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Research Article

Weighted Norm Inequalities for Solutions to the Nonhomogeneous A-Harmonic Equation

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We first prove the local and global two-weight norm inequalities for solutions to the nonhomogeneous A-harmonic equation $A(x, g + du) = h + d^{\star}v$ for differential forms. Then, we obtain some weighed Lipschitz norm and BMO norm inequalities for differential forms satisfying the different nonhomogeneous A-harmonic equations.

1. Introduction

In the recent years, the A-harmonic equations for differential forms have been widely investigated, see [1], and many interesting and important results have been found, such as some weighted integral inequalities for solutions to the A-harmonic equations; see [2–7]. Those results are important for studying the theory of differential forms and both qualitative and quantitative properties of the solutions to the different versions of A-harmonic equation. In the different versions of A-harmonic equation, the nonhomogeneous A-harmonic equation $A(x, g + du) = h + d^{\star}v$ has received increasing attentions, in [8] Ding has presented some estimates to such equation. In this paper, we extend some estimates that Ding has presented in [8] into the two-weight case. Our results are more general, so they can be used broadly.

It is well-known that the Lipschitz norm $\sup_{Q\subset\Omega}|Q|^{-1-(k/n)}\|u - u_Q\|_{1,Q}$, where the supremum is over all local cubes Q, as $k \to 0$ is the BMO norm $\sup_{Q\subset\Omega}|Q|^{-1}\|u - u_Q\|_{1,Q}$, so the natural limit of the space $\operatorname{locLip}k(\Omega)$ as $k \to 0$ is the space $\mathrm{BMO}(\Omega)$. In Section 3, we establish a relation between these two norms and L^p-norm. We first present the local two-weight Poincaré inequality for A-harmonic tensors. Then, as the application of this inequality and the result in [8], we prove some weighted Lipschitz norm inequalities and BMO norm inequalities for differential forms satisfying the different nonhomogeneous A-harmonic equations. These results can be used to study the basic properties of the solutions to the nonhomogeneous A-harmonic equations.

Now, we first introduce related concepts and notations.

Throughout this paper we assume that Ω is a bounded connected open subset of $\mathbb{R}^n$. We assume that B is a ball in Ω with diameter $\operatorname{diam}(B)$ and σB is the ball with the same center as B with $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$. We use $|E|$ to denote the Lebesgue measure of E. We denote w a weight if $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $w > 0$ a.e.. Also in general $d\mu = w dx$. For $0 < p < \infty$, we write $f \in L^p(E, w^\alpha)$ if the weighted L^p-norm of f over E satisfies $\|f\|_{p,E,w^\alpha} = \left(\int_E |f(x)|^p w(x)^\alpha dx\right)^{1/p} < \infty$, where α is a real number. A differential l-form ω on Ω is a schwartz distribution on Ω with value in $\Lambda^l(\mathbb{R}^n)$, we denote the space of differential l-forms by $D'(\Omega, \Lambda^l)$. We write $L^p(\Omega, \Lambda^l)$ for the l-forms $w(x) = \sum_I w_I(x) dx_I = \sum w_{i_1 i_2 \cdots i_l}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$ with $w_I \in L^p(\Omega, \mathbb{R})$ for all ordered l-tuples $I = (i_1, i_2, \ldots, i_l)$, $1 \le i_1 < i_2 < \cdots < i_l \le n$, $l = 0, 1, \ldots, n$. Thus $L^p(\Omega, \Lambda^l)$ is a Banach space with norm $\|w\|_{p,\Omega} = \left(\int_\Omega |w(x)|^p dx\right)^{1/p} = \left(\int_\Omega \left(\sum_I |w_I(x)|^2\right)^{p/2} dx\right)^{1/p}$. We denote the exterior derivative by $d : D'(\Omega, \Lambda^l) \to D'(\Omega, \Lambda^{l+1})$ for $l = 0, 1, \ldots, n-1$. Its formal adjoint operator $d^\star : D'(\Omega, \Lambda^{l+1}) \to D'(\Omega, \Lambda^l)$ is given by $d^\star = (-1)^{nl+1} \star d \star$ on $D'(\Omega, \Lambda^{l+1})$, $l = 0, 1, 2, \ldots, n-1$. A differential l-form $u \in D'(\Omega, \Lambda^l)$ is called a closed form if $du = 0$ in Ω. Similarly, a differential $(l+1)$-form $v \in D'(\Omega, \Lambda^{l+1})$ is called a coclosed form if $d^\star v = 0$. The l-form $\omega_B \in D'(B, \Lambda^l)$ is defined by $\omega_B = |B|^{-1} \int_B \omega(y) dy$, $l = 0$ and $\omega_B = d(T\omega)$, $l = 1, 2, \ldots, n$, for all $\omega \in L^p(B, \Lambda^l)$, $1 \le p < \infty$, here T is a homotopy operator, for its definition, see [8].

Then, we introduce some A-harmonic equations.

In this paper we consider solutions to the nonhomogeneous A-harmonic equation

$$A(x, g + du) = h + d^\star v \tag{1.1}$$

for differential forms, where $g, h \in D'(\Omega, \Lambda^l)$ and $A : \Omega \times \Lambda^l(\mathbb{R}^n) \to \Lambda^l(\mathbb{R}^n)$ satisfies the following conditions:

$$|A(x, \xi)| \le a|\xi|^{p-1}, \qquad \langle A(x, \xi), \xi \rangle \ge |\xi|^p, \tag{1.2}$$

for almost every $x \in \Omega$ and all $\xi \in \Lambda^l(\mathbb{R}^n)$. Here $a > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with (1.1) and $p^{-1} + q^{-1} = 1$. Note that if we choose $g = h = 0$ in (1.1), then (1.1) will reduce to the conjugate A-harmonic equation $A(x, du) = d^\star v$.

Definition 1.1. We call u and v a pair of conjugate A-harmonic tensor in Ω if u and v satisfy the conjugate A-harmonic equation

$$A(x, du) = d^\star v \tag{1.3}$$

in Ω, and A^{-1} exists in Ω, we call u and v conjugate A-harmonic tensors in Ω.

We also consider solutions to the equation of the form

$$d^\star A(x, dw) = B(x, dw), \tag{1.4}$$

here $A : \Omega \times \Lambda^l(\mathbb{R}^n) \to \Lambda^l(\mathbb{R}^n)$ and $B : \Omega \times \Lambda^l(\mathbb{R}^n) \to \Lambda^{l-1}(\mathbb{R}^n)$ satisfy the conditions:

$$|A(x,\xi)| \le a|\xi|^{p-1}, \qquad \langle A(x,\xi),\xi\rangle \ge |\xi|^p, \qquad |B(x,\xi)| \le b|\xi|^{p-1}, \tag{1.5}$$

for almost every $x \in \Omega$ and all $\xi \in \Lambda^l(\mathbb{R}^n)$. Here $a, b > 0$ are constants and $1 < p < \infty$ is a fixed exponent associated with (1.4). A solution to (1.4) is an element of the Sobolev space $W^1_{p,\mathrm{loc}}(\Omega, \Lambda^{l-1})$ such that

$$\int_\Omega \langle A(x,dw), d\varphi\rangle + \langle B(x,dw),\varphi\rangle = 0 \tag{1.6}$$

for all $\varphi \in W^1_{p,\mathrm{loc}}(\Omega, \Lambda^{l-1})$, with compact support.

Definition 1.2. We call u an A-harmonic tensor in Ω if u satisfies the A-harmonic equation (1.4) in Ω.

2. The Local and Global $A_{r,\lambda}(\Omega)$-Weighted Estimates

In this section, we will extend Lemma 2.3, see in [8], to new version with $A_{r,\lambda}(\Omega)$ weight both locally and globally.

Definition 2.1. We say a pair of weights $(w_1(x), w_2(x))$ satisfies the $A_{r,\lambda}(\Omega)$-condition in a domain Ω and write $(w_1(x), w_2(x)) \in A_{r,\lambda}(\Omega)$ for some $\lambda \ge 1$ and $1 < r < \infty$ with $1/r + 1/r' = 1$, if

$$\sup_{B\subset\Omega}\left(\frac{1}{|B|}\int_B (w_1)^\lambda dx\right)^{1/\lambda r}\left(\frac{1}{|B|}\int_B\left(\frac{1}{w_2}\right)^{\lambda r'/r} dx\right)^{1/\lambda r'} < \infty, \tag{2.1}$$

for any ball $B \subset \Omega$.

See [9] for properties of $A_{r,\lambda}(\Omega)$-weights. We will need the following generalized Hölder's inequality.

Lemma 2.2. *Let $0 < \alpha < \infty$, $0 < \beta < \infty$, and $s^{-1} = \alpha^{-1} + \beta^{-1}$, if f and g are measurable functions on $\mathbb{R}^n$, then*

$$\|fg\|_{s,\Omega} \le \|f\|_{\alpha,\Omega} \cdot \|g\|_{\beta,\Omega}, \tag{2.2}$$

for any $\Omega \in \mathbb{R}^n$.

We also need the following lemma; see [8].

Lemma 2.3. *Let u and v be a pair of solutions to the nonhomogeneous A-harmonic equation* (1.1) *in a domain $\Omega \subset \mathbb{R}^n$. If $g \in L^p(B, \Lambda^l)$ and $h \in L^q(B, \Lambda^l)$, then $du \in L^p(B, \Lambda^l)$ if and only if $d^\star v \in L^q(B, \Lambda^l)$. Moreover, there exist constants C_1 and C_2, independent of u and v, such that*

$$\begin{aligned} \|d^\star v\|_{q,B}^q &\le C_1\left(\|h\|_{q,B}^q + \|g\|_{p,B}^p + \|du\|_{p,B}^p\right), \\ \|du\|_{p,B}^p &\le C_2\left(\|h\|_{q,B}^q + \|g\|_{p,B}^p + \|d^\star v\|_{q,B}^q\right), \end{aligned} \tag{2.3}$$

for all balls B with $B \subset \Omega \subset \mathbb{R}^n$.

Theorem 2.4. *Let u and v be a pair of solutions to the nonhomogeneous A-harmonic equation* (1.1) *in a domain $\Omega \subset \mathbb{R}^n$. Assume that $(w_1(x), w_2(x)) \in A_{r,\lambda}(\Omega)$ for some $\lambda \ge 1$ and $1 < r < \infty$ with $1/r + 1/r' = 1$. Then, there exists a constants C, independent of u and v, such that*

$$\|d^\star v\|_{s,B,w_1^\alpha} \le C|B|^{\alpha r/s\lambda}\left(\|h\|_{t,B,w_2^{\alpha t/s}} + \left\||g|^{p/q}\right\|_{t,B,w_2^{\alpha t/s}} + \left\||du|^{p/q}\right\|_{t,B,w_2^{\alpha t/s}}\right), \tag{2.4}$$

for all balls B with $B \subset \Omega \subset \mathbb{R}^n$. Here α is any positive constant with $\lambda > \alpha r$, $s = q(\lambda - \alpha)/\lambda$, and $t = s\lambda/(\lambda - \alpha r) = qs\lambda/(s\lambda - q\alpha(r-1))$. Note that (2.4) *can be written as the following symmetric form:*

$$|B|^{-1/s}\|d^\star v\|_{s,B,w_1^\alpha} \le C|B|^{-1/t}\left(\|h\|_{t,B,w_2^{\alpha t/s}} + \left\||g|^{p/q}\right\|_{t,B,w_2^{\alpha t/s}} + \left\||du|^{p/q}\right\|_{t,B,w_2^{\alpha t/s}}\right). \tag{2.5}$$

Proof. Choose $s = q(\lambda - \alpha)/\lambda < q$, since $1/s = 1/q + (q-s)/qs$, using Hölder inequality, we find that

$$\begin{aligned} \|d^\star v\|_{s,B,w_1^\alpha} &= \left(\int_B |d^\star v|^s w_1^\alpha(x)\,dx\right)^{1/s} \\ &= \left(\int_B \left(|d^\star v| w_1^{\alpha/s}\right)^s dx\right)^{1/s} \\ &\le \left(\int_B |d^\star v|^q dx\right)^{1/q}\left(\int_B \left(w_1^{\alpha/s}\right)^{qs/(q-s)} dx\right)^{(q-s)/qs} \\ &\le \|d^\star v\|_{q,B}\left(\int_B w_1^\lambda dx\right)^{\alpha/\lambda s}. \end{aligned} \tag{2.6}$$

Applying the elementary inequality $|\sum_{i=1}^{N} t_i|^T \le N^{T-1}\sum_{i=1}^{N}|t_i|^T$ and Lemma 2.3, we obtain

$$\|d^\star v\|_{q,B} \le C_1\Big(\|h\|_{q,B} + \|g\|_{p,B}^{p/q} + \|du\|_{p,B}^{p/q}\Big). \tag{2.7}$$

Choose $t = qs\lambda/(s\lambda - q\alpha(r-1)) > q$, using Hölder inequality with $1/q = 1/t + (t-q)/qt$ again yields

$$\begin{aligned}
\|h\|_{q,B} &= \left(\int_B \left(|h| w_2^{\alpha/s} w_2^{-\alpha/s}\right)^q dx\right)^{1/q} \\
&\le \left(\int_B |h|^t w_2^{\alpha t/s} dx\right)^{1/t} \left(\int_B \left(\frac{1}{w_2}\right)^{\alpha q t/s(t-q)} dx\right)^{(t-q)/qt} \\
&= \|h\|_{t,B,w_2^{\alpha t/s}} \left(\int_B \left(\frac{1}{w_2}\right)^{\lambda/(r-1)} dx\right)^{\alpha(r-1)/\lambda s}.
\end{aligned} \tag{2.8}$$

Then, choosing $k = p + \alpha p t(r-1)/s\lambda > p$, using Hölder inequality once again, we have

$$\begin{aligned}
\|g\|_{p,B} &= \left(\int_B |g|^p w_2^{\alpha t/ks} w_2^{-\alpha t/ks} dx\right)^{1/p} \\
&\le \left(\int_B |g|^k w_2^{\alpha t/s} dx\right)^{1/k} \left(\int_B \left(\frac{1}{w_2}\right)^{\alpha t p/s(k-q)} dx\right)^{(k-q)/kp} \\
&= \|g\|_{k,B,w_2^{\alpha t/s}} \left(\int_B \left(\frac{1}{w_2}\right)^{\lambda/(r-1)} dx\right)^{k-p/kp}.
\end{aligned} \tag{2.9}$$

We know that

$$\begin{aligned}
\frac{k-p}{kp} &= \frac{\alpha t(r-1)}{s\lambda} \cdot \frac{s\lambda}{s\lambda p + \alpha p t(r-1)} \\
&= \frac{\alpha(r-1)}{sp} \cdot \frac{st}{s\lambda + \alpha t(r-1)} \\
&= \frac{\alpha(r-1)q}{sp\lambda},
\end{aligned} \tag{2.10}$$

and hence

$$\|g\|_{p,B}^{p/q} \le \|g\|_{k,B,w_2^{\alpha t/s}}^{p/q} \cdot \left(\int_B \left(\frac{1}{w_2}\right)^{\lambda/(r-1)} dx\right)^{\alpha(r-1)/s\lambda}. \tag{2.11}$$

Note that

$$
\begin{aligned}
\|g\|_{k,B,w_2^{\alpha t/s}}^{p/q} &= \left(\int_B |g|^k w_2^{\alpha t/s} dx\right)^{p/kq} \\
&= \left(\int_B |g|^{(ps\lambda+\alpha pt(r-1))/s\lambda} w_2^{\alpha t/s} dx\right)^{ps\lambda/(pqs\lambda+\alpha pqt(r-1))} \\
&= \left(\int_B |g|^{p(s\lambda+\alpha t(r-1))/s\lambda} w_2^{\alpha t/s} dx\right)^{s\lambda/(qs\lambda+\alpha qt(r-1))}.
\end{aligned} \tag{2.12}
$$

Since

$$
(r-1)\alpha t + s\lambda = \frac{s\lambda t}{q}, \tag{2.13}
$$

then,

$$
\begin{aligned}
\|g\|_{k,B,w_2^{\alpha t/s}}^{p/q} &= \left(\int_B |g|^{pt/q} w_2^{\alpha t/s} dx\right)^{1/t} \\
&= \left\| |g|^{p/q} \right\|_{t,B,w_2^{\alpha t/s}}.
\end{aligned} \tag{2.14}
$$

Combining (2.11) and (2.14), we obtain

$$
\|g\|_{p,B}^{p/q} \le \left\| |g|^{p/q} \right\|_{t,B,w_2^{\alpha t/s}} \cdot \left(\int_B \left(\frac{1}{w_2}\right)^{\lambda/(r-1)} dx\right)^{\alpha(r-1)/s\lambda}. \tag{2.15}
$$

Using the similar method, we can easily get that

$$
\|du\|_{p,B}^{p/q} \le \left\| |du|^{p/q} \right\|_{t,B,w_2^{\alpha t/s}} \cdot \left(\int_B \left(\frac{1}{w_2}\right)^{\lambda/(r-1)} dx\right)^{\alpha(r-1)/s\lambda}. \tag{2.16}
$$

Combining (2.6) and (2.7) gives

$$
\|d^\star v\|_{s,B,w_1^\alpha} \le C_1\left(\|h\|_{q,B} + \|g\|_{p,B}^{p/q} + \|du\|_{p,B}^{p/q}\right)\left(\int_B w_1^\lambda dx\right)^{\alpha/s\lambda}. \tag{2.17}
$$

Substituting (2.8), (2.15), and (2.16) into (2.17), we have

$$
\begin{aligned}
\|d^\star v\|_{s,B,w_1^\alpha} \le C_1&\left(\|h\|_{t,B,w_2^{\alpha t/s}} + \left\| |g|^{p/q} \right\|_{t,B,w_2^{\alpha t/s}} + \left\| |du|^{p/q} \right\|_{t,B,w_2^{\alpha t/s}}\right) \\
&\cdot \left(\int_B w_1^\lambda dx\right)^{\alpha/s\lambda} \left(\int_B \left(\frac{1}{w_2}\right)^{\lambda/(r-1)} dx\right)^{\alpha(r-1)/s\lambda}.
\end{aligned} \tag{2.18}
$$

Since $(w_1, w_2) \in A_{r,\lambda}(\Omega)$, then

$$\begin{aligned}
&\left(\int_B w_1^{\lambda} dx\right)^{\alpha/s\lambda}\left(\int_B\left(\frac{1}{w_2}\right)^{\lambda/(r-1)} dx\right)^{\alpha(r-1)/s\lambda} \\
&\quad = \left(\left(\int_B w_1^{\lambda} dx\right)\left(\int_B\left(\frac{1}{w_2}\right)^{\lambda/(r-1)} dx\right)^{(r-1)}\right)^{\alpha/s\lambda} \\
&\quad = \left(|B|^{1/\lambda r}\left(\frac{1}{|B|}\int_B w_1^{\lambda} dx\right)^{1/\lambda r}|B|^{1/\lambda r'}\left(\frac{1}{|B|}\int_B\left(\frac{1}{w_2}\right)^{\lambda r'/r} dx\right)^{1/\lambda r'}\right)^{\alpha r/s} \\
&\quad \leq C_2|B|^{\alpha r/s\lambda}.
\end{aligned} \tag{2.19}$$

Putting (2.19) into (2.18), we obtain the desired result

$$\|d^{\star}v\|_{s,B,w_1^{\alpha}} \leq C_3|B|^{\alpha r/s\lambda}\left(\|h\|_{t,B,w_2^{\alpha t/s}} + \left\||g|^{p/q}\right\|_{t,B,w_2^{\alpha t/s}} + \left\||du|^{p/q}\right\|_{t,B,w_2^{\alpha t/s}}\right). \tag{2.20}$$

The proof of Theorem 2.4 has been completed. □

Using the same method, we have the following two-weighted L^s-estimate for du.

Theorem 2.5. *Let u and v be a pair of solutions to the nonhomogeneous A-harmonic equation* (1.1) *in a domain $\Omega \subset \mathbb{R}^n$. Assume that $(w_1(x), w_2(x)) \in A_{r,\lambda}(\Omega)$ for some $\lambda \geq 1$ and $1 < r < \infty$ with $1/r + 1/r' = 1$. Then, there exists a constants C, independent of u and v, such that*

$$\|du\|_{s,B,w_1^{\alpha}} \leq C|B|^{\alpha r/s\lambda}\left(\|g\|_{t,B,w_2^{\alpha t/s}} + \left\||h|^{q/p}\right\|_{t,B,w_2^{\alpha t/s}} + \left\||d^{\star}v|^{q/p}\right\|_{t,B,w_2^{\alpha t/s}}\right), \tag{2.21}$$

for all balls B with $B \subset \Omega \subset \mathbb{R}^n$. Here α is any positive constant with $\lambda > \alpha r$, $s = p(\lambda - \alpha)/\lambda$, and $t = s\lambda/(\lambda - \alpha r) = ps\lambda/(s\lambda - p\alpha(r-1))$.

It is easy to see that the inequality (2.21) *is equivalent to*

$$|B|^{-1/s}\|du\|_{s,B,w_1^{\alpha}} \leq C|B|^{-1/t}\left(\|g\|_{t,B,w_2^{\alpha t/s}} + \left\||h|^{q/p}\right\|_{t,B,w_2^{\alpha t/s}} + \left\||d^{\star}v|^{q/p}\right\|_{t,B,w_2^{\alpha t/s}}\right). \tag{2.22}$$

As applications of the local results, we prove the following global norm comparison theorem.

Lemma 2.6. *Each Ω has a modified Whitney cover of cubes $\mathcal{V} = \{Q_i\}$ such that*

$$\begin{gathered}
\bigcup_i Q_i = \Omega, \\
\sum_{Q \in \mathcal{V}} \chi_{\sqrt{5/4}Q} \leq N\chi_{\Omega},
\end{gathered} \tag{2.23}$$

for all $x \in \mathbb{R}^n$ and some $N > 1$ and if $Q_i \cap Q_j \neq \emptyset$, then there exists a cube $\mathbb{R}$ (this cube does not need be a member of $\mathcal{V}$) in $Q_i \cap Q_j$ such that $Q_i \cap Q_j \subset N\mathbb{R}$.

Theorem 2.7. *Let u and v be a pair of solutions to the nonhomogeneous A-harmonic equation* (1.1) *in a bounded domain $\Omega \subset \mathbb{R}^n$. Assume that $(w_1(x), w_2(x)) \in A_{r,\lambda}(\Omega)$ for some $\lambda \geq 1$ and $1 < r < \infty$ with $1/r + 1/r' = 1$. Then, there exist constants C_1 and C_2, independent of u and v, such that*

$$\|d^\star v\|_{s,\Omega,w_1^\alpha} \leq C_1\left(\|h\|_{t,\Omega,w_2^{\alpha t/s}} + \left\| |g|^{p/q} \right\|_{t,\Omega,w_2^{\alpha t/s}} + \left\| |du|^{p/q} \right\|_{t,\Omega,w_2^{\alpha t/s}} \right). \tag{2.24}$$

Here α is any positive constant with $\lambda > \alpha r$, $s = q(\lambda - \alpha)/\lambda$, $t = s\lambda/(\lambda - \alpha r) = qs\lambda/(s\lambda - q\alpha(r-1))$, and

$$\|du\|_{s,\Omega,w_1^\alpha} \leq C_2\left(\|g\|_{t,\Omega,w_2^{\alpha t/s}} + \left\| |h|^{q/p} \right\|_{t,\Omega,w_2^{\alpha t/s}} + \left\| |d^\star v|^{q/p} \right\|_{t,\Omega,w_2^{\alpha t/s}} \right), \tag{2.25}$$

for $s = p(\lambda - \alpha)/\lambda$ and $t = s\lambda/(\lambda - \alpha r) = ps\lambda/(s\lambda - p\alpha(r-1))$.

Proof. Applying Theorem 2.4 and Lemma 2.6, we have

$$\begin{aligned}
\|d^\star v\|_{s,\Omega,w_1^\alpha} &= \left(\int_\Omega |d^\star v|^s w_1^\alpha dx \right)^{1/s} \\
&\leq \sum_{B\in\mathcal{V}} \left(\int_B |d^\star v|^s w_1^\alpha dx \right)^{1/s} \\
&\leq \sum_{B\in\mathcal{V}} \left(\int_B |d^\star v|^s w_1^\alpha dx \right)^{1/s} \chi_{\sqrt{5/4}B} \\
&\leq C_1 \sum_{B\in\mathcal{V}} |B|^{\alpha r/s\lambda} \left(\|h\|_{t,B,w_2^{\alpha t/s}} + \left\| |g|^{p/q} \right\|_{t,B,w_2^{\alpha t/s}} + \left\| |du|^{p/q} \right\|_{t,B,w_2^{\alpha t/s}} \right) \chi_{\sqrt{5/4}B} \\
&\leq C_1 \sum_{B\in\mathcal{V}} |\Omega|^{\alpha r/s\lambda} \left(\|h\|_{t,\Omega,w_2^{\alpha t/s}} + \left\| |g|^{p/q} \right\|_{t,\Omega,w_2^{\alpha t/s}} + \left\| |du|^{p/q} \right\|_{t,\Omega,w_2^{\alpha t/s}} \right) \chi_{\sqrt{5/4}B} \\
&\leq C_2 \left(\|h\|_{t,\Omega,w_2^{\alpha t/s}} + \left\| |g|^{p/q} \right\|_{t,\Omega,w_2^{\alpha t/s}} + \left\| |du|^{p/q} \right\|_{t,\Omega,w_2^{\alpha t/s}} \right) \sum_{B\in\mathcal{V}} \chi_{\sqrt{5/4}B} \\
&\leq C_3 \left(\|h\|_{t,\Omega,w_2^{\alpha t/s}} + \left\| |g|^{p/q} \right\|_{t,\Omega,w_2^{\alpha t/s}} + \left\| |du|^{p/q} \right\|_{t,\Omega,w_2^{\alpha t/s}} \right).
\end{aligned} \tag{2.26}$$

Since Ω is bounded. The proof of inequality (2.24) has been completed. Similarly, using Theorem 2.5 and Lemma 2.6, inequality (2.25) can be proved immediately. This ends the proof of Theorem 2.7. □

Definition 2.8. We say the weight $w(x)$ satisfies the $A_r(\Omega)$-condition in a domain Ω write $w(x) \in A_r(\Omega)$ for some $1 < r < \infty$ with $1/r + 1/r' = 1$, if

$$\sup_{B\subset\Omega}\left(\frac{1}{|B|}\int_B w\,dx\right)^{1/r}\left(\frac{1}{|B|}\int_B\left(\frac{1}{w}\right)^{r'/r}dx\right)^{1/r'} < \infty, \tag{2.27}$$

for any ball $B \subset \Omega$.

We see that $A_{r,\lambda}(\Omega)$-weight reduce to the usual $A_r(\Omega)$-weight if $w_1(x) = w_2(x)$ and $\lambda = 1$; see [10].

And, if $w_1(x) = w_2(x)$ and $\lambda = 1$ in Theorem 2.7, it is easy to obtain Theorems 4.2 and 4.4 in [8].

3. Estimates for Lipschitz Norms and BMO Norms

In [11] Ding has presented some estimates for the Lipchitz norms and BMO norms. In this section, we will prove another estimates for the Lipchitz norms and BMO norms.

Definition 3.1. Let $\omega \in L^1_{\mathrm{loc}}(\Omega, \Lambda^l)$, $l = 0, 1, 2, \ldots, n$. We write $\omega \in \mathrm{locLip}_k(\Omega, \Lambda^l)$, $0 \le k \le 1$, if

$$\|\omega\|_{\mathrm{locLip}_k,\Omega} = \sup_{\sigma B\subset\Omega}|B|^{-(n+k)/n}\|\omega - \omega_B\|_{1,B} < \infty, \tag{3.1}$$

for some $\sigma \ge 1$.

Similarly, we write $\omega \in \mathrm{BMO}(\Omega, \Lambda^l)$ if

$$\|\omega\|_{\star,\Omega} = \sup_{\sigma B\subset\Omega}|B|^{-1}\|\omega - \omega_B\|_{1,B} < \infty, \tag{3.2}$$

for some $\sigma \ge 1$. When ω is a o-form, (3.2) reduces to the classical definition of $\mathrm{BMO}(\Omega)$.

We also discuss the weighted Lipschitz and BMO norms.

Definition 3.2. Let $\omega \in L^1_{\mathrm{loc}}(\Omega, \Lambda^l, w^\alpha)$, $l = 0, 1, 2, \ldots, n$. We write $\omega \in \mathrm{locLip}_k(\Omega, \Lambda^l, w^\alpha)$, $0 \le k \le 1$, if

$$\|\omega\|_{\mathrm{locLip}_k,\Omega,w^\alpha} = \sup_{\sigma B\subset\Omega}\left(\mu(B)\right)^{-(n+k)/n}\|\omega - \omega_B\|_{1,B,w^\alpha} < \infty. \tag{3.3}$$

Similarly, for $\omega \in L^1_{\mathrm{loc}}(\Omega, \Lambda^l, w^\alpha)$, $l = 0, 1, 2, \ldots, n$. We write $\omega \in \mathrm{BMO}(\Omega, \Lambda^l, w^\alpha)$, if

$$\|\omega\|_{\star,\Omega,w^\alpha} = \sup_{\sigma B\subset\Omega}\left(\mu(B)\right)^{-1}\|\omega - \omega_B\|_{1,B,w^\alpha} < \infty, \tag{3.4}$$

for some $\sigma > 1$, where Ω is a bounded domain, the measure μ is defined by $d\mu = w(x)^\alpha dx$, w is a weight, and α is a real number.

We need the following classical Poincaré inequality; see [10].

Lemma 3.3. *Let $u \in D'(\Omega, \Lambda^l)$ and $du \in L^q(B, \Lambda^{l+1})$, then $u - u_B$ is in $W_q^1(B, \Lambda^l)$ with $1 < q < \infty$ and*

$$\|u - u_B\|_{q,B} \leq C(n,q)|B||B|^{1/n}\|du\|_{q,B}. \tag{3.5}$$

We also need the following lemma; see [2].

Lemma 3.4. *Suppose that u is a solution to (1.4), $\sigma > 1$ and $q > 0$. There exists a constant C, depending only on σ, n, p, a, b, and q, such that*

$$\|du\|_{p,B} \leq C|B|^{(q-p)/pq}\|du\|_{q,\sigma B}, \tag{3.6}$$

for all balls B with $\sigma B \subset \Omega$.

We need the following local weighted Poincaré inequality for A-harmonic tensors.

Theorem 3.5. *Let $u \in D'(\Omega, \Lambda^l)$ be an A-harmonic tensor in a domain $\Omega \subset \mathbb{R}^n$ and $du \in L^s(\Omega, \Lambda^{l+1})$, $l = 0, 1, 2, \ldots, n$. Assume that $\sigma > 1$, $1 < s < \infty$, and $(w_1(x), w_2(x)) \in A_{r,\lambda}(\Omega)$ for some $\lambda \geq 1$ and $1 < r < \infty$ with $1/r + 1/r' = 1$. Then, there exists a constant C, independent of u, such that*

$$\|u - u_B\|_{s,B,w_1^\alpha} \leq C|B||B|^{1/n}\|du\|_{s,\sigma B,w_2^\alpha}, \tag{3.7}$$

for all balls B with $\sigma B \subset \Omega$. Here α is any constant with $0 < \alpha < \lambda$.

Proof. Choose $t = \lambda s/(\lambda - \alpha)$, since $1/s = 1/t + (t-s)/st$, using Hölder inequality, we find that

$$\begin{aligned}
\|u - u_B\|_{s,B,w_1^\alpha} &= \left(\int_B |u - u_B|^s w_1^\alpha dx\right)^{1/s} \\
&= \left(\int_B \left(|u - u_B| w_1^{\alpha/s}\right)^s dx\right)^{1/s} \\
&\leq \left(\int_B |u - u_B|^t dx\right)^{1/t} \left(\int_B \left(w_1^{\alpha/s}\right)^{st/(t-s)} dx\right)^{(t-s)/st} \\
&= \|u - u_B\|_{t,B} \left(\int_B w_1^\lambda dx\right)^{\alpha/\lambda s}.
\end{aligned} \tag{3.8}$$

Taking $m = \lambda s/(\lambda + \alpha(r-1))$, then $m < s < t$, using Lemmas 3.4 and 3.3 and the same method as [2, Proof of Theorem 2.12], we obtain

$$\|u - u_B\|_{s,B,w_1^\alpha} \leq C_2|B|^{1+1/n}|B|^{(m-t)/mt}\|du\|_{m,\sigma B}\|w_1\|_{\lambda,B}^{\alpha/s}, \tag{3.9}$$

where $\sigma > 1$. Using Hölder inequality with $1/m = 1/s + (s-m)/sm$ again yields

$$\begin{aligned}
\|du\|_{m,\sigma B} &= \left(\int_{\sigma B} |du|^m w_2^{\alpha m/s} w_2^{-\alpha m/s} dx\right)^{1/m} \\
&= \left(\int_{\sigma B} \left(|du| w_2^{\alpha/s} w_2^{-\alpha/s}\right)^m dx\right)^{1/m} \\
&\le \left(\int_{\sigma B} |du|^s w_2^{\alpha} dx\right)^{1/s} \left(\int_{\sigma B} \left(\frac{1}{w_2}\right)^{\lambda/(r-1)} dx\right)^{\alpha(r-1)/\lambda s}.
\end{aligned} \tag{3.10}$$

Substituting (3.10) in (3.9), we have

$$\|u - u_B\|_{s,B,w_1^\alpha} \le C_2 |B|^{1+1/n+(m-t)/mt} \|du\|_{s,\sigma B,w_2^\alpha} \|w_1\|_{\lambda,B}^{\alpha/s} \left\|\frac{1}{w_2}\right\|_{\lambda/(r-1),\sigma B}^{\alpha/s}. \tag{3.11}$$

Since $(w_1(x), w_2(x)) \in A_{r,\lambda}(\Omega)$, then

$$\begin{aligned}
&\|w_1\|_{\lambda,B}^{\alpha/s} \left\|\frac{1}{w_2}\right\|_{\lambda/(r-1),\sigma B}^{\alpha/s} \\
&\quad\le \left(\left(\int_{\sigma B} w_1^{\lambda} dx\right)\left(\int_{\sigma B} \left(\frac{1}{w_2}\right)^{\lambda/(r-1)} dx\right)^{r-1}\right)^{\alpha/\lambda s} \\
&\quad= \left(|\sigma B|^{1/\lambda} \left(\frac{1}{|\sigma B|}\int_{\sigma B} w_1^{\lambda} dx\right)^{1/\lambda r} \left(\frac{1}{|\sigma B|}\int_{\sigma B} \left(\frac{1}{w_2}\right)^{\lambda r'/r} dx\right)^{1/\lambda r'}\right)^{r\alpha/s} \\
&\quad\le C_3 |B|^{r\alpha/\lambda s}.
\end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12) gives

$$\|u - u_B\|_{s,B,w_1^\alpha} \le C_4 |B|^{1+1/n+(m-t)/mt+r\alpha/\lambda s} \|du\|_{s,\sigma B,w_2^\alpha}. \tag{3.13}$$

Note that

$$\frac{m-t}{mt} + \frac{r\alpha}{\lambda s} = \frac{\lambda - \alpha}{\lambda s} - \frac{\lambda + \alpha(r-1)}{\lambda s} + \frac{r\alpha}{\lambda s} = 0. \tag{3.14}$$

Finally, we obtain the desired result

$$\|u - u_B\|_{s,B,w_1^\alpha} \le C_4 |B|^{1+1/n} \|du\|_{s,\sigma B,w_2^\alpha}. \tag{3.15}$$

This ends the proof of Theorem 3.5. □

Similarly, if setting $w_1(x) = w_2(x)$ and $\lambda = 1$ in Theorem 3.5, we obtain Theorem 2.12 in [2]. And we choose $w_1(x) = w_2(x) = 1$ in Theorem 3.5, we have the classical Poincaré inequality (3.5).

Lemma 3.6 (see [8]). *Let u and v be a pair of solution to the conjugate A-harmonic tensor in Ω. Assume $w(x) \in A_r(\Omega)$ for some $r \geq 1$. Then, there exists a constant C, independent of u, such that*

$$\|du\|_{s,\Omega,w^{\alpha}} \leq C\|d^{\star}v\|^{q/p}_{qt/p,\Omega,w^{\alpha t/s}}. \tag{3.16}$$

Here α is any positive constant with $1 > \alpha r$, $s = (1-\alpha)p$ and $t = s/(1-\alpha r) = ps/(s - \alpha p(r-1))$.

Theorem 3.7. *Let $u \in D'(\Omega, \Lambda^l)$ be an A-harmonic tensor in a domain $\Omega \subset \mathbb{R}^n$, and all $c \in D'(\Omega, \Lambda^l)$ with $dc = 0$, and $du \in L^s(\Omega, \Lambda^{l+1})$, $l = 0, 1, 2, \ldots, n-1$. Assume that $1 < s < \infty$ and $(w_1(x), w_2(x)) \in A_{r,\lambda}(\Omega)$ for some $\lambda \geq 1$ and $1 < r < \infty$ with $w_1(x) \geq \epsilon > 0$ for any $x \in \Omega$. Then, there exist constants C and C', independent of u, such that*

$$\|u - c\|_{locLip_k,\Omega,w_1^{\alpha}} \leq C\|du\|_{s,\Omega,w_2^{\alpha}}, \tag{3.17}$$

$$\|u - c\|_{\star,\Omega,w_1^{\alpha}} \leq C^{'}\|du\|_{s,\Omega,w_2^{\alpha}}, \tag{3.18}$$

where k and α are constants with $0 \leq k \leq 1$ and $0 < \alpha < \lambda$.

Proof. We note that $\mu_1(B) = \int_B w_1^{\alpha} dx \geq \int_B \epsilon^{\alpha} dx = C_1|B|$ implies that

$$\frac{1}{\mu_1(B)} \leq \frac{C_2}{|B|}, \tag{3.19}$$

for any ball B. Using (3.7) and the Hölder inequality with $1 = 1/s + (s-1)/s$, we have

$$\begin{aligned} \|u - u_B\|_{1,B,w_1^{\alpha}} &= \int_B |u - u_B| d\mu_1 \\ &\leq \left(\int_B |u - u_B|^s d\mu_1\right)^{1/s} \left(\int_B 1^{s/(s-1)} d\mu_1\right)^{(s-1)/s} \\ &= \left(\mu_1(B)\right)^{(s-1)/s} \|u - u_B\|_{s,B,w_1^{\alpha}} \\ &\leq \left(\mu_1(B)\right)^{1-1/s} \left(C_3|B|^{1+1/n} \|du\|_{s,\sigma B,w_2^{\alpha}}\right). \end{aligned} \tag{3.20}$$

From the definition of the Lipschitz norm (3.3), (3.19), and (3.20), we obtain

$$\begin{aligned}
\|u-c\|_{\operatorname{locLip}_k,\Omega,w_1^\alpha} &= \sup_{\sigma B\subset\Omega} \left(\mu_1(B)\right)^{-(n+k)/n}\left(\|u-c-(u-c)_B\|_{1,B,w_1^\alpha}\right) \\
&= \sup_{\sigma B\subset\Omega} \left(\mu_1(B)\right)^{-1-k/n}\left(\|u-u_B\|_{1,B,w_1^\alpha}\right) \\
&\le C_3 \sup_{\sigma B\subset\Omega} \left(\mu_1(B)\right)^{-1/s-k/n}\left(|B|^{1+1/n}\|du\|_{s,\sigma B,w_2^\alpha}\right) \\
&\le C_4 \sup_{\sigma B\subset\Omega} \left(|B|^{-1/s-k/n+1+1/n}\|du\|_{s,\sigma B,w_2^\alpha}\right) \\
&\le C_4 \sup_{\sigma B\subset\Omega} \left(|\Omega|^{-1/s-k/n+1+1/n}\|du\|_{s,\sigma B,w_2^\alpha}\right) \\
&\le C_5 \sup_{\sigma B\subset\Omega} \left(\|du\|_{s,\sigma B,w_2^\alpha}\right) \\
&\le C_5 \|du\|_{s,\Omega,w_2^\alpha}.
\end{aligned} \tag{3.21}$$

Since $1-1/s+1/n-k/n>0$ and $|\Omega|<\infty$. The desired result for Lipschitz norm has been completed.

Then, we prove the theorem for BMO norm

$$\begin{aligned}
\|u-c\|_{\star,\Omega,w_1^\alpha} &= \sup_{\sigma B\subset\Omega} \left(\mu_1(B)\right)^{-1}\left(\|u-c-(u-c)_B\|_{1,B,w_1^\alpha}\right) \\
&\le \sup_{\sigma B\subset\Omega} \left(\mu_1(\Omega)\right)^{k/n}\left(\left(\mu_1(B)\right)^{-(n+k)/n}\|u-u_B\|_{1,B,w_1^\alpha}\right) \\
&\le \left(\mu_1(\Omega)\right)^{k/n} \sup_{\sigma B\subset\Omega}\left(\left(\mu_1(B)\right)^{-(n+k)/n}\|u-u_B\|_{1,B,w_1^\alpha}\right).
\end{aligned} \tag{3.22}$$

From (3.21) we find

$$\|u-c\|_{\star,\Omega,w_1^\alpha} \le C_1\|u-c\|_{\operatorname{locLip}_k,\Omega,w_1^\alpha}. \tag{3.23}$$

Using (3.17) we have

$$\|u-c\|_{\star,\Omega,w_1^\alpha} \le C_2\|du\|_{s,\Omega,w_2^\alpha}. \tag{3.24}$$

Now, we have completed the proof of Theorem 3.7. □

Similarly, if setting $w_1(x)=w_2(x)=w(x)$ and $\lambda=1$ in Theorem 3.7, we obtain the following theorem.

Theorem 3.8. *Let $u\in D'(\Omega,\Lambda^l)$ be an A-harmonic tensor in a domain $\Omega\subset\mathbb{R}^n$, and all $c\in D'(\Omega,\Lambda^l)$ with $dc=0$, and $du\in L^s(\Omega,\Lambda^{l+1})$, $l=0,1,2,\ldots,n-1$. Assume that $1<s<\infty$*

and $w(x) \in A_r(\Omega)$ for $r > 1$ with $w(x) \geq \epsilon > 0$ for any $x \in \Omega$. Then, there exist constants C and C', independent of u, such that

$$\|u - c\|_{locLip_k,\Omega,w^\alpha} \leq C\|du\|_{s,\Omega,w^\alpha}, \tag{3.25}$$

$$\|u - c\|_{\star,\Omega,w^\alpha} \leq C^{'}\|du\|_{s,\Omega,w^\alpha}, \tag{3.26}$$

where k and α are constants with $0 \leq k \leq 1$ and $0 \leq \alpha \leq 1$.

If $w \equiv 1$, we have

$$\begin{gathered} \|u - c\|_{\mathrm{locLip}_k,\Omega} \leq C\|du\|_{s,\Omega}, \\ \|u - c\|_{\star,\Omega} \leq C^{'}\|du\|_{s,\Omega}. \end{gathered} \tag{3.27}$$

Using Lemma 3.6, we can also obtain the following theorem.

Theorem 3.9. *Let u and v be a pair of conjugate A-harmonic tensor in a domain $\Omega \subset \mathbb{R}^n$, then $du \in L^p(\Omega, \Lambda^l, \mu)$ if and only if $d^\star v \in L^q(\Omega, \Lambda^l, \mu)$ where the measure μ is defined by $d\mu = w(x)^\alpha dx$, and all $c \in D'(\Omega, \Lambda^l)$ with $dc = 0$. Assume that $w(x) \in A_r(\Omega)$ for $r > 1$ with $w(x) \geq \epsilon > 0$ for any $x \in \Omega$. Then, there exist constants C and C', independent of u and v, such that*

$$\begin{gathered} \|u - c\|_{locLip_k,\Omega,w^\alpha} \leq C\|d^\star v\|^{q/p}_{qt/p,\Omega,w^{\alpha t/s}}, \\ \|u - c\|_{\star,\Omega,w^\alpha} \leq C^{'}\|d^\star v\|^{q/p}_{qt/p,\Omega,w^{\alpha t/s}}, \end{gathered} \tag{3.28}$$

where k and α are positive constants with $0 \leq k \leq 1$ and $\alpha r < 1$, for $s = (1-\alpha)p$, $t = s/(1-\alpha r) = ps/(s - \alpha p(r-1))$.

Proof. From (3.25), we have

$$\|u - c\|_{\mathrm{locLip}_k,\Omega,w^\alpha} \leq C_1\|du\|_{s,\Omega,w^\alpha}. \tag{3.29}$$

Choose $s = (1-\alpha)p$, $t = s/(1-\alpha r) = ps/(s - \alpha p(1-r))$, using Lemma 3.6, it is easy to obtain the desire result

$$\|u - c\|_{\mathrm{locLip}_k,\Omega,w^\alpha} \leq C_2\|d^\star v\|^{q/p}_{qt/p,\Omega,w^{\alpha t/s}}. \tag{3.30}$$

Using the similar method for BMO norm, we have

$$\|u - c\|_{\star,\Omega,w^\alpha} \leq C_3\|du\|_{s,\Omega,w^\alpha} \leq C_4\|d^\star v\|^{q/p}_{qt/p,\Omega,w^{\alpha t/s}}. \tag{3.31}$$

If $w \equiv 1$, we have

$$\|u - c\|_{\mathrm{locLip}_k,\Omega} \leq C\|d^\star v\|_{q,\Omega}^{q/p},$$
$$\|u - c\|_{\star,\Omega} \leq C\|d^\star v\|_{q,\Omega}^{q/p}. \tag{3.32}$$

□

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