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Research Article

A New Iterative Algorithm for Approximating Common Fixed Points for Asymptotically Nonexpansive Mappings

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Suppose that K is a nonempty closed convex subset of a real uniformly convex and smooth Banach space E with P as a sunny nonexpansive retraction. Let $T_1, T_2 : K \to E$ be two weakly inward and asymptotically nonexpansive mappings with respect to P with sequences $\{K_n\}, \{l_n\} \subset [1,\infty)$, $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} l_n = 1$, $F(T_1) \cap F(T_2) = \{x \in K : T_1x = T_2x = x\} \neq \varnothing$, respectively. Suppose that $\{x_n\}$ is a sequence in K generated iteratively by $x_1 \in K$, $x_{n+1} = \alpha_n x_n + \beta_n (PT_1)^n x_n + \gamma_n (PT_2)^n x_n$, for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon > 0$ which satisfy condition $\alpha_n + \beta_n + \gamma_n = 1$. Then, we have the following. (1) If one of T_1 and T_2 is completely continuous or demicompact and $\sum_{n=1}^{\infty}(k_n - 1) < \infty, \sum_{n=1}^{\infty}(l_n - 1) < \infty$, then the strong convergence of $\{x_n\}$ to some $q \in F(T_1) \cap F(T_2)$ is established. (2) If E is a real uniformly convex Banach space satisfying Opial's condition or whose norm is Fréchet differentiable, then the weak convergence of $\{x_n\}$ to some $q \in F(T_1) \cap F(T_2)$ is proved.

1. Introduction

Let K be a nonempty closed convex subset of a real uniformly convex Banach space E. A self-mapping $T : K \to K$ is said to be nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in K$. A self-mapping $T : K \to K$ is called asymptotically nonexpansive if there exist sequences $\{k_n\} \subset [1,\infty)$, $k_n \to 1$ as $n \to \infty$ such that

$$\left\|T^n(x) - T^n(y)\right\| \leq k_n \|x - y\|, \quad \forall x, y \in K,\ n \geq 1. \tag{1.1}$$

A self-mapping $T : K \to K$ is said to be uniformly L-Lipschitzian if there exists constant $L > 0$ such that

$$||T^n(x) - T^n(y)|| \le L\|x - y\|, \quad \forall x, y \in K,\ n \ge 1. \tag{1.2}$$

A self-mapping $T : K \to K$ is called asymptotically quasi-nonexpansive if $F(T) \neq \varnothing$ and there exist sequences $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$||T^n(x) - p|| \le k_n\|x - p\|, \quad \forall x \in K,\ p \in F(T),\ n \ge 1. \tag{1.3}$$

It is clear that, if T is an asymptotically nonexpansive mapping from K into itself with a fixed point in K, then T is asymptotically quasi-nonexpansive, but the converse may be not true.

As a generalization of the class of nonexpansive maps, the class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [1] in 1972, who proved that if K is a nonempty bounded closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping of K, then T has a fixed point.

In 1978, Bose [2] first proved that if K is a nonempty bounded closed convex subset of a real uniformly convex Banach space E satisfying Opial's condition and $T : K \to K$ is an asymptotically nonexpansive mapping, then the sequence $\{T^n x\}$ converges weakly to a fixed point of T, provided that T is asymptotically regular at $x \in K$, that is,

$$\lim_{n\to\infty} ||T^n x - T^{n+1}x|| = 0. \tag{1.4}$$

In 1982, Passty [3] proved that Bose's weak convergence theorem still holds if Opial's condition is replaced by the condition that E has a Fréchet differentiable norm.

Furthermore, Tan and Xu [4, 5] later proved that the asymptotic regularity of T at x can be weakened to the weakly asymptotic regularity of T at x, that is,

$$\omega - \lim_{n\to\infty} (T^n x - T^{n+1}x) = 0. \tag{1.5}$$

In all the above results ($x_n = T^n x$), the asymptotic regularity of T at $x \in K$ is equivalent to $x_n - Tx_n \to 0$ as $n \to \infty$. We wish that the later is a conclusion rather than an assumption.

In 1991, Schu [6, 7] introduced a modified Mann iterative algorithm to approximate fixed points of asymptotically nonexpansive maps without assuming the asymptotic regularity of T at $x \in K$. Schu established the conclusion that $x_n - Tx_n \to 0$ as $n \to \infty$ by choosing properly iterative parameters $\{\alpha_n\}$.

Schu's iterative algorithm was defined as follows:

$$\begin{gathered} x_1 \in K, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \ge 1. \end{gathered} \tag{1.6}$$

Since then, many authors have developed Schu's algorithm and results. Rhoades [8] and Tan and Xu [4] generalized Schu's iterative algorithm to the modified Ishikawa iterative algorithm and extended the main results of Schu to uniformly convex Banach spaces.

Furthermore, Osilike and Aniagbosor [9] improved the main results of Schu [6]. Schu [7] and Rhoades [8], without assuming the boundedness condition, imposed on K. Recently, Chang et al. [10] established a more general demiclosed principle and improved the corresponding results of Bose [2], Górnicki [11], Passty [3], Reich [12], Schu [6, 7], and Tan and Xu [4, 5].

Some iterative algorithms for approximating fixed points of nonself nonexpansive mappings have been studied by various authors (see [13–18]). However, iterative algorithms for approximating fixed points of nonself asymptotically nonexpansive mappings have not been paid too much attention. The main reason is the fact that when T is not a self-mapping, the mapping T^n is nonsensical. Recently, in order to establish the convergence theorems for non-self-asymptotically nonexpansive mappings, Chidume et al. [19] introduced the following definition.

Definition 1.1. Let K be a nonempty subset of real-normed linear space E. Let $P: E \to K$ be the nonexpansive retraction of E onto K.

(1) A non-self-mapping $T: K \to E$ is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\left\|T(PT)^{n-1}(x) - T(PT)^{n-1}(y)\right\| \le k_n\|x-y\|, \quad \forall x,y \in K,\ n \ge 1. \tag{1.7}$$

(2) T is said to be *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\left\|T(PT)^{n-1}(x) - T(PT)^{n-1}(y)\right\| \le L\|x-y\|, \quad \forall x,y \in K,\ n \ge 1. \tag{1.8}$$

By using the following iterative algorithm:

$$\begin{gathered} x_1 \in K, \\ x_{n+1} = P\big((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n\big), \quad \forall n \ge 1, \end{gathered} \tag{1.9}$$

Chidume et al. [19] established the following demiclosed principle, strong and weak convergence theorems for non-self-asymptotically nonexpansive mappings in uniformly convex Banach spaces.

Theorem 1.2 [19]. *Let E be a uniformly convex Banach space, K a nonempty closed convex subset of E. Let $T: K \to E$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1,\infty)$ and $k_n \to 1$ as $n \to \infty$. Then $I - T$ is demiclosed at zero.*

Theorem 1.3 [19]. *Let E be a uniformly convex Banach space and let K be a nonempty closed convex subset of E. Let $T: K \to E$ be completely continuous and asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}, (k_n^2 - 1) < \infty$, and $F(T) \neq \emptyset$. Let $\{\alpha_n\} \subset (0,1)$ be a sequence such that $\epsilon \le 1-\alpha_n \le 1-\epsilon$ for all $n \ge 1$ and some $\epsilon > 0$. For an arbitrary point $x_1 \in K$, define the sequence $\{x_n\}$ by (1.9). Then, $\{x_n\}$ converges strongly to some fixed point of T.*

Theorem 1.4 [19]. *Let E be a uniformly convex Banach space which has a Fréchet differentiable norm and let K be a nonempty closed convex subset of E. Let $T: K \to E$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\} \subset (0,1)$ be a sequence such that $\epsilon \le 1-\alpha_n \le 1-\epsilon$ for all $n \ge 1$*

and some $\epsilon > 0$. *For an arbitrary point* $x_1 \in K$, *let* $\{x_n\}$ *be the sequence defined by (1.9). Then* $\{x_n\}$ *converges weakly to some fixed point of* T.

We now introduce the following definition.

Definition 1.5. Let K be a nonempty subset of real normed linear space E. Let $P : E \to K$ be a nonexpansive retraction of E onto K.

(1) A non-self-mapping $T : K \to E$ is called *asymptotically nonexpansive* with respect to P if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|(PT)^n x - (PT)^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in K,\ n \geq 1. \tag{1.10}$$

(2) T is said to be *uniformly L-Lipschitzian* with respect to P if there exists a constant $L > 0$ such that

$$\|(PT)^n x - (PT)^n y\| \leq L\|x - y\|, \quad \forall x, y \in K,\ n \geq 1. \tag{1.11}$$

Remark 1.6. If T is self-mapping, then P becomes the identity mapping, so that (1.7), (1.8), and (1.9) reduce to (1.1), (1.2), and (1.6), respectively.

We remark in the passing that if $T : K \to E$ is asymptotically nonexpansive in light of (1.7) and $P : E \to K$ is a nonexpansive retraction, then $PT : K \to K$ is asymptotically nonexpansive in light of (1.1). Indeed, by definition (1.7), we have

$$\begin{aligned} &\|(PT)^n x - (PT)^n y\| \\ &\quad = \|PT(PT)^{n-1}x - PT(PT)^{n-1}y\| \\ &\quad \leq \|T(PT)^{n-1}x - T(PT)^{n-1}y\| \\ &\quad \leq k_n\|x - y\|, \quad \forall x, y \in K,\ n \geq 1. \end{aligned} \tag{1.12}$$

Conversely, it may not be true.

It is our purpose in this paper to introduce a new iterative algorithm (see (2.6)) for approximating common fixed points of two non-self-asymptotically nonexpansive mappings with respect to P and to prove some strong and weak convergence theorems for such mappings in uniformly convex Banach spaces. As a consequence, the main results of Chidume et al. [19] are deduced.

2. Preliminaries

In this section, we will introduce a new iterative algorithm and prove a new demiclosedness principle for a non-self-asymptotically nonexpansive mapping in the sense of (1.10).

Let E be a Banach space with dimension $E \geq 2$. The modulus of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x + y)\right\| : \|x\| = 1,\ \|y\| = 1,\ \epsilon = \|x - y\|\right\}. \tag{2.1}$$

A Banach space E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0,2]$.

A subset K of E is said to be retract if there exists a continuous mapping $P: E \to K$ such that $Px = x$ for all $x \in K$. Every closed convex subset of a uniformly convex Banach space is a retraction. A mapping $P: E \to E$ is said to be a retraction if $P^2 = P$. Note that if a mapping P is a retraction, then $Pz = z$ for all $z \in R(P)$, the range of P.

Let E be a Banach space and let C, D be subsets of E. Then, a mapping $P: C \to D$ is said to be sunny if

$$P(Px + t(x - Px)) = Px, \tag{2.2}$$

whenever $Px + t(x - Px) \in C$ for all $x \in C$ and $t \geq 0$.

Let K be a subset of a Banach space E. For all $x \in K$, define a set $I_K(x)$ by

$$I_K(x) = \{x + \lambda(y - x) : \lambda > 0,\ y \in K\}. \tag{2.3}$$

A non-self-mapping $T: K \to E$ is said to be inward if $Tx \in I_k(x)$ for all $x \in K$ and T is said to be weakly inward if $Tx \in \overline{I_K(x)}$ for all $x \in K$.

The following facts are well known (see [20, 18]).

Lemma 2.1. *Let C be a nonempty convex subset of a smooth Banach space E, $C_0 \subset C$, let $J: E \to E^*$ be the normalized duality mapping of E, and let $P: C \to C_0$ be a retraction. Then, the following statements are equivalent:*

(1) $\langle x - Px, J(y - Px)\rangle \leq 0$ *for all* $x \in C$ *and* $y \in C_0$*;*

(2) *P is both sunny and nonexpansive.*

Lemma 2.2. *Let E be a real smooth Banach space, let K be a nonempty closed convex subset of E with P as a sunny nonexpansive retraction, and let $T: K \to E$ be a mapping satisfying weakly inward condition. Then $F(PT) = F(T)$.*

A Banach space E is said to satisfy Opial's condition if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y|| \tag{2.4}$$

for all $y \in E$ with $y \neq x$, where $x_n \rightharpoonup x$ denotes that $\{x_n\}$ converges weakly to x. It is well known that Hilbert space and l^p $(1 < p < \infty)$ admit Opial's property, while L^p does not unless $p = 2$.

Let E be a Banach space and $S(E) = \{x \in E : \|x\| = 1\}$. The space E is said to be smooth if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.5}$$

exists for all $x, y \in S(E)$. For any $x, y \in E$ $(x \neq 0)$, we denote this limit by (x, y). The norm $\|\cdot\|$ of E is said to be Fréchet differentiable if for all $x \in S(E)$, the limit (x, y) exists uniformly for all $y \in S(E)$.

A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be demiclosed at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges to $x^* \in D(T)$ and $\{Tx_n\}$ converges strongly to p, $Tx^* = p$.

Let E be a real normed linear space, let K be a nonempty closed convex subset of E which is also a nonexpansive retraction of E with a retraction P. Let $T_1 : K \to E$ and $T_2 : K \to E$ be two non-self-asymptotically nonexpansive mappings with respect to P. For approximating the common fixed points of two non-self-asymptotically nonexpansive mappings, we introduce the following iterative algorithm:

$$\begin{gathered} x_1 \in K, \\ x_{n+1} = \alpha_n x_n + \beta_n (PT_1)^n x_n + \gamma_n (PT_2)^n x_n, \quad \forall n \geq 1, \end{gathered} \tag{2.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three real sequences in $(0,1)$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$.

Lemma 2.3 [21]. *Let $\{\alpha_n\}$ and $\{t_n\}$ be two nonnegative real sequences satisfying*

$$\alpha_{n+1} \leq \alpha_n + t_n, \quad \forall n \geq 1. \tag{2.7}$$

If $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} \alpha_n$ exists.

The following lemma can be found in Zhou et al. [22].

Lemma 2.4 [22]. *Let E be a real uniformly convex Banach space and let $B_r(0)$ be the closed ball of E with centre at the origin and radius $r > 0$. Then, there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda \|x\|^2 + \mu \|y\|^2 + \gamma \|z\|^2 - \lambda\mu g(\|x - y\|) \tag{2.8}$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0,1]$ with $\lambda + \mu + \gamma = 1$.

The following demiclosedness principle for non-self-mapping follows from [10, Theorem 1].

Lemma 2.5. *Let E be a real smooth and uniformly convex Banach space and K a nonempty closed convex subset of E with P as a sunny nonexpansive retraction. Let $T : K \to E$ be a weakly inward and asymptotically nonexpansive mapping with respect to P with a sequence $\{k_n\} \subset [1,\infty)$ such that $\{k_n\} \to 1$ as $n \to \infty$. Then $I - T$ is demiclosed at zero, that is, $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$ imply that $Tx = x$.*

Proof. Suppose that $\{x_n\} \subset K$ converges weakly to $x^* \in K$ and $x_n - Tx_n \to 0$ as $n \to \infty$. We will prove that $Tx^* = x^*$. Indeed, since $\{x_n\} \subset K$, by the property of P, we have $Px_n = x_n$ for all $n \geq 1$ and so $x_n - PTx_n \to 0$ as $n \to \infty$. By Chang et al. [10, Theorem 1], we conclude that $x^* = PTx^*$. Since $F(PT) = F(T)$ by Lemma 2.2, we have $Tx^* = x^*$. This completes the proof. □

Remark 2.6. Lemma 2.5 extends Chang et al. [10, Theorem 1] to non-self-mapping case.

Using the proof lines of Reich [12, Proposition], then we can prove the following lemma.

Lemma 2.7. *Let K be a closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm and let $\{T_n : 1 \leq n \leq \infty\}$ be a family of Lipschitzian self-mappings of K with a nonempty common fixed point set F and a Lipschitzian constant* *sequence* $\{L_n\}$ *such that* $\sum_{n=1}^{\infty}(L_n - 1) < \infty$. *If* $x_1 \in K$ *and* $x_{n+1} = T_n x_n$ *for* $n \geq 1$, *then* $\lim_{n\to\infty}(f_1 - f_2, x_n)$ *exists for all* $f_1 \neq f_2 \in F$.

Remark 2.8. Lemma 2.7 is an extension of a proposition due to Reich [12].

3. Main results

In this section, we present some several strong and weak convergence theorems for two non-self-asymptotically nonexpansive mappings with respect to P.

Lemma 3.1. *Let* K *be a nonempty closed convex subset of a normed linear space* E. *Let* $T_1, T_2 : K \to E$ *be two non-self-asymptotically nonexpansive mappings with respect to* P *with sequences* $\{k_n\}, \{l_n\} \subset [1,\infty)$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, *respectively. Suppose that* $\{x_n\}$ *is the sequence defined by (2.6). If* $F(T_1) \cap F(T_2) \neq \emptyset$, *then* $\lim_{n\to\infty}\|x_n - q\|$ *and* $\lim_{n\to\infty}\|y_n - q\|$ *exist for any* $q \in F(T_1) \cap F(T_2)$.

Proof. For any $q \in F(T_1) \cap F(T_2)$, using the fact that P is nonexpansive and (2.6), then we have

$$\begin{aligned} \|x_{n+1} - q\| &= \|(\alpha_n x_n + \beta_n (PT_1)^n x_n + \gamma_n (PT_2)^n x_n) - Pq\| \\ &\leq \alpha_n \|x_n - q\| + \beta_n k_n \|x_n - q\| + \gamma_n l_n \|x_n - q\| \\ &\leq m_n \|x_n - q\|, \end{aligned} \tag{3.1}$$

where $m_n = \max\{k_n, l_n\}$ for all $n \geq 1$. It is clear that $\sum_{n=1}^{\infty}(m_n - 1) < \infty$ by the assumptions on $\{k_n\}$ and $\{l_n\}$. It follows from Lemma 2.3 that $\lim_{n\to\infty}\|x_n - q\|$ exists. This completes the proof. □

Lemma 3.2. *Let* K *be a nonempty closed convex subset of a real uniformly convex Banach space* E. *Let* $T_1, T_2 : K \to E$ *be two non-self-asymptotically nonexpansive mappings with respect to* P *with sequences* $\{k_n\}, \{l_n\} \subset [1,\infty)$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, *respectively. Suppose that* $\{x_n\}$ *is the sequence defined by (2.6), where* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are three sequences in* $[\epsilon, 1-\epsilon]$ *for some* $\epsilon > 0$. *If* $F(T_1) \cap F(T_2) \neq \emptyset$, *then*

$$\lim_{n\to\infty}\|x_n - (PT_1)x_n\| = \lim_{n\to\infty}\|x_n - (PT_2)x_n\| = 0. \tag{3.2}$$

Proof. From (2.6), by the property of P, and Lemma 2.4, we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \|\alpha_n x_n + \beta_n (PT_1)^n x_n + \gamma_n (PT_2)^n x_n - q\|^2 \\ &= \|\alpha_n (x_n - q) + \beta_n ((PT_1)^n x_n - q) + \gamma_n ((PT_2)^n x_n - q)\|^2 \\ &\leq \alpha_n \|x_n - q\|^2 + \beta_n \|(PT_1)^n x_n - q\|^2 + \gamma_n \|(PT_2)^n x_n - q\|^2 \\ &\quad - \alpha_n \beta_n g(\|x_n - (PT_1)^n x_n\|) \\ &\leq m_n^2 \|x_n - q\|^2 - \epsilon^2 g(\|x_n - (PT_1)^n x_n\|), \end{aligned} \tag{3.3}$$

which implies that $g(\|x_n - (PT_1)^n x_n\|) \to 0$ as $n \to \infty$. Since $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ being a continuous strictly increasing convex function, we have $x_n - (PT_1)^n x_n \to 0$ as

$n \to \infty$. Consequently, $x_n - (PT_1)x_n \to 0$ as $n \to \infty$. Similarly, we can prove that $x_n - (PT_2)x_n \to 0$ as $n \to \infty$. This completes the proof. □

THEOREM 3.3. *Let K be a nonempty closed convex subset of a real smooth uniformly convex Banach space E with P as a sunny nonexpansive retraction. Let $T_1, T_2 : K \to E$ be two weakly inward and asymptotically nonexpansive mappings with respect to P with sequences $\{k_n\}, \{l_n\} \subset [1,\infty)$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, respectively. Let $\{x_n\} \subset K$ be the sequence defined by (2.6), where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $[\epsilon, 1-\epsilon)$ for some $\epsilon > 0$. If one of T_1 and T_2 is completely continuous and $F(T_1) \cap F(T_2) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.*

Proof. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - q\|$ exists for any $q \in F$. It is sufficient to show that $\{x_n\}$ has a subsequence which converges strongly to a common fixed point of T_1 and T_2. By Lemma 3.2, $\lim_{n\to\infty} \|x_n - PT_1x_n\| = \lim_{n\to\infty} \|x_n - PT_2x_n\| = 0$. Suppose that T_1 is completely continuous. Noting that P is nonexpansive, we conclude that there exists subsequence $\{PT_1x_{n_j}\}$ of $\{PT_1x_n\}$ such that $PT_1x_{n_j} \to q$, and hence $x_{n_j} \to q$ as $j \to \infty$. By the continuity of P, T_1, and T_2, we have $q = PT_1q = PT_2q$, and so $q \in F(T_1) \cap F(T_2)$ by Lemma 2.2. Thus, $\{x_n\}$ converges strongly to a common fixed point q of T_1 and T_2. This completes the proof. □

THEOREM 3.4. *Let K be a nonempty closed convex subset of a real smooth and uniformly convex Banach space E with P as a sunny nonexpansive retraction. Let $T_1, T_2 : K \to E$ be two weakly inward asymptotically nonexpansive mappings with respect to P with sequences $\{k_n\}, \{l_n\} \subset [1,\infty)$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, respectively. Let $\{x_n\} \subset K$ be the sequence defined by (2.6), where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $[\epsilon, 1-\epsilon)$ for some $\epsilon > 0$. If one of T_1 and T_2 is demicompact and $F(T_1) \cap F(T_2) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of T_1 and T_2.*

Proof. Since one of T_1 and T_2 is demicompact, so is one of PT_1 and PT_2. Suppose that PT_1 is demicompact. Noting that $\{x_n\}$ is bounded, we assert that there exists a subsequence $\{PT_1x_{n_j}\}$ of $\{PT_1x_n\}$ such that $PT_1x_{n_j}$ converges strongly to q. By Lemma 3.2, we have $x_{n_j} \to q$ as $j \to \infty$. Since P, T_1, and T_2 are all continuous, we have $q = PT_1q = PT_2q$ and $q \in F(T_1) \cap F(T_2)$ by Lemma 2.2. By Lemma 3.1, we know that $\lim_{n\to\infty} \|x_n - q\|$exists. Therefore, $\{x_n\}$ converges strongly to q as $n \to \infty$. This completes the proof. □

THEOREM 3.5. *Let K be a nonempty closed convex subset of a real smooth and uniformly convex Banach space E satisfying Opial's condition or whose norm is Fréchet differentiable. Let $T_1, T_2 : K \to E$ be two weakly inward and asymptotically nonexpansive mappings with respect to P with sequences $\{k_n\}, \{l_n\} \subset [1,\infty)$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, respectively. Let $\{x_n\} \subset K$ be the sequence defined by (2.6), where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $[\epsilon, 1-\epsilon)$ for some $\epsilon > 0$. If $F(T_1) \cap F(T_2) \neq \emptyset$, then $\{x_n\}$ converges weakly to a common fixed point of T_1 and T_2.*

Proof. For any $q \in F(T_1) \cap F(T_2)$, by Lemma 3.1, we know that $\lim_{n\to\infty} \|x_n - q\|$ exists. We now prove that $\{x_n\}$ has a unique weakly subsequential limit in $F(T_1) \cap F(T_2)$. First of all, Lemmas 2.2, 2.5, and 3.2 guarantee that each weakly subsequential limit of $\{x_n\}$ is a common fixed point of T_1 and T_2. Secondly, Opial's condition and Lemma 2.7 guarantee that the weakly subsequential limit of $\{x_n\}$ is unique. Consequently, $\{x_n\}$ converges weakly to a common fixed point of T_1 and T_2.This completes the proof. □

Remark 3.6. The main results of this paper can be extended to a finite family of non-self-asymptotically nonexpansive mappings $\{T_i : 1 \le i \le m\}$, where m is a fixed positive integer, by introducing the following iterative algorithm:

$$\begin{gathered} x_1 \in K, \\ x_{n+1} = \alpha_{n1}x_n + \alpha_{n2}(PT_1)^n x_n + \alpha_{n3}(PT_2)^n x_n + \cdots + \alpha_{n(m+1)}(PT_m)^n x_n, \end{gathered} \tag{3.4}$$

where $\{\alpha_{n1}\}, \{\alpha_{n2}\}, \ldots,$ and $\{\alpha_{n(m+1)}\}$ are $m+1$ real sequences in $(0,1)$ satisfying $\alpha_{n1} + \alpha_{n2} + \cdots + \alpha_{n(m+1)} = 1$.

We close this section with the following open question.

How to devise an iterative algorithm for approximating common fixed points of an infinite family of non-self-asymptotically nonexpansive mappings?

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