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Research Article

A Common Fixed Point Theorem in D^*-Metric Spaces

Shaban Sedghi, Nabi Shobe, and Haiyun Zhou

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We give some new definitions of D^*-metric spaces and we prove a common fixed point theorem for a class of mappings under the condition of weakly commuting mappings in complete D^*-metric spaces. We get some improved versions of several fixed point theorems in complete D^*-metric spaces.

1. Introduction

The concept of fuzzy sets was introduced initially by Zadeh [1] in 1965. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and applications. Especially, Deng [2], Erceg [3], Kaleva and Seikkala [4], and Kramosil and Michálek [5] have introduced the concepts of fuzzy metric spaces in different ways. George and Veeramani [6] and Kramosil and Michálek [5] have introduced the concept of fuzzy topological spaces induced by fuzzy metric which have very important applications in quantum particle physics particularly in connection with both string and E-infinity theories which were given and studied by El Naschie [7–10]. Many authors [11–17] have studied the fixed point theory in fuzzy (probabilistic) metric spaces. On the other hand, there have been a number of generalizations of metric spaces. One of such generalizations is generalized metric space (or D-metric space) initiated by Dhage [18] in 1992. He proved the existence of unique fixed point of a self-map satisfying a contractive condition in complete and bounded D-metric spaces. Dealing with D-metric space, Ahmad et al. [19], Dhage [18, 20], Dhage et al. [21], Rhoades [22], Singh and Sharma [23], and others made a significant contribution in fixed point theory of D-metric space. Unfortunately, almost all theorems in D-metric spaces are not valid (see [24–26]).

In this paper, we introduce D^*-metric which is a probable modification of the definition of D-metric introduced by Dhage [18, 20] and prove some basic properties in D^*-metric spaces.

In what follows (X,D^*) will denote a D^*-metric space, $\mathbb{N}$ the set of all natural numbers, and $\mathbb{R}^+$ the set of all positive real numbers.

Definition 1.1. Let X be a nonempty set. A generalized metric (or D^*-metric) on X is a function, $D^* : X^3 \to [0,\infty)$, that satisfies the following conditions for each $x,y,z,a \in X$:

(1) $D^*(x,y,z) \geq 0$,
(2) $D^*(x,y,z) = 0$ if and only if $x = y = z$,
(3) $D^*(x,y,z) = D^*(p\{x,y,z\})$, (symmetry) where p is a permutation function,
(4) $D^*(x,y,z) \leq D^*(x,y,a) + D^*(a,z,z)$.

The pair (X,D^*) is called a generalized metric (or D^*-metric) space.

Immediate examples of such a function are

(a) $D^*(x,y,z) = \max\{d(x,y),d(y,z),d(z,x)\}$,
(b) $D^*(x,y,z) = d(x,y)+d(y,z)+d(z,x)$.

Here, d is the ordinary metric on X.

(c) If $X = \mathbb{R}^n$ then we define

$$D^*(x,y,z) = \left(\|x-y\|^p + \|y-z\|^p + \|z-x\|^p\right)^{1/p} \tag{1.1}$$

for every $p \in \mathbb{R}^+$.

(d) If $X = \mathbb{R}$, then we define

$$D^*(x,y,z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x,y,z\} & \text{otherwise.} \end{cases} \tag{1.2}$$

Remark 1.2. In a D^*-metric space, we prove that $D^*(x,x,y) = D^*(x,y,y)$. For

(i) $D^*(x,x,y) \leq D^*(x,x,x) + D^*(x,y,y) = D^*(x,y,y)$ and similarly
(ii) $D^*(y,y,x) \leq D^*(y,y,y) + D^*(y,x,x) = D^*(y,x,x)$.

Hence by (i), (ii) we get $D^*(x,x,y) = D^*(x,y,y)$.

Let (X,D^*) be a D^*-metric space. For $r > 0$, define

$$B_{D^*}(x,r) = \{y \in X : D^*(x,y,y) < r\}. \tag{1.3}$$

Example 1.3. Let $X = \mathbb{R}$. Denote $D^*(x,y,z) = |x-y| + |y-z| + |z-x|$ for all $x,y,z \in \mathbb{R}$. Thus

$$\begin{aligned} B_{D^*}(1,2) &= \{y \in \mathbb{R} : D^*(1,y,y) < 2\} \\ &= \{y \in \mathbb{R} : |y-1| + |y-1| < 2\} \\ &= \{y \in \mathbb{R} : |y-1| < 1\} = (0,2). \end{aligned} \tag{1.4}$$

Definition 1.4. Let (X,D^*) be a D^*-metric space and $A\subset X$.

(1) If for every $x\in A$, there exists $r>0$ such that $B_{D^*}(x,r)\subset A$, then subset A is called open subset of X.

(2) Subset A of X is said to be D^*-bounded if there exists $r>0$ such that $D^*(x,y,y)<r$ for all $x,y\in A$.

(3) A sequence $\{x_n\}$ in X converges to x if and only if $D^*(x_n,x_n,x)=D^*(x,x,x_n)\to 0$ as $n\to\infty$. That is, for each $\epsilon>0$ there exists $n_0\in\mathbb{N}$ such that

$$\forall n\geq n_0 \Longrightarrow D^*(x,x,x_n)<\epsilon(*). \tag{1.5}$$

This is equivalent; for each $\epsilon>0$, there exists $n_0\in\mathbb{N}$ such that

$$\forall n,m\geq n_0 \Longrightarrow D^*(x,x_n,x_m)<\epsilon(**). \tag{1.6}$$

Indeed, if $(*)$ holds, then

$$D^*(x_n,x_m,x)=D^*(x_n,x,x_m)\leq D^*(x_n,x,x)+D^*(x,x_m,x_m)<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\varepsilon. \tag{1.7}$$

Conversely, set $m=n$ in $(**)$, then we have $D^*(x_n,x_n,x)<\epsilon$.

(4) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\epsilon>0$, there exists $n_0\in\mathbb{N}$ such that $D^*(x_n,x_n,x_m)<\epsilon$ for each $n,m\geq n_0$. The D^*-metric space (X,D^*) is said to be complete if every Cauchy sequence is convergent.

Let τ be the set of all $A\subset X$ with $x\in A$ if and only if there exists $r>0$ such that $B_{D^*}(x,r)\subset A$. Then τ is a topology on X (induced by the D^*-metric D^*).

Lemma 1.5. *Let (X,D^*) be a D^*-metric space. If $r>0$, then ball $B_{D^*}(x,r)$ with center $x\in X$ and radius r is open ball.*

Proof. Let $z\in B_{D^*}(x,r)$, hence $D^*(x,z,z)<r$. Let $D^*(x,z,z)=\delta$ and $r'=r-\delta$. Let $y\in B_{D^*}(z,r')$, by triangular inequality we have $D^*(x,y,y)=D^*(y,y,x)\leq D^*(y,y,z)+D^*(z,x,x)<r'+\delta=r$. Hence $B_{D^*}(z,r')\subseteq B_{D^*}(x,r)$. Hence the ball $B_{D^*}(x,r)$ is an open ball. □

Definition 1.6. Let (X,D^*) be a D^*-metric space. D^* is said to be a continuous function on X^3 if

$$\lim_{n\to\infty} D^*(x_n,y_n,z_n)=D^*(x,y,z) \tag{1.8}$$

whenever a sequence $\{(x_n,y_n,z_n)\}$ in X^3 converges to a point $(x,y,z)\in X^3$, that is,

$$\lim_{n\to\infty} x_n=x,\qquad \lim_{n\to\infty} y_n=y,\qquad \lim_{n\to\infty} z_n=z. \tag{1.9}$$

Lemma 1.7. *Let (X,D^*) be a D^*-metric space. Then D^* is a continuous function on X^3.*

Proof. Suppose the sequence $\{(x_n,y_n,z_n)\}$ in X^3 converges to a point $(x,y,z)\in X^3$, that is,

$$\lim_{n\to\infty} x_n=x,\qquad \lim_{n\to\infty} y_n=y,\qquad \lim_{n\to\infty} z_n=z. \tag{1.10}$$

Then for each $\epsilon > 0$ there exist n_1, n_2, and $n_3 \in \mathbb{N}$ such that $D^*(x,x,x_n) < \epsilon/3 \,\forall n \geq n_1$, $D^*(y,y,y_n) < \epsilon/3$ for all $n \geq n_2$, and $D^*(z,z,z_n) < \epsilon/3 \,\forall n \geq n_3$.

If we set $n_0 = \max\{n_1,n_2,n_3\}$, then for all $n \geq n_0$ by triangular inequality we have

$$\begin{aligned} D^*(x_n,y_n,z_n) &\leq D^*(x_n,y_n,z) + D^*(z,z_n,z_n) \\ &\leq D^*(x_n,z,y) + D^*(y,y_n,y_n) + D^*(z,z_n,z_n) \\ &\leq D^*(z,y,x) + D^*(x,x_n,x_n) + D^*(y,y_n,y_n) + D^*(z,z_n,z_n) \\ &< D^*(x,y,z) + \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = D^*(x,y,z) + \epsilon. \end{aligned} \tag{1.11}$$

Hence we have

$$\begin{gathered} D^*(x_n,y_n,z_n) - D^*(x,y,z) < \epsilon, \\ \begin{aligned} D^*(x,y,z) &\leq D^*(x,y,z_n) + D^*(z_n,z,z) \\ &\leq D^*(x,z_n,y_n) + D^*(y_n,y,y) + D^*(z_n,z,z) \\ &\leq D^*(z_n,y_n,x_n) + D^*(x_n,x,x) + D^*(y_n,y,y) + D^*(z_n,z,z) \\ &< D^*(x_n,y_n,z_n) + \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = D^*(x_n,y_n,z_n) + \epsilon. \end{aligned} \end{gathered} \tag{1.12}$$

That is,

$$D^*(x,y,z) - D^*(x_n,y_n,z_n) < \epsilon. \tag{1.13}$$

Therefore we have $|D^*(x_n,y_n,z_n) - D^*(x,y,z)| < \epsilon$, that is,

$$\lim_{n\to\infty} D^*(x_n,y_n,z_n) = D^*(x,y,z). \tag{1.14}$$

□

Lemma 1.8. *Let (X,D^*) be a D^*-metric space. If sequence $\{x_n\}$ in X converges to x, then x is unique.*

Proof. Let $x_n \to y$ and $y \neq x$. Since $\{x_n\}$ converges to x and y, for each $\epsilon > 0$ there exist $n_1, n_2 \in \mathbb{N}$ such that $D^*(x,x,x_n) < \epsilon/2 \,\forall n \geq n_1$ and $D^*(y,y,x_n) < \epsilon/2 \,\forall n \geq n_2$.

If we set $n_0 = \max\{n_1,n_2\}$, then for every $n \geq n_0$ by triangular inequality we have

$$D^*(x,x,y) \leq D^*(x,x,x_n) + D^*(x_n,y,y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{1.15}$$

Hence $D^*(x,x,y) = 0$ which is a contradiction. So, $x = y$. □

Lemma 1.9. *Let (X,D^*) be a D^*-metric space. If sequence $\{x_n\}$ in X is convergent to x, then sequence $\{x_n\}$ is a Cauchy sequence.*

Proof. Since $x_n \to x$, for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $D^*(x_n, x_n, x) < \epsilon/2 \, \forall n \geq n_0$. Then for every $n, m \geq n_0$, by triangular inequality, we have

$$D^*(x_n, x_n, x_m) \leq D^*(x_n, x_n, x) + D^*(x, x_m, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{1.16}$$

Hence sequence $\{x_n\}$ is a Cauchy sequence. □

Definition 1.10. Let A and S be two mappings from a D^*-metric space (X, D^*) into itself. Then $\{A, S\}$ is said to be weakly commuting pair if

$$D^*(ASx, SAx, SAx) \leq D^*(Ax, Sx, Sx), \tag{1.17}$$

for all $x \in X$. Clearly, a commuting pair is weakly commuting, but not conversely as shown in the following example.

Example 1.11. Let (X, D^*) be a D^*-metric space, where $X = [0,1]$ and

$$D^*(x, y, z) = |x - y| + |y - z| + |x - z|. \tag{1.18}$$

Define self-maps A and S on X as follows:

$$Sx = \frac{x}{2}, \qquad Ax = \frac{x}{x+2} \qquad \forall x \in X. \tag{1.19}$$

Then for all x in X one gets

$$\begin{aligned} D^*(SAx, ASx, ASx) &= \left| \frac{x}{x+4} - \frac{x}{2x+4} \right| + \left| \frac{x}{x+4} - \frac{x}{x+4} \right| + \left| \frac{x}{x+4} - \frac{x}{2x+4} \right| \\ &= \frac{2x^2}{(x+4)(2x+4)} \leq \frac{2x^2}{2x+4} \\ &= \left| \frac{x}{2} - \frac{x}{x+2} \right| + \left| \frac{x}{2} - \frac{x}{x+2} \right| + 0 \\ &= D^*(Sx, Ax, Ax). \end{aligned} \tag{1.20}$$

So $\{A, S\}$ is a weakly commuting pair.

However, for any nonzero $x \in X$ we have

$$SAx = \frac{x}{x+4} > \frac{x}{2x+4} = ASx. \tag{1.21}$$

Thus A and S are not commuting mappings.

2. The main results

A class of implicit relation. Throughout this section (X, D^*) denotes a D^*-metric space and Φ denotes a family of mappings such that each $\varphi \in \Phi$, $\varphi : (\mathbb{R}^+)^5 \to \mathbb{R}^+$, and φ is continuous and increasing in each coordinate variable. Also $\gamma(t) = \varphi(t, t, a_1 t, a_2 t, t) < t$ for every $t \in \mathbb{R}^+$ where $a_1 + a_2 = 3$.

Example 2.1. Let $\varphi : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ be defined by

$$\varphi(t_1,t_2,t_3,t_4,t_5) = \frac{1}{7}(t_1+t_2+t_3+t_4+t_5). \tag{2.1}$$

The following lemma is the key in proving our result.

Lemma 2.2. *For every $t>0$, $\gamma(t)<t$ if and only if $\lim_{n\to\infty}\gamma^n(t)=0$, where γ^n denotes the composition of γ with itself n times.*

Our main result, for a complete D^*-metric space X, reads as follows.

Theorem 2.3. *Let A be a self-mapping of complete D^*-metric space (X,D^*), and let S,T be continuous self-mappings on X satisfying the following conditions:*
(i) *$\{A,S\}$ and $\{A,T\}$ are weakly commuting pairs such that $A(X)\subset S(X)\cap T(X)$;*
(ii) *there exists a $\varphi\in\Phi$ such that for all $x,y\in X$,*

$$\begin{aligned} &D^*(Ax,Ay,Az)\\ &\quad\le \varphi(D^*(Sx,Ty,Tz),D^*(Sx,Ax,Ax),D^*(Sx,Ay,Ay),D^*(Ty,Ax,Ax),D^*(Ty,Ay,Ay)). \end{aligned} \tag{2.2}$$

Then A, S, and T have a unique common fixed point in X.

Proof. Let $x_0\in X$ be an arbitrary point in X. Then $Ax_0\in X$. Since $A(X)$ is contained in $S(X)$, there exists a point $x_1\in X$ such that $Ax_0=Sx_1$. Since $A(X)$ is also contained in $T(X)$, we can choose a point $x_2\in X$ such that $Ax_1=Tx_2$. Continuing this way, we define by induction a sequence $\{x_n\}$ in X such that

$$\begin{aligned} Sx_{2n+1} &= Ax_{2n} = y_{2n}, \quad n=0,1,2,\dots,\\ Tx_{2n+2} &= Ax_{2n+1} = y_{2n+1}, \quad n=0,1,2,\dots. \end{aligned} \tag{2.3}$$

For simplicity, we set

$$d_n = D^*(y_n,y_{n+1},y_{n+1}), \quad n=0,1,2\dots. \tag{2.4}$$

We prove that $d_{2n}\le d_{2n-1}$. Now, if $d_{2n}>d_{2n-1}$ for some $n\in\mathbb{N}$, since φ is an increasing function, then

$$\begin{aligned} d_{2n} &= D^*(y_{2n},y_{2n+1},y_{2n+1}) = D^*(Ax_{2n},Ax_{2n+1},Ax_{2n+1}) = D^*(Ax_{2n+1},Ax_{2n},Ax_{2n})\\ &\le \varphi\begin{pmatrix} D^*(Sx_{2n+1},Tx_{2n},Tx_{2n}), & D^*(Sx_{2n+1},Ax_{2n+1},Ax_{2n+1}),D^*(Sx_{2n+1},Ax_{2n},Ax_{2n})\\ D^*(Tx_{2n},Ax_{2n+1},Ax_{2n+1}), & D^*(Tx_{2n},Ax_{2n},A_{2n}) \end{pmatrix}\\ &= \varphi\begin{pmatrix} D^*(y_{2n},y_{2n-1},y_{2n-1}), & D^*(y_{2n},y_{2n+1},y_{2n+1}),D^*(y_{2n},y_{2n},y_{2n})\\ D^*(y_{2n-1},y_{2n+1},y_{2n+1}), & D^*(y_{2n-1},y_{2n},y_{2n}) \end{pmatrix}. \end{aligned} \tag{2.5}$$

Since

$$D^*(y_{2n-1}, y_{2n+1}, y_{2n+1}) \le D^*(y_{2n-1}, y_{2n-1}, y_{2n}) + D^*(y_{2n}, y_{2n+1}, y_{2n+1}) = d_{2n-1} + d_{2n}, \tag{2.6}$$

hence by the above inequality we have

$$d_{2n} \le \varphi(d_{2n-1}, d_{2n}, 0, d_{2n-1} + d_{2n}, d_{2n-1}) \le \varphi(d_{2n}, d_{2n}, d_{2n}, 2d_{2n}, d_{2n}) < d_{2n}, \tag{2.7}$$

a contradiction. Hence $d_{2n} \le d_{2n-1}$. Similarly, one can prove that $d_{2n+1} \le d_{2n}$ for $n = 0, 1, 2, \ldots$. Consequently, $\{d_n\}$ is a nonincreasing sequence of nonnegative reals. Now,

$$\begin{aligned} d_1 &= D^*(y_1, y_2, y_2) = D^*(Ax_1, Ax_2, Ax_2) \\ &\le \varphi\begin{pmatrix} D^*(Sx_1, Tx_2, Tx_2), & D^*(Sx_1, Ax_1, Ax_1), D^*(Sx_1, Ax_2, Ax_2) \\ D^*(Tx_2, Ax_1, Ax_1), & D^*(Tx_2, Ax_2, A_2) \end{pmatrix} \\ &= \varphi\begin{pmatrix} D^*(y_0, y_1, y_1), & D^*(y_0, y_1, y_1), D^*(y_0, y_2, y_2) \\ D^*(y_1, y_1, y_1), & D^*(y_1, y_2, y_2) \end{pmatrix} \\ &= \varphi(d_0, d_0, d_0 + d_1, 0, d_0) \\ &\le \varphi(d_0, d_0, 2d_0, d_0, d_0) = \gamma(d_0). \end{aligned} \tag{2.8}$$

In general, we have $d_n \le \gamma^n(d_0)$. So if $d_0 > 0$, then Lemma 2.2 gives $\lim_{n\to\infty} d_n = 0$. For $d_0 = 0$, we clearly have $\lim_{n\to\infty} d_n = 0$, since then $d_n = 0$ for each n. Now we prove that sequence $\{Ax_n = y_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d_n = 0$, it is sufficient to show that the sequence $\{Ax_{2n} = y_{2n}\}$ is a Cauchy sequence. Suppose that $\{Ax_{2n} = y_{2n}\}$ is not a Cauchy sequence. Then there is an $\epsilon > 0$ such that for each even integer $2k$, for $k = 0, 1, 2, \ldots$, there exist even integers $2n(k)$ and $2m(k)$ with $2k \le 2n(k) < 2m(k)$ such that

$$D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}) > \epsilon. \tag{2.9}$$

Let, for each even integer $2k, 2m(k)$ be the least integer exceeding $2n(k)$ satisfying (2.9). Therefore

$$D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)-2}) \le \epsilon, \qquad D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}) > \epsilon. \tag{2.10}$$

Then, for each even integer $2k$ we have

$$\begin{aligned} \epsilon &< D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}) \\ &\le D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)-2}) + D^*(Ax_{2m(k)-2}, Ax_{2m(k)-2}, Ax_{2m(k)-1}) \\ &\quad + D^*(Ax_{2m(k)-1}, Ax_{2m(k)-1}, Ax_{2m(k)}) \\ &= D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1}. \end{aligned} \tag{2.11}$$

So, by (2.10) and $d_n \to 0$, we obtain

$$\lim_{k\to\infty} D^*\left(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}\right) = \epsilon. \tag{2.12}$$

It follows immediately from the triangular inequality that

$$\begin{gathered}
\left| D^*\left(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)-1}\right) - D^*\left(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}\right) \right| \le d_{2m(k)-1}, \\
\left| D^*\left(Ax_{2n(k)+1}, Ax_{2n(k)+1}, Ax_{2m(k)-1}\right) - D^*\left(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}\right) \right| < d_{2m(k)-1} + d_{2n(k)}.
\end{gathered} \tag{2.13}$$

Hence by (2.10), as $k \to \infty$,

$$\begin{gathered}
D^*\left(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)-1}\right) \longrightarrow \epsilon, \\
D^*\left(Ax_{2n(k)+1}, Ax_{2n(k)+1}, Ax_{2m(k)-1}\right) \longrightarrow \epsilon.
\end{gathered} \tag{2.14}$$

Now

$$\begin{aligned}
& D^*\left(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}\right) \\
& \le D^*\left(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2n(k)+1}\right) + D^*\left(Ax_{2n(k)+1}, Ax_{2m(k)}, Ax_{2m(k)}\right) \\
& \le d_{2n(k)} + \varphi\begin{pmatrix} D^*\left(Ax_{2n(k)}, Ax_{2m(k)-1}, Ax_{2m(k)-1}\right), & d_{2n(k)}, D^*\left(Ax_{2n(k)}, Ax_{2m(k)}, Ax_{2m(k)}\right) \\ D^*\left(Ax_{2m(k)-1}, Ax_{2n(k)+1}, Ax_{2n(k)+1}\right), & d_{2m(k)-1} \end{pmatrix}.
\end{aligned} \tag{2.15}$$

Using (2.14), $\lim_{k\to\infty} d_n = 0$, and continuity and nondecreasing property of φ in each coordinate variable, we have

$$\epsilon \le \varphi(\epsilon, 0, \epsilon, \epsilon, 0) \le \varphi(\epsilon, \epsilon, 2\epsilon, \epsilon, \epsilon) = \gamma(\epsilon) < \epsilon \tag{2.16}$$

as $k \to \infty$, which is a contradiction. Thus $\{Ax_n = y_n\}$ is a Cauchy sequence and hence by completeness of X, it converges to $z \in X$. That is,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} y_n = z. \tag{2.17}$$

Since the sequences $\{Sx_{2n+1} = y_{2n+1}\}$ and $\{Tx_{2n} = y_{2n}\}$ are subsequences of $\{Ax_n = y_n\}$; they have the same limit z. As S and T are continuous, we have $STx_{2n} \to Sz$ and $TSx_{2n+1} \to Tz$.

Now consider

$$\begin{aligned}
D^*\left(STx_{2n}, TSx_{2n+1}, TSx_{2n+1}\right) &= D^*\left(SAx_{2n-1}, TAx_{2n}, TAx_{2n}\right) \\
&\le D^*\left(SA_{2n-1}, ASx_{2n-1}, ASx_{2n-1}\right) \\
&\quad + D^*\left(ASx_{2n-1}, ASx_{2n-1}, ATx_{2n}\right) \\
&\quad + D^*\left(ATx_{2n}, ATx_{2n}, TAx_{2n}\right).
\end{aligned} \tag{2.18}$$

Using (ii) and the weak commutativity of $\{A,S\}$ and $\{A,T\}$, we get

$$\begin{aligned}
&D^*(STx_{2n},TSx_{2n+1},TSx_{2n+1})\\
&\quad\le D^*(Sx_{2n-1},Ax_{2n-1},Ax_{2n-1})+D^*(ASx_{2n-1},ATx_{2n},ATx_{2n})+D^*(Ax_{2n},Ax_{2n},Tx_{2n})\\
&\quad\le D^*(Sx_{2n-1},Ax_{2n-1},Ax_{2n-1})\\
&\qquad+\varphi\begin{pmatrix} D^*(S^2x_{2n-1},T^2x_{2n},T^2x_{2n}), & D^*(S^2x_{2n-1},ASx_{2n-1},ASx_{2n-1}),\\ & D^*(S^2x_{2n-1},ATx_{2n},ATx_{2n})\\ D^*(T^2x_{2n},ASx_{2n-1},ASx_{2n-1}), & D^*(T^2x_{2n},ATx_{2n},ATx_{2n})\end{pmatrix}\\
&\qquad+D^*(Ax_{2n},Ax_{2n},Tx_{2n})\\
&\quad\le D^*(Sx_{2n-1},Ax_{2n-1},Ax_{2n-1})\\
&\qquad+\varphi\begin{pmatrix} D^*(S^2x_{2n-1},T^2x_{2n},T^2x_{2n}),D^*(S^2x_{2n-1},S^2x_{2n-1},SAx_{2n-1})\\ +D^*(Sx_{2n-1},Sx_{2n-1},Ax_{2n-1}),\\ D^*(S^2x_{2n-1},TAx_{2n},TAx_{2n})+D^*(Tx_{2n},Tx_{2n},Ax_{2n}),\\ D^*(T^2x_{2n},SAx_{2n-1},SAx_{2n-1})+D^*(Sx_{2n-1},Sx_{2n-1},Ax_{2n-1}),\\ D^*(T^2x_{2n},TAx_{2n},TAx_{2n})+D^*(Tx_{2n},Ax_{2n},Ax_{2n})\end{pmatrix}\\
&\qquad+D^*(Ax_{2n},Ax_{2n},Tx_{2n}).
\end{aligned}\tag{2.19}$$

If $D^*(Sz,Tz,Tz)>0$, then as $n\to\infty$ we have

$$\begin{aligned}
&D^*(Sz,Tz,Tz)\\
&\quad\le D^*(z,z,z)+\varphi\begin{pmatrix} D^*(Sz,Tz,Tz), & D^*(Sz,Sz,Sz)+0,D^*(Sz,Tz,Tz)+0\\ D^*(Tz,Sz,Sz)+0, & D^*(Tz,Tz,Tz)+0\end{pmatrix}+0\\
&\quad\le\gamma(D^*(Sz,Tz,Tz))< D^*(Sz,Tz,Tz),
\end{aligned}\tag{2.20}$$

a contradiction.Therefore, $Sz=Tz$.

Now we will prove that $Az=Sz$. To end this, consider the inequality

$$D^*(SAx_{2n+1},Az,Az)\le D^*(SAx_{2n+1},ASx_{2n+1},ASx_{2n+1})+D^*(Az,Az,ASx_{2n+1}).\tag{2.21}$$

Again using (ii) and the weak commutativity of $\{A,S\}$, we have

$$\begin{aligned}
D^*(SAx_{2n+1},Az,Az)&\le D^*(Sx_{2n+1},Ax_{2n+1},Ax_{2n+1})\\
&\quad+\varphi\begin{pmatrix} D^*(Sz,Tz,TSx_{2n+1}), & D^*(Sz,Az,Az),D^*(Sz,Az,Az)\\ D^*(Tz,Az,Az), & D^*(Tz,Az,Az)\end{pmatrix}.
\end{aligned}\tag{2.22}$$

Taking $n\to\infty$, we have

$$\begin{aligned} D^*(Sz,Az,Az) &\le D^*(z,z,z)+\varphi\begin{pmatrix} D^*(Sz,Tz,Tz),D^*(Sz,Az,Az),D^*(Sz,Az,Az) \\ D^*(Tz,Az,Az),D^*(Tz,Az,Az) \end{pmatrix} \\ &= \varphi(0,D^*(Sz,Az,Az),D^*(Sz,Az,Az),D^*(Sz,Az,Az),D^*(Sz,Az,Az)) \\ &\le \delta(D^*(Sz,Az,Az))< D^*(Sz,Az,Az) \end{aligned} \tag{2.23}$$

given there by $Sz = Az$. Thus $Az = Sz = Tz$. It now follows that

$$D^*(Az,Ax_{2n},Ax_{2n}) \le \varphi\begin{pmatrix} D^*(Sz,Tx_{2n},Tx_{2n}), & D^*(Sz,Az,Az),D^*(Sz,Ax_{2n},Ax_{2n}) \\ D^*(Tx_{2n},Az,Az), & D^*(Tx_{2n},Ax_{2n},Ax_{2n}) \end{pmatrix}. \tag{2.24}$$

Then as $n\to\infty$, we get

$$\begin{aligned} D^*(Az,z,z) &\le \varphi(D^*(Sz,z,z),0,D^*(Sz,z,z),D^*(z,Az,Az),0) \\ &\le \gamma(D^*(Az,z,z))< D^*(Az,z,z), \end{aligned} \tag{2.25}$$

a contradiction, and therefore $Az = z = Sz = Tz$. Thus z is a common fixed point of A,S, and T. The unicity of the common fixed point is not hard to verify. This completes the proof of the theorem. □

Example 2.4. Let (X,D^*) be a D^*-metric space, where $X = [0,1]$ and

$$D^*(x,y,z) = |x-y|+|y-z|+|x-z|. \tag{2.26}$$

Define self-maps A,T, and S on X as follows:

$$Sx = x, \qquad Ax = 1, \qquad Tx = \frac{x+1}{2}, \tag{2.27}$$

for all $x \in X$.

Let

$$\varphi(t_1,t_2,t_3,t_4,t_5) = \frac{1}{7}(t_1+t_2+t_3+t_4+t_5). \tag{2.28}$$

Then

$$A(X) = \{1\} \subset [0,1] \cap \left[\frac{1}{2},1\right] = S(X) \cap T(X), \tag{2.29}$$

and for every $x \in X$, we have

$$\begin{aligned} D^*(ATx,TAx,TAx) = D^*(1,1,1) = 0 \le D^*(Ax,Tx,Tx), \\ D^*(ASx,SAx,SAx) = D^*(1,1,1) = 0 \le D^*(Ax,Sx,Sx). \end{aligned} \tag{2.30}$$

That is, the pairs (A,S) and (A,T) are weakly commuting.

Also for all $x,y,z \in X$, we have

$$\begin{aligned}&D^*(Ax,Ay,Az) = 0\\&\leq \varphi(D^*(Sx,Ty,Tz),D^*(Sx,Ax,Ax),D^*(Sx,Ay,Ay),D^*(Ty,Ax,Ax),D^*(Ty,Ay,Ay)).\end{aligned} \tag{2.31}$$

That is, all conditions of Theorem 2.3 hold and 1 is the unique common fixed point of A,S, and T.

Corollary 2.5. *Let A,R,S,T, and H be self-mappings of complete D^*-metric space (X,D^*), and let SR,TH be continuous self-mappings on X satisfying the following conditions:*

(i) *$\{A,SR\}$ and $\{A,TH\}$ are weakly commuting pairs such that $A(X) \subset SR(X) \cap TH(X)$;*

(ii) *there exists a $\varphi \in \Phi$ such that for all $x,y \in X$,*

$$D^*(Ax,Ay,Az) \leq \varphi\begin{pmatrix}D^*(SRx,THy,THz),D^*(SRx,Ax,Ax),D^*(SRx,Ay,Ay),\\D^*(THy,Ax,Ax),D^*(THy,Ay,Ay)\end{pmatrix}. \tag{2.32}$$

If $SR = RS, TH = HT, AH = HA$, and $AR = RA$, then A,S,R,H, and T have a unique common fixed point in X.

Proof. By Theorem 2.3, A,TH, and SR have a unique common fixed point in X. That is, there exists $a \in X$, such that $A(a) = TH(a) = SR(a) = a$. We prove that $R(a) = a$. By (ii), we get

$$D^*(ARa,Aa,Aa) \leq \varphi\begin{pmatrix}D^*(SRRa,THa,THa),D^*(SRRa,ARa,ARa),D^*(SRRa,Aa,Aa),\\D^*(THa,ARa,ARa),D^*(THa,Aa,Aa)\end{pmatrix}. \tag{2.33}$$

Hence if $Ra \neq a$, then we have

$$\begin{aligned}D^*(Ra,a,a) &\leq \varphi(D^*(Ra,a,a),D^*(Ra,Ra,Ra),D^*(Ra,a,a),D^*(a,Ra,Ra),D^*(a,a,a))\\&\leq \varphi(D^*(Ra,a,a),D^*(Ra,a,a),D^*(Ra,a,a),2D^*(Ra,a,a),D^*(Ra,a,a))\\&< D^*(Ra,a,a),\end{aligned} \tag{2.34}$$

a contradiction. Therefore it follows that $Ra = a$. Hence $S(a) = SR(a) = a$. Similarly, we get that $T(a) = H(a) = a$. □

Corollary 2.6. *Let A_i be a sequence self-mapping of complete D^*-metric space (X,D^*) for $i \in \mathbb{N}$, and let S,T be continuous self-mappings on X satisfying the following conditions:*

(i) *there exists $i_0 \in \mathbb{N}$ such that $\{A_{i_0},S\}$ and $\{A_{i_0},T\}$ are weakly commuting pairs such that $A_{i_0}(X) \subset S(X) \cap T(X)$;*

(ii) *there exists a $\varphi \in \Phi$ and $i, j, k \in \mathbb{N}$ such that for all $x, y \in X$,*

$$D^*(A_i x, A_j y, A_k z) \le \varphi\begin{pmatrix} D^*(Sx, Ty, Tz), D^*(Sx, A_i x, A_i x), D^*(Sx, A_j y, A_j y), \\ D^*(Ty, A_i x, A_i x), D^*(Ty, A_j y, A_j y) \end{pmatrix}. \tag{2.35}$$

Then A_i, S, and T have a unique common fixed point in X for every $i \in \mathbb{N}$.

Proof. By Theorem 2.3, S, T, and A_{i_0}, for some $i = j = k = i_0 \in \mathbb{N}$, have a unique common fixed point in X. That is, there exists a unique $a \in X$ such that

$$S(a) = T(a) = A_{i_0}(a) = a. \tag{2.36}$$

Suppose there exists $i \in \mathbb{N}$ such that $i \neq i_0$ and $j = i_0, k = i_0$. Then we have

$$D^*(A_i a, A_{i_0} a, A_{i_0} a) \le \varphi\begin{pmatrix} D^*(Sa, Ta, Ta), D^*(Sa, A_i a, A_i a), D^*(Sa, A_{i_0} a, A_{i_0} a), \\ D^*(Ta, A_i a, A_i a), D^*(Ta, A_{i_0} a, A_{i_0} a) \end{pmatrix}. \tag{2.37}$$

Hence if $A_i a \neq a$, then we get

$$\begin{aligned} D^*(A_i a, a, a) &\le \varphi\begin{pmatrix} D^*(a,a,a), D^*(a, A_i a, A_i a), D^*(a,a,a), \\ D^*(a, A_i a, A_i a), D^*(a,a,a) \end{pmatrix} \\ &\le \varphi\begin{pmatrix} D^*(A_i a, a, a), D^*(A_i a, a, a), D^*(A_i a, a, a), \\ 2D^*(A_i a, a, a), D^*(A_i a, a, a) \end{pmatrix} \\ &< D^*(A_i a, a, a), \end{aligned} \tag{2.38}$$

a contradiction. Hence for every $i \in \mathbb{N}$ it follows that $A_i(a) = a$ for every $i \in \mathbb{N}$. □

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Shaban Sedghi: Department of Mathematics, Islamic Azad University-Ghaemshahr Branch, P.O. Box 163, Ghaemshahr, Iran
Email address: sedghi_gh@yahoo.com

Nabi Shobe: Department of Mathematics, Islamic Azad University-Babol Branch, Babol, Iran
Email address: nabi_shobe@yahoo.com

Haiyun Zhou: Department of Mathematics, Shijiazhuang Mechnical Engineering University, Shijiazhuang 050003, China
Email address: witman66@yahoo.com.cn