# PICARD ITERATION CONVERGES FASTER THAN MANN ITERATION FOR A CLASS OF QUASI-CONTRACTIVE OPERATORS

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*Received 20 November 2003 and in revised form 6 February 2004*

In the class of quasi-contractive operators satisfying Zamfirescu's conditions, the most used fixed point iterative methods, that is, the Picard, Mann, and Ishikawa iterations, are all known to be convergent to the unique fixed point. In this paper, the comparison of the first two methods with respect to their convergence rate is obtained.

## 1. Introduction

In the last three decades many papers have been published on the iterative approximation of fixed points for certain classes of operators, using the Mann and Ishikawa iteration methods, see [4], for a recent survey. These papers were motivated by the fact that, under weaker contractive type conditions, the Picard iteration (or the method of successive approximations), need not converge to the fixed point of the operator in question.

However, there exist large classes of operators, as for example that of quasi-contractive type operators introduced in [4, 7, 10, 11], for which not only the Picard iteration, but also the Mann and Ishikawa iterations can be used to approximate the fixed points. In such situations, it is of theoretical and practical importance to compare these methods in order to establish, if possible, which one converges faster.

As far as we know, there are only a few papers devoted to this very important numerical problem: the one due to Rhoades [11], in which the Mann and Ishikawa iterations are compared for the class of continuous and nondecreasing functions $f : [0,1] \to [0,1]$, and also the author's papers [1, 3, 5], concerning the Picard and Krasnoselskij iterative procedures in the class of Lipschitzian and generalized pseudocontractive operators.

An empirical comparison of Newton, Mann, and Ishikawa iterations over two families of decreasing functions was also reported in [13]. In [4] some conclusions of an empirical numerical study of Krasnoselskij, Mann, and Ishikawa iterations for some Lipschitz strongly pseudocontractive mappings, for which the Picard iteration does not converge, were also presented.


Fixed Point Theory and Applications 2004:2 (2004) 97–105
2000 Mathematics Subject Classification: 47H10, 54H25
URL: http://dx.doi.org/10.1155/S1687182004311058

It is the main purpose of this paper to compare the Picard and Mann iterations over a class of quasi-contractive mappings, that is, the ones satisfying the Zamfirescu's conditions [15]. Theorem 3.1 in the present paper shows that for the aforementioned class of operators, considered in uniformly convex Banach spaces, the Picard iteration always converges faster than the Mann iterative procedure. Moreover, Theorem 3.3 extends this result to arbitrary Banach spaces and also to Mann iterations defined by weaker assumptions on the sequence $\{\alpha_n\}$.

## 2. Some fixed point iteration procedures

Let $E$ be a normed linear space and $T : E \to E$ a given operator. Let $x_0 \in E$ be arbitrary and $\{\alpha_n\} \subset [0,1]$ a sequence of real numbers. The sequence $\{x_n\}_{n=0}^{\infty} \subset E$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n = 0,1,2,\ldots, \tag{2.1}$$

is called the *Mann iteration* or *Mann iterative procedure*.

The sequence $\{x_n\}_{n=0}^{\infty} \subset E$ defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \quad n = 0,1,2,\ldots, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \quad n = 0,1,2,\ldots, \end{aligned} \tag{2.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $[0,1]$, and $x_0 \in E$ is arbitrary, is called the *Ishikawa iteration* or *Ishikawa iterative procedure*.

*Remarks 2.1.* For $\alpha_n = \lambda$ (constant), the iteration (2.1) reduces to the so-called *Krasnoselskij iteration* while for $\alpha_n \equiv 1$ we obtain the *Picard iteration* (2.3), or the method of successive approximations, as it is commonly known. Obviously, for $\beta_n \equiv 0$ the Ishikawa iteration (2.2) reduces to (2.1).

*Example 2.2* [4]. Let $K = [1/2,2] \subset \mathbb{R}$ be endowed with the usual norm and $T : K \to K$, defined by $Tx = 1/x$, $x \in K$. Then,

(1) $T$ has a unique fixed point, that is, $F_T = \{1\}$;
(2) the Picard iteration (2.3) does not converge to 1, for any $x_0 \neq 1$ in $[1/2,2]$;
(3) the Krasnoselskij iteration converges to the fixed point of $T$, for $\lambda$ satisfying $0 < \lambda < 2(1-r)/(17-2r)$, where $0 < r < 1$.

It is well known that the Krasnoselskij, Mann, and Ishikawa iterative procedures have been introduced mainly in order to approximate fixed points of those operators for which the Picard iteration does not converge. But, as we already mentioned, there exist important classes of contractive mappings, that is, the class of quasi-contractions, for which all Picard, Krasnoselskij, Mann, and Ishikawa iterations converge. The next two theorems refer to the Picard and Mann iterations.

Theorem 2.3 [15]. *Let* $(X,d)$ *be a complete metric space and* $T : X \to X$ *a map for which there exist the real numbers a, b, and c satisfying* $0 < a < 1$, $0 < b,c < 1/2$ *such that for each pair* $x, y$ *in* $X$, *at least one of the following is true:*

($z_1$) $d(Tx,Ty) \le a\,d(x,y)$;
($z_2$) $d(Tx,Ty) \le b[d(x,Tx)+d(y,Ty)]$;
($z_3$) $d(Tx,Ty) \le c[d(x,Ty)+d(y,Tx)]$.

*Then* $T$ *has a unique fixed point* $p$ *and the Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *defined by*

$$x_{n+1} = Tx_n, \quad n = 0,1,2,\ldots, \tag{2.3}$$

*converges to* $p$, *for any* $x_0 \in X$.

Theorem 2.4 [10]. *Let E be a uniformly convex Banach space, K a closed convex subset of E, and* $T : K \to K$ *a Zamfirescu mapping. Then the Mann iteration* $\{x_n\}$ *given by (2.1) with* $\{\alpha_n\}$ *satisfying the conditions*

(i) $\alpha_1 = 1$;
(ii) $0 \le \alpha_n < 1$ *for* $n > 1$;
(iii) $\sum \alpha_n(1-\alpha_n) = \infty$;

*converges to the fixed point of* $T$.

In order to compare two fixed point iteration procedures $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ that converge to a certain fixed point $p$ of a given operator $T$, Rhoades [11] considered that $\{u_n\}$ is *better* than $\{v_n\}$ if

$$\|u_n - p\| \le \|v_n - p\|, \quad \forall n. \tag{2.4}$$

In the following we will adopt the terminology from our papers [3, 4, 5], which is slightly different from that of Rhoades, but more suitable for our purposes here.

*Definition 2.5.* Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be two sequences of real numbers that converge to $a$ and $b$, respectively, and assume that there exists

$$l = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}. \tag{2.5}$$

(a) If $l = 0$, then it can be said that $\{a_n\}_{n=0}^{\infty}$ converges *faster* to $a$ than $\{b_n\}_{n=0}^{\infty}$ to $b$.
(b) If $0 < l < \infty$, then it can be said that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ *have the same rate of convergence*.

*Remarks 2.6.* (1) In the case (a) we use the notation $a_n - a = o(b_n - b)$.
(2) If $l = \infty$, then the sequence $\{b_n\}_{n=0}^{\infty}$ converges faster than $\{a_n\}_{n=0}^{\infty}$, that is

$$b_n - b = o(a_n - a). \tag{2.6}$$

Suppose that for two fixed point iteration procedures $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$, both converging to the same fixed point $p$, the error estimates

$$\|u_n - p\| \le a_n, \quad n = 0,1,2,\dots, \tag{2.7}$$

$$\|v_n - p\| \le b_n, \quad n = 0,1,2,\dots, \tag{2.8}$$

are available, where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences of positive numbers (converging to zero).

Then, in view of Definition 2.5, we will adopt the following concept.

*Definition 2.7.* Let $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be two fixed point iteration procedures that converge to the same fixed point $p$ and satisfy (2.7) and (2.8), respectively. If $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$, then it can be said that $\{u_n\}_{n=0}^{\infty}$ *converges faster* than $\{v_n\}_{n=0}^{\infty}$ to $p$.

*Example 2.8.* If we take $p = 0$, $u_n = 1/(n+1)$, $v_n = 1/n$, $n \ge 1$, then $\{u_n\}$ is better than $\{v_n\}$, but $\{u_n\}$ does not converge faster than $\{v_n\}$. Indeed, we have

$$\lim_{n\to\infty} \frac{u_n}{v_n} = 1, \tag{2.9}$$

and hence $\{u_n\}$ and $\{v_n\}$ have the same rate of convergence.

The previous example shows that Definition 2.7 introduces a sharper concept of rate of convergence than the one considered by Rhoades [11].

Using Theorems 2.3 and 2.4 and based on Definition 2.7, the next section compares the Picard and Mann iterations in the class of Zamfirescu operators. The conclusion will be that the Picard iteration always converges faster than the Mann iteration, as was observed empirically on some numerical tests in [4].

## 3. Comparing Picard and Mann iterations

The main result of this paper is given by the next theorem.

THEOREM 3.1. *Let $E$ be a uniformly convex Banach space, $K$ a closed convex subset of $E$, and $T : K \to K$ a Zamfirescu operator; that is, an operator that satisfies ($z_1$), ($z_2$), and ($z_3$). Let $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration associated with $T$, starting from $x_0 \in K$, given by (2.3), and $\{y_n\}_{n=0}^{\infty}$ the Mann iteration given by (2.1), where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence satisfying*

(i) $\alpha_1 = 1$;
(ii) $0 \le \alpha_n < 1$ *for* $n \ge 1$;
(iii) $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$.

*Then,*

(1) *$T$ has a unique fixed point in $E$, that is, $F_T = \{p\}$;*
(2) *the Picard iteration $\{x_n\}$ converges to $p$ for any $x_0 \in K$;*
(3) *the Mann iteration $\{y_n\}$ converges to $p$ for any $y_0 \in K$ and $\{\alpha_n\}$ satisfying (i), (ii), and (iii);*
(4) *Picard iteration is faster than any Mann iteration.*

*Proof.* Conclusions (1), (2), and (3) follow by Theorems 2.3 and 2.4.

(4) First of all, we prove that any Zamfirescu operator satisfies

$$\|Tx - Ty\| \le \delta \cdot \|x - y\| + 2\delta \cdot \|x - Tx\|, \tag{3.1}$$

$$\|Tx - Ty\| \le \delta \cdot \|x - y\| + 2\delta \cdot \|y - Tx\|, \tag{3.2}$$

for all $x, y \in K$, where $\delta$ is given by (3.6).

Indeed, choose $x, y \in K$. Then at least one of $(z_1)$, $(z_2)$, or $(z_3)$ is true. If $(z_1)$ is satisfied, then (3.1) and (3.2) obviously hold with $\delta = a$.

If $(z_2)$ holds, then

$$\begin{aligned} \|Tx - Ty\| &\le b[\|x - Tx\| + \|y - Ty\|] \\ &\le b\{\|x - Tx\| + [\|y - x\| + \|x - Tx\| + \|Tx - Ty\|]\}, \end{aligned} \tag{3.3}$$

which yields

$$\|Tx - Ty\| \le \frac{b}{1-b}\|x - y\| + \frac{2b}{1-b}\|x - Tx\|. \tag{3.4}$$

If $(z_3)$ holds, then we similarly get

$$\|Tx - Ty\| \le \frac{c}{1-c}\|x - y\| + \frac{2c}{1-c}\|x - Tx\|. \tag{3.5}$$

Therefore, by denoting

$$\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}, \tag{3.6}$$

then in view of the assumptions $0 \le a < 1$; $0 \le b < 1/2$; $0 \le c < 1/2$ it follows that $0 \le \delta < 1$ and so, for all $x, y \in K$, inequality (3.1) is true. Inequality (3.2) is obtained similarly.

Taking $y := x_n$; $x := p$ in (3.1), we obtain

$$||x_{n+1} - p|| \le \delta \cdot ||x_n - p||, \tag{3.7}$$

which inductively yields

$$||x_{n+1} - p|| \le \delta^n \cdot ||x_1 - p||, \quad n \ge 0. \tag{3.8}$$

Now let $y_0 \in K$ and let $\{y_n\}_{n=0}^{\infty}$ be the Mann iteration associated with $T$, $y_0$, and the sequence $\{\alpha_n\}$. Then by (2.1) we have

$$\begin{aligned} ||y_{n+1} - p|| &= ||(1-\alpha_n)y_n + \alpha_n T y_n - [(1-\alpha_n) + \alpha_n]p|| \\ &\le (1-\alpha_n)||y_n - p|| + \alpha_n||Ty_n - p||. \end{aligned} \tag{3.9}$$

Using (3.2) with $y := y_n$, $x := p$, we get

$$||Ty_n - p|| \le \delta \cdot ||y_n - p|| + 2\delta||y_n - p|| = 3\delta||y_n - p|| \tag{3.10}$$

and therefore

$$||y_{n+1} - p|| \le [1 - \alpha_n + 3\delta\alpha_n] \cdot ||y_n - p||, \quad n = 0, 1, 2, \dots, \tag{3.11}$$

which implies that

$$\|y_{n+1} - p\| \leq \prod_{k=1}^{n} [1 - \alpha_k + 3\delta\alpha_k] \cdot \|y_1 - p\|, \quad n = 0,1,2,\dots. \tag{3.12}$$

In order to compare $\{x_n\}$ and $\{y_n\}$, we must compare $\delta^n$ and $\prod_{k=1}^{n}(1 - \alpha_k + 3\delta\alpha_k)$.

First, note that $1 - \alpha_k + 3\delta\alpha_k > 0$, for all $\delta \in [0,1)$ and $\{\alpha_k\}_{k=1}^{\infty}$ satisfying (ii). Moreover, if $\delta \in [0,1/3)$, then

$$1 - \alpha_k + 3\delta\alpha_k < 1, \tag{3.13}$$

while for $\delta \in [1/3,1)$ we have

$$1 - \alpha_k + 3\delta\alpha_k \geq 1. \tag{3.14}$$

Thus, for $\delta \in [1/3,1)$ we have

$$0 \leq \lim_{n\to\infty} \frac{\delta^n}{\prod_{k=1}^{n}(1 - \alpha_k + 3\delta\alpha_k)} \leq \lim_{n\to\infty} \delta^n = 0, \tag{3.15}$$

which shows that, in this case, the Picard iteration converges faster than the Mann iteration.

If $\delta \in [0,1/3)$, then it is easy to verify that, for any $\{\alpha_k\} \subset [0,1]$,

$$\alpha_k \leq 1 < \frac{1 - 2\delta}{3\delta^2 - 4\delta + 1}, \tag{3.16}$$

which yields

$$\frac{\delta}{1 - \alpha_k + 3\delta\alpha_k} < 1 - \delta. \tag{3.17}$$

Hence

$$\frac{\delta^n}{\prod_{k=1}^{n}(1 - \alpha_k + 3\delta\alpha_k)} < (1 - \delta)^n, \quad \forall n \geq 1, \tag{3.18}$$

and therefore

$$\lim_{n\to\infty} \frac{\delta^n}{\prod_{k=1}^{n}(1 - \alpha_k + 3\delta\alpha_k)} = 0. \tag{3.19}$$

This shows that the Picard iteration converges faster than the Mann iteration for $\delta \in [0,1/3)$. □

*Remarks 3.2.* (1) Theorem 3.1 shows that, to efficiently approximate fixed points of Zamfirescu operators, one should always use the Picard iteration.

(2) Since strict contractions, Kannan mappings [9], Hardy and Rogers generalized contractions [8], as well as Chatterjea mappings [6] belong to the class of Zamfirescu operators, by Theorem 3.1, we obtain similar results for these classes of contractive mappings.

(3) Some numerical tests performed with the aid of the software package fixpoint [4] raise the following open problem: for the class of Zamfirescu operators, does the Mann iteration converge faster than the Ishikawa iteration?

(4) The uniform convexity of $E$ is not necessary for the conclusion of Theorem 2.4 to hold. (See [2], where the author extended Theorem 2.4 to arbitrary Banach spaces and also to Mann iterations defined by weaker assumptions on the sequence $\{\alpha_n\}$.)

The following question then naturally arises: is conclusion (4) in Theorem 3.1 still valid under these weaker hypotheses?

A positive answer is provided by the next theorem.

Theorem 3.3. *Let $E$ be an arbitrary Banach space, $K$ a closed convex subset of $E$, and $T : K \to K$ an operator satisfying Zamfirescu's conditions. Let $\{y_n\}_{n=0}^{\infty}$ be defined by (2.1) and $y_0 \in K$, with $\{\alpha_n\} \subset [0,1]$ satisfying*

(iv) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

*Then $\{y_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of $T$ and, moreover, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by (2.3) and $x_0 \in K$ converges faster than the Mann iteration.*

*Proof.* Similar to the proof of Theorem 3.1 we get

$$||y_{n+1} - p|| \leq (1-\alpha_n)||y_n - p|| + \alpha_n ||Ty_n - p||. \tag{3.20}$$

Take $x := p$ and $y := y_n$ in (3.1) to obtain

$$||Ty_n - p|| \leq \delta \cdot ||y_n - p||, \tag{3.21}$$

and then

$$||y_{n+1} - p|| \leq [1-(1-\delta)\alpha_n]||y_n - p||, \quad n = 0,1,2,\dots. \tag{3.22}$$

By induction, we get

$$||y_{n+1} - p|| \leq \prod_{k=0}^{n} [1-(1-\delta)\alpha_k] \cdot ||y_0 - p||, \quad n = 0,1,2,\dots. \tag{3.23}$$

As $\delta < 1$, $\alpha_k \in [0,1]$, and $\sum_{k=0}^{\infty} \alpha_k = \infty$, it follows that

$$\lim_{n\to\infty} \prod_{k=0}^{n} [1-(1-\delta)\alpha_k] = 0, \tag{3.24}$$

which by (3.23) implies that

$$\lim_{n\to\infty} ||y_{n+1} - p|| = 0, \tag{3.25}$$

that is, $\{y_n\}_{n=0}^{\infty}$ converges strongly to $p$.

The proof of the second part of the theorem is similar to that of Theorem 3.1. □

*Remarks 3.4.* (1) Condition (iv) in Theorem 3.3 is weaker than conditions (i), (ii), and (iii) in Theorems 2.4 and 3.1. Indeed, in view of the inequality

$$0 < \alpha_k(1-\alpha_k) < \alpha_k, \tag{3.26}$$

valid for all $\alpha_k$ satisfying (i) and (ii), condition (iii) implies (iv).

There also exist values of $\{\alpha_n\}$, for example, $\alpha_n \equiv 1$, such that (iv) is satisfied, but (iii) is not.

(2) The main merit of this paper consists not only in the results given by Theorems 3.1 and 3.3, but also in the fact that these theoretical results were suggested by some empirical tests on contractive-type operators, see [4, Chapter 9].

(3) The class of mappings $T$ satisfying Zamfirescu's conditions coincides (see [12]) with the class of operators for which there exists a real number $0 < h < 1$ such that

$$d(Tx,Ty) \le h\max\left\{d(x,y), \frac{[d(x,Tx)+d(y,Ty)]}{2}, \frac{[d(x,Ty)+d(y,Tx)]}{2}\right\}, \tag{3.27}$$

so, our results are valid for all fixed point theorems obtained for these operators as well.

(4) For the larger class of quasi-contractions introduced by Ćirić [7], both Picard [7] and Mann [10] (and also Ishikawa [14]) iterations are known to converge to the unique fixed point. It remains to answer the natural question whether or not Picard iteration converges faster than the Mann iteration for this class of mappings.

## Acknowledgment

The author would like to thank one of the anonymous referees for his helpful comments and some stylistic changes suggested, as well as for pointing to reference [8].

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