)$ is nonempty and proved a strong convergence theorem. Very
recently, S. Takahashi and W. Takahashi [3] also introduced a new iterative scheme:

$$
\begin{gather*}
B\left(y_{n}, u\right)+\frac{1}{r_{n}}\left\langle u-y_{n}, y_{n}-x_{n}\right\rangle \geq 0, \quad \forall u \in C  \tag{1.2}\\
x_{n+1}=\alpha_{n} f\left(x_{n}\right)+\left(1-\alpha_{n}\right) S y_{n},
\end{gather*}
$$

for approximating a common element of the set of fixed points of a nonself nonexpansive mapping and the set of solutions of the equilibrium problem and obtained a strong convergence theorem in a real Hilbert space.

Recall that a linear bounded operator $A$ is strongly positive if there is a constant $\bar{\gamma}>0$ with property $\langle A x, x\rangle \geq \bar{\gamma}\|x\|^{2}, \forall x \in H$.

Recently iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, for example, [4-7] and the references therein. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space $H$ :

$$
\begin{equation*}
\min _{x \in C} \frac{1}{2}\langle A x, x\rangle-\langle x, b\rangle \tag{1.3}
\end{equation*}
$$

where $C$ is the fixed point set of a nonexpansive mapping $S$ and $b$ is a given point in $H$. In [6], it is proved that the sequence $\left\{x_{n}\right\}$ defined by the iterative method below, with the initial guess $x_{0} \in H$ chosen arbitrarily, $x_{n+1}=\left(I-\alpha_{n} A\right) S x_{n}+\alpha_{n} b, n \geq 0$, converges strongly to the unique solution of the minimization problem (1.3) provided the sequence $\left\{\alpha_{n}\right\}$ satisfies certain conditions. Recently, Marino and Xu [8] introduced a new iterative scheme by the viscosity approximation method [9]:

$$
\begin{equation*}
x_{n+1}=\left(I-\alpha_{n} A\right) S x_{n}+\alpha_{n} \gamma f\left(x_{n}\right), \quad n \geq 0 \tag{1.4}
\end{equation*}
$$

They proved the sequence $\left\{x_{n}\right\}$ generated by above iterative scheme converges strongly to the unique solution of the variational inequality $\left\langle(A-\gamma f) x^{*}, x-x^{*}\right\rangle \geq 0, x \in C$, which is the optimality condition for the problem $\min _{x \in C}(1 / 2)\langle A x, x\rangle-h(x)$, where $C$ is the fixed point set of a nonexpansive mapping $S, h$ is a potential function for $\gamma f$ (i.e., $h^{\prime}(x)=\gamma f(x)$ for $x \in H$ ).

In this paper, motivated by Combettes and Hirstoaga [1], Moudafi [9], S. Takahashi and W. Takahashi [3], Marino and Xu [8], and Wittmann [10], we introduce a general iterative scheme as following:

$$
\begin{gather*}
B\left(y_{n}, u\right)+\frac{1}{r_{n}}\left\langle u-y_{n}, y_{n}-x_{n}\right\rangle \geq 0, \quad \forall u \in C  \tag{1.5}\\
x_{n+1}=\alpha_{n} \gamma f\left(x_{n}\right)+\left(I-\alpha_{n} A\right) S y_{n}
\end{gather*}
$$

We will prove that the sequence $\left\{x_{n}\right\}$ generated by (1.5) converges strongly to a common element of the set of fixed points of nonexpansive mapping $S$ and the set of solutions of equilibrium problem (1.1), which is the unique solution of the variational inequality $\langle\gamma f(q)-A q, q-p\rangle \leq 0, \forall p \in F$, where $F=F(S) \cap E P(B)$ and is also the optimality condition for the minimization problem $\min _{x \in F}(1 / 2)\langle A x, x\rangle-h(x)$, where $h$ is a potential function for $\gamma f$ (i.e., $h^{\prime}(x)=\gamma f(x)$ for $x \in H$ ).

## 2. Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle\cdot, \cdot \cdot\rangle$ and norm $\|\cdot\|$, respectively. It is well known that for all $x, y \in H$ and $\lambda \in[0,1]$, there holds

$$
\begin{equation*}
\|\lambda x+(1-\lambda) y\|^{2}=\lambda\|x\|^{2}+(1-\lambda)\|y\|^{2}-\lambda(1-\lambda)\|x-y\|^{2} . \tag{2.1}
\end{equation*}
$$

A space $X$ is said to satisfy Opial's condition [11] if for each sequence $\left\{x_{n}\right\}_{n=1}^{\infty}$ in $X$ which converges weakly to point $x \in X$, we have

$$
\begin{equation*}
\liminf _{n \rightarrow \infty}\left\|x_{n}-x\right\|<\liminf _{n \rightarrow \infty}\left\|x_{n}-y\right\|, \quad \forall y \in X, y \neq x . \tag{2.2}
\end{equation*}
$$

For solving the equilibrium problem for a bifunction $B: C \times C \rightarrow \mathbb{R}$, let us assume that $B$ satisfies the following conditions:
(A1) $B(x, x)=0$ for all $x \in C$;
(A2) $B$ is monotone, that is, $B(x, y)+B(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C, \lim _{t \downarrow 0} B(t z+(1-t) x, y) \leq B(x, y)$;.
(A4) for each $x \in C, y \mapsto B(x, y)$ is convex and lower semicontinuous.
Lemma 2.1 [5]. Assume $\left\{\alpha_{n}\right\}$ is a sequence of nonnegative real numbers such that

$$
\begin{equation*}
\alpha_{n+1} \leq\left(1-\gamma_{n}\right) \alpha_{n}+\delta_{n}, \quad n \geq 0, \tag{2.3}
\end{equation*}
$$

where $\left\{\gamma_{n}\right\}$ is a sequence in $(0,1)$ and $\left\{\delta_{n}\right\}$ is a sequence in $\mathbb{R}$ such that
(i) $\sum_{n=1}^{\infty} \gamma_{n}=\infty$;
(ii) limsup ${ }_{n \rightarrow \infty} \delta_{n} / \gamma_{n} \leq 0$ or $\sum_{n=1}^{\infty}\left|\delta_{n}\right|<\infty$.

Then $\lim _{n \rightarrow \infty} \alpha_{n}=0$.
Lemma 2.2 [12]. Let C be a nonempty closed convex subset of $H$ and let $B$ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r>0$ and $x \in H$. Then, there exists $z \in C$ such that $B(z, y)+(1 / r)\langle y-z, z-x\rangle \geq 0, \forall y \in C$.

Lemma 2.3 [1]. Assume that $B: C \times C \rightarrow \mathbf{R}$ satisfies (A1)-(A4). For $r>0$ and $x \in H$, define a mapping $T_{r}: H \rightarrow C$ as follows:

$$
\begin{equation*}
T_{r}(x)=\left\{z \in C: B(z, y)+\frac{1}{r}\langle y-z, z-x\rangle \geq 0, \forall y \in C\right\} \tag{2.4}
\end{equation*}
$$

for all $z \in H$. Then, the following hold:
(1) $T_{r}$ is single-valued;
(2) $T_{r}$ is firmly nonexpansive, that is, for any $x, y \in H$,

$$
\begin{equation*}
\left\|T_{r} x-T_{r} y\right\|^{2} \leq\left\langle T_{r} x-T_{r} y, x-y\right\rangle ; \tag{2.5}
\end{equation*}
$$

(3) $F\left(T_{r}\right)=E P(B)$;
(4) $E P(B)$ is closed and convex.

Lemma 2.4. In a real Hilbert space $H$, there holds the the inequality $\|x+y\|^{2} \leq\|x\|^{2}+$ $2\langle y, x+y\rangle$, for all $x, y \in H$.

Lemma 2.5 [8]. Assume that $A$ is a strong positive linear bounded operator on a Hilbert space $H$ with coefficient $\bar{\gamma}>0$ and $0<\rho \leq\|A\|^{-1}$. Then $\|I-\rho A\| \leq 1-\rho \bar{\gamma}$.

## 3. Main results

Theorem 3.1. Let C be a nonempty closed convex subset of a Hilbert space H. Let B be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)-(A4) and let $S$ be a nonexpansive mapping of $C$ into $H$ such that $F(S) \cap E P(B) \neq \varnothing$ and a strongly positive linear bounded operator $A$ with coefficient $\bar{\gamma}>0$. Assume that $0<\gamma<\bar{\gamma} / \alpha$. Let $f$ be a contraction of $H$ into itself with a coefficient $\alpha(0<\alpha<1)$ and let $\left\{x_{n}\right\}$ and $\left\{y_{n}\right\}$ be sequences generated by $x_{1} \in H$ and

$$
\begin{gather*}
B\left(y_{n}, u\right)+\frac{1}{r_{n}}\left\langle u-y_{n}, y_{n}-x_{n}\right\rangle \geq 0, \quad \forall u \in C,  \tag{3.1}\\
x_{n+1}=\alpha_{n} \gamma f\left(x_{n}\right)+\left(I-\alpha_{n} A\right) S y_{n}
\end{gather*}
$$

for all $n$, where $\left\{\alpha_{n}\right\} \subset[0,1]$ and $\left\{r_{n}\right\} \subset(0, \infty)$ satisfy
(C1) $\lim _{n \rightarrow \infty} \alpha_{n}=0$;
(C2) $\sum_{n=1}^{\infty} \alpha_{n}=\infty$;
(C3) $\sum_{n=1}^{\infty}\left|\alpha_{n+1}-\alpha_{n}\right|<\infty$ and $\sum_{n=1}^{\infty}\left|r_{n+1}-r_{n}\right|<\infty$;
(C4) $\liminf _{n \rightarrow \infty} r_{n}>0$.
Then, both $\left\{x_{n}\right\}$ and $\left\{y_{n}\right\}$ converge strongly to $q \in F(S) \cap E P(B)$, where $q=P_{F(S) \cap E P(B)}(\gamma f+$ $(I-A))(q)$, which solves some variation inequality:

$$
\begin{equation*}
\langle\gamma f(q)-A q, q-p\rangle \leq 0, \quad \forall p \in F(S) \cap E P(B) . \tag{3.2}
\end{equation*}
$$

Proof. Since $\alpha_{n} \rightarrow 0$ by the condition (C1), we may assume, with no loss of generality, that $\alpha_{n}<\|A\|^{-1}$ for all $n$. From Lemma 2.5, we know that if $0<\rho \leq\|A\|^{-1}$, then $\|I-\rho A\| \leq$ $1-\rho \bar{\gamma}$. We will assume that $\|I-A\| \leq 1-\bar{\gamma}$.

Now, we observe that $\left\{x_{n}\right\}$ is bounded. Indeed, pick $p \in F(S) \cap E P(B)$. Since $y_{n}=$ $T_{r_{n}} x_{n}$, we have

$$
\begin{equation*}
\left\|y_{n}-p\right\|=\left\|T_{r_{n}} x_{n}-T_{r_{n}} p\right\| \leq\left\|x_{n}-p\right\| . \tag{3.3}
\end{equation*}
$$

It follows that

$$
\begin{align*}
\left\|x_{n+1}-p\right\| & =\left\|\alpha_{n}\left(\gamma f\left(x_{n}\right)-A p\right)+\left(I-\alpha_{n} A\right)\left(S y_{n}-p\right)\right\| \\
& \leq\left[1-(\bar{\gamma}-\gamma \alpha) \alpha_{n}\right]\left\|x_{n}-p\right\|+\alpha_{n}\|\gamma f(p)-A p\|, \tag{3.4}
\end{align*}
$$

which gives that $\left\|x_{n}-p\right\| \leq \max \left\{\left\|x_{0}-p\right\|,\|\gamma f(p)-A p\| /(\bar{\gamma}-\gamma \alpha)\right\}, n \geq 0$. Therefore, we obtain that $\left\{x_{n}\right\}$ is bounded. So is $\left\{y_{n}\right\}$. Next, we show that

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|x_{n+1}-x_{n}\right\|=0 \tag{3.5}
\end{equation*}
$$

Observing that $y_{n}=T_{r_{n}} x_{n}$ and $y_{n+1}=T_{r_{n+1}} x_{n+1}$, we have

$$
\begin{gather*}
B\left(y_{n}, u\right)+\frac{1}{r_{n}}\left\langle u-y_{n}, y_{n}-x_{n}\right\rangle \geq 0, \quad \forall u \in C,  \tag{3.6}\\
B\left(y_{n+1}, u\right)+\frac{1}{r_{n+1}}\left\langle u-y_{n+1}, y_{n+1}-x_{n+1}\right\rangle \geq 0, \quad \forall u \in C . \tag{3.7}
\end{gather*}
$$

Putting $u=y_{n+1}$ in (3.6) and $u=y_{n}$ in (3.7), we have

$$
\begin{equation*}
B\left(y_{n}, y_{n+1}\right)+\frac{1}{r_{n}}\left\langle y_{n+1}-y_{n}, y_{n}-x_{n}\right\rangle \geq 0 \tag{3.8}
\end{equation*}
$$

and $B\left(y_{n+1}, y_{n}\right)+\left(1 / r_{n+1}\right)\left\langle y_{n}-y_{n+1}, y_{n+1}-x_{n+1}\right\rangle \geq 0$. It follows from (A2) that

$$
\begin{equation*}
\left\langle y_{n+1}-y_{n}, \frac{y_{n}-x_{n}}{r_{n}}-\frac{y_{n+1}-x_{n+1}}{r_{n+1}}\right\rangle \geq 0 . \tag{3.9}
\end{equation*}
$$

That is, $\left\langle y_{n+1}-y_{n}, y_{n}-y_{n+1}+y_{n+1}-x_{n}-\left(r_{n} / r_{n+1}\right)\left(y_{n+1}-x_{n+1}\right)\right\rangle \geq 0$. Without loss of generality, let us assume that there exists a real number $m$ such that $r_{n}>m>0$ for all $n$. It follows that

$$
\begin{equation*}
\left\|y_{n+1}-y_{n}\right\|^{2} \leq\left\|y_{n+1}-y_{n}\right\|\left(\left\|x_{n+1}-x_{n}\right\|+\left|1-\frac{r_{n}}{r_{n+1}}\right|\left\|y_{n+1}-x_{n+1}\right\|\right) \tag{3.10}
\end{equation*}
$$

It follows that

$$
\begin{equation*}
\left\|y_{n+1}-y_{n}\right\| \leq\left\|x_{n+1}-x_{n}\right\|+M_{1}\left|r_{n+1}-r_{n}\right| \tag{3.11}
\end{equation*}
$$

where $M_{1}$ is an appropriate constant such that $M_{1} \geq \sup _{n \geq 1}\left\|y_{n}-x_{n}\right\|$. Observe that

$$
\begin{align*}
\left\|x_{n+2}-x_{n+1}\right\| \leq & \left(1-\alpha_{n+1} \bar{\gamma}\right)\left\|y_{n+1}-y_{n}\right\|+\left|\alpha_{n+1}-\alpha_{n}\right|\left\|A S y_{n}\right\| \\
& +\gamma\left[\alpha_{n+1} \alpha\left\|x_{n+1}-x_{n}\right\|+\left|\alpha_{n+1}-\alpha_{n}\right|\left\|f\left(x_{n}\right)\right\|\right] . \tag{3.12}
\end{align*}
$$

Substitute (3.11) into (3.12) yields that

$$
\begin{equation*}
\left\|x_{n+2}-x_{n+1}\right\| \leq\left[1-(\bar{\gamma}-\gamma \alpha) \alpha_{n+1}\right]\left\|x_{n+1}-x_{n}\right\|+M_{2}\left(2\left|\alpha_{n+1}-\alpha_{n}\right|+\left|r_{n+1}-r_{n}\right|\right), \tag{3.13}
\end{equation*}
$$

where $M_{2}$ is an appropriate constant. An application of Lemma 2.1 to (3.13) implies that

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|x_{n+1}-x_{n}\right\|=0 \tag{3.14}
\end{equation*}
$$

Observing (3.11), (3.14), and condition (C3), we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|y_{n+1}-y_{n}\right\|=0 \tag{3.15}
\end{equation*}
$$

Since $x_{n}=\alpha_{n-1} \gamma f\left(x_{n-1}\right)+\left(I-\alpha_{n-1} A\right) S y_{n-1}$, we have

$$
\begin{equation*}
\left\|x_{n}-S y_{n}\right\| \leq \alpha_{n-1}\left\|\gamma f\left(x_{n}\right)-A S y_{n-1}\right\|+\left\|y_{n-1}-y_{n}\right\|, \tag{3.16}
\end{equation*}
$$

which combines with $\alpha_{n} \rightarrow 0$, and (3.15) gives that

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|x_{n}-S y_{n}\right\|=0 \tag{3.17}
\end{equation*}
$$

For $p \in F(S) \cap E P(B)$, we have

$$
\begin{align*}
\left\|y_{n}-p\right\|^{2} & =\left\|T_{r_{n}} x_{n}-T_{r_{n}} p\right\|^{2} \leq\left\langle T_{r_{n}} x_{n}-T_{r_{n}} p, x_{n}-p\right\rangle=\left\langle y_{n}-p, x_{n}-p\right\rangle \\
& =\frac{1}{2}\left(\left\|y_{n}-p\right\|^{2}+\left\|x_{n}-p\right\|^{2}-\left\|x_{n}-y_{n}\right\|^{2}\right) \tag{3.18}
\end{align*}
$$

and hence $\left\|y_{n}-p\right\|^{2} \leq\left\|x_{n}-p\right\|^{2}-\left\|x_{n}-y_{n}\right\|^{2}$. It follows that

$$
\begin{align*}
\left\|x_{n+1}-p\right\|^{2}= & \left\|\alpha_{n}\left(\gamma f\left(x_{n}\right)-A p\right)+\left(I-\alpha_{n} A\right)\left(S y_{n}-p\right)\right\|^{2} \\
\leq & \alpha_{n}\left\|\gamma f\left(x_{n}\right)-A p\right\|^{2}+\left\|x_{n}-p\right\|^{2}-\left(1-\alpha_{n} \bar{\gamma}\right)\left\|x_{n}-y_{n}\right\|^{2}  \tag{3.19}\\
& +2 \alpha_{n}\left(1-\alpha_{n} \bar{\gamma}\right)\left\|\gamma f\left(x_{n}\right)-A p\right\|\left\|y_{n}-p\right\| .
\end{align*}
$$

That is,

$$
\begin{align*}
\left(1-\alpha_{n} \bar{\gamma}\right)\left\|x_{n}-y_{n}\right\|^{2} \leq & \alpha_{n}\left\|\gamma f\left(x_{n}\right)-A p\right\|^{2} \\
& +\left(\left\|x_{n}-p\right\|+\left\|x_{n+1}-p\right\|\right)\left\|x_{n}-x_{n+1}\right\|  \tag{3.20}\\
& +2 \alpha_{n}\left(1-\alpha_{n} \bar{\gamma}\right)\left\|\gamma f\left(x_{n}\right)-A p\right\|\left\|y_{n}-p\right\| .
\end{align*}
$$

It follows from $\lim _{n \rightarrow \infty} \alpha_{n}=0$ that

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|x_{n}-y_{n}\right\|=0 \tag{3.21}
\end{equation*}
$$

Observe from $\left\|S y_{n}-y_{n}\right\| \leq\left\|S y_{n}-x_{n}\right\|+\left\|x_{n}-y_{n}\right\|$, which combines with (3.17) and (3.21), that

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|S y_{n}-y_{n}\right\|=0 \tag{3.22}
\end{equation*}
$$

On the other hand, we have

$$
\begin{equation*}
\left\|x_{n}-S x_{n}\right\|=\left\|S x_{n}-S y_{n}\right\|+\left\|S y_{n}-x_{n}\right\| \leq\left\|x_{n}-y_{n}\right\|+\left\|S y_{n}-x_{n}\right\| . \tag{3.23}
\end{equation*}
$$

It follows from (3.17) and (3.21) that $\lim _{n \rightarrow \infty}\left\|S x_{n}-x_{n}\right\|=0$. Observe that $P_{F(S) \cap E P(B)}(\gamma f+$ $(I-A))$ is a contraction. Indeed, $\forall x, y \in H$, we have

$$
\begin{align*}
& \left\|P_{F(S) \cap E P(B)}(\gamma f+(I-A))(x)-P_{F(S) \cap E P(B)}(\gamma f+(I-A))(y)\right\| \\
& \quad \leq \gamma\|f(x)-f(y)\|+\|I-A\|\|x-y\|  \tag{3.24}\\
& \quad \leq \gamma \alpha\|x-y\|+(1-\bar{\gamma})\|x-y\|<\|x-y\| .
\end{align*}
$$

Banach's contraction mapping principle guarantees that $P_{F(S) \cap E P(B)}(\gamma f+(I-A))$ has a unique fixed point, say $q \in H$. That is, $q=P_{F(S) \cap E P(B)}(\gamma f+(I-A))(q)$. Next, we show that

$$
\begin{equation*}
\underset{n \rightarrow \infty}{\limsup }\left\langle\gamma f(q)-A q, x_{n}-q\right\rangle \leq 0 \tag{3.25}
\end{equation*}
$$

To see this, we choose a subsequence $\left\{x_{n_{i}}\right\}$ of $\left\{x_{n}\right\}$ such that

$$
\begin{equation*}
\limsup _{n \rightarrow \infty}\left\langle\gamma f(q)-A q, x_{n}-q\right\rangle=\lim _{i \rightarrow \infty}\left\langle\gamma f(q)-A q, x_{n_{i}}-q\right\rangle \tag{3.26}
\end{equation*}
$$

Correspondingly, there exists a subsequence $\left\{y_{n_{i}}\right\}$ of $\left\{y_{n}\right\}$. Since $\left\{y_{n_{i}}\right\}$ is bounded, there exists a subsequence $\left\{y_{n_{i j}}\right\}$ of $\left\{y_{n_{i}}\right\}$ which converges weakly to $w$. Without loss of generality, we can assume that $y_{n_{i}}$ harpoonupw. From (3.22), we obtain $S y_{n_{i}}$ harpoonupw.

Next, we show $w \in F(S) \cap E P(B)$. First, we prove $w \in E P(B)$. Since $y_{n}=T_{r_{n}} x_{n}$, we have $B\left(y_{n}, u\right)+\left(1 / r_{n}\right)\left\langle u-y_{n}, y_{n}-x_{n}\right\rangle \geq 0$ for all $u \in C$. It follows from (A2) that $\langle u-$ $\left.y_{n},\left(y_{n}-x_{n}\right) / r_{n}\right\rangle \geq B\left(u, y_{n}\right)$. Since $\left(y_{n_{i}}-x_{n_{i}}\right) / r_{n_{i}} \rightarrow 0$, $y_{n_{i}}$ harpoonupw, and (A4), we have $B(u, w) \leq 0$ for all $u \in C$. For $t$ with $0<t \leq 1$ and $u \in C$, let $u_{t}=t u+(1-t) w$. Since $u \in C$ and $w \in C$, we have $u_{t} \in C$ and hence $B\left(u_{t}, w\right) \leq 0$. So, from (A1) and (A4), we have $0=B\left(u_{t}, u_{t}\right) \leq t B\left(u_{t}, u\right)+(1-t) B\left(u_{t}, w\right) \leq t B\left(u_{t}, u\right)$. That is, $B\left(u_{t}, u\right) \geq 0$. It follows from (A3) that $B(w, u) \geq 0$ for all $u \in C$ and hence $w \in E P(B)$. Since Hilbert spaces are Opial's spaces, from (3.22), we have

$$
\begin{equation*}
\liminf _{n \rightarrow \infty}\left\|y_{n_{i}}-w\right\| \leq \liminf _{n \rightarrow \infty}\left\|S y_{n_{i}}-S w\right\| \leq \liminf _{n \rightarrow \infty}\left\|y_{n_{i}}-w\right\|<\liminf _{n \rightarrow \infty}\left\|y_{n_{i}}-S w\right\| \tag{3.27}
\end{equation*}
$$

which derives a contradiction. Thus, we have $w \in F(S)$. That is, $w \in F(S) \cap E P(B)$. Since $q=P_{F(S) \cap E P(B)} f(q)$, we have

$$
\begin{equation*}
\limsup _{n \rightarrow \infty}\left\langle\gamma f(q)-A q, x_{n}-q\right\rangle=\lim _{i \rightarrow \infty}\left\langle\gamma f(q)-A q, x_{n_{i}}-q\right\rangle=\langle\gamma f(q)-A q, w-q\rangle \leq 0 \tag{3.28}
\end{equation*}
$$

That is, (3.25) holds. Next, it follows Lemma 2.4 that

$$
\begin{align*}
& \left\|x_{n+1}-q\right\|^{2} \\
& \quad \leq\left(1-\alpha_{n} \bar{\gamma}\right)^{2}\left\|x_{n}-q\right\|^{2}+\alpha_{n} \gamma \alpha\left(\left\|x_{n}-q\right\|^{2}+\left\|x_{n+1}-q\right\|^{2}\right)+2 \alpha_{n}\left\langle\gamma f(q)-A q, x_{n+1}-q\right\rangle \tag{3.29}
\end{align*}
$$

which implies that

$$
\begin{align*}
\left\|x_{n+1}-q\right\|^{2} \leq & {\left[1-\frac{2 \alpha_{n}(\bar{\gamma}-\alpha \gamma)}{1-\alpha_{n} \gamma \alpha}\right]\left\|x_{n}-q\right\|^{2} } \\
& +\frac{2 \alpha_{n}(\bar{\gamma}-\alpha \gamma)}{1-\alpha_{n} \gamma \alpha}\left[\frac{1}{\bar{\gamma}-\alpha \gamma}\left\langle\gamma f(q)-A q, x_{n+1}-q\right\rangle+\frac{\alpha_{n} \bar{\gamma}^{2}}{2(\bar{\gamma}-\alpha \gamma)} M_{3}\right] \tag{3.30}
\end{align*}
$$

where $M_{3}$ is an appropriate constant such that $M_{3}=\sup _{n \rightarrow \infty}\left\|x_{n}-q\right\|$ for all $n$. Put $l_{n}=2 \alpha_{n}\left(\bar{\gamma}-\alpha_{n} \gamma\right) /\left(1-\alpha_{n} \alpha \gamma\right)$ and $t_{n}=(1 /(\bar{\gamma}-\alpha \gamma))\left\langle\gamma f(q)-A q, x_{n+1}-q\right\rangle+\left(\alpha_{n} \bar{\gamma}^{2} / 2(\bar{\gamma}-\right.$ $\alpha \gamma)) M_{3}$. That is,

$$
\begin{equation*}
\left\|x_{n+1}-q\right\|^{2} \leq\left(1-l_{n}\right)\left\|x_{n}-q\right\|+l_{n} t_{n} . \tag{3.31}
\end{equation*}
$$

It follows from condition (C1), (C2), and (3.25) that $\lim _{n \rightarrow \infty} l_{n}=0, \sum_{n=1}^{\infty} l_{n}=\infty$, and $\limsup { }_{n \rightarrow \infty} t_{n} \leq 0$. Apply Lemma 2.1 to (3.31) to conclude $x_{n} \rightarrow q$.

## 4. Applications

Theorem 4.1. Let C be a nonempty closed convex subset of a Hilbert space $H$. and let $S$ be a nonexpansive mapping of $C$ into $H$ such that $F(S) \neq \varnothing$. Let A be a strongly positive linear bounded operator with coefficient $\bar{\gamma}>0$. Assume that $0<\gamma<\bar{\gamma} / \alpha$. Let $f$ be a contraction of $H$ into itself with a coefficient $\alpha(0<\alpha<1)$ and let $\left\{x_{n}\right\}$ be a sequence generated by $x_{1} \in H$ and

$$
\begin{equation*}
x_{n+1}=\alpha_{n} \gamma f\left(x_{n}\right)+\left(I-\alpha_{n} A\right) S P_{C} x_{n} \tag{4.1}
\end{equation*}
$$

for all $n$, where $\alpha_{n} \subset[0,1]$ and $\left\{r_{n}\right\} \subset(0, \infty)$ satisfy
(C1) $\lim _{n \rightarrow \infty} \alpha_{n}=0$;
(C2) $\sum_{n=1}^{\infty} \alpha_{n}=\infty$;
(C3) $\sum_{n=1}^{\infty}\left|\alpha_{n+1}-\alpha_{n}\right|<\infty$.
Then $\left\{x_{n}\right\}$ converges strongly to $q \in F(S)$, where $q=P_{F(S)}(\gamma f+(I-A))(q)$.
Proof. Put $B(x, y)=0$ for all $x, y \in C$ and $\left\{r_{n}\right\}=1$ for all $n$ in Theorem 3.1. Then we have $y_{n}=P_{C} x_{n}$. So, the sequence $\left\{x_{n}\right\}$ converges strongly to $q \in F(S)$, where $q=P_{F(S)}(\gamma f+$ $(I-A))(q)$.

Remark 4.2. It is very clear that our algorithm with a variational regularization parameter $\left\{r_{n}\right\}$ has certain advantages over the algorithm with a fixed regularization parameter $r$. In some setting, when the regularization parameter $\left\{r_{n}\right\}$ depends on the iterative step $n$, the algorithm may converge to some solution $Q$-superlinearly, that is, the algorithm has a faster convergence rate when the regularization parameter $\left\{r_{n}\right\}$ depends on $n$, see [13] and the references therein for more information.

## Acknowledgment

This project is supported by the National Natural Science Foundation of China under Grant no. 10771050.

## References

[1] P. L. Combettes and S. A. Hirstoaga, "Equilibrium programming in Hilbert spaces," Journal of Nonlinear and Convex Analysis, vol. 6, no. 1, pp. 117-136, 2005.
[2] S. D. Flåm and A. S. Antipin, "Equilibrium programming using proximal-like algorithms," Mathematical Programming, vol. 78, no. 1, pp. 29-41, 1997.
[3] S. Takahashi and W. Takahashi, "Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces," Journal of Mathematical Analysis and Applications, vol. 331, no. 1, pp. 506-515, 2007.
[4] F. Deutsch and I. Yamada, "Minimizing certain convex functions over the intersection of the fixed point sets of nonexpansive mappings," Numerical Functional Analysis and Optimization, vol. 19, no. 1-2, pp. 33-56, 1998.
[5] H.-K. Xu, "Iterative algorithms for nonlinear operators," Journal of the London Mathematical Society, vol. 66, no. 1, pp. 240-256, 2002.
[6] H. K. Xu, "An iterative approach to quadratic optimization," Journal of Optimization Theory and Applications, vol. 116, no. 3, pp. 659-678, 2003.
[7] I. Yamada, "The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings," in Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications (Haifa, 2000), D. Butnariu, Y. Censor, and S. Reich, Eds., vol. 8 of Studies in Computational Mathematics, pp. 473-504, North-Holland, Amsterdam, The Netherlands, 2001.
[8] G. Marino and H.-K. Xu, "A general iterative method for nonexpansive mappings in Hilbert spaces," Journal of Mathematical Analysis and Applications, vol. 318, no. 1, pp. 43-52, 2006.
[9] A. Moudafi, "Viscosity approximation methods for fixed-points problems," Journal of Mathematical Analysis and Applications, vol. 241, no. 1, pp. 46-55, 2000.
[10] R. Wittmann, "Approximation of fixed points of nonexpansive mappings," Archiv der Mathematik, vol. 58, no. 5, pp. 486-491, 1992.
[11] Z. Opial, "Weak convergence of the sequence of successive approximations for nonexpansive mappings," Bulletin of the American Mathematical Society, vol. 73, pp. 591-597, 1967.
[12] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," The Mathematics Student, vol. 63, no. 1-4, pp. 123-145, 1994.
[13] M. V. Solodov and B. F. Svaiter, "A truly globally convergent Newton-type method for the monotone nonlinear complementarity problem," SIAM Journal on Optimization, vol. 10, no. 2, pp. 605-625, 2000.

Meijuan Shang: Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China; Department of Mathematics, Shijiazhuang University, Shijiazhuang 050035, China
Email address: meijuanshang@yahoo.com.cn
Yongfu Su: Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China
Email address: suyongfu@tjpu.edu.cn
Xiaolong Qin: Department of Mathematics, Gyeongsang National University, Chinju 660-701, Korea Email address: qxlxajh@163.com

