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Research Article

On a Li-Stević Integral-Type Operators between Different Weighted Bloch-Type Spaces

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Let φ be an analytic self-map of the unit disk $\mathbb{D}$, let $H(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$, and let $g \in H(\mathbb{D})$. Recently, Li and Stević defined the following operator: $C_\varphi^g f(z) = \int_0^z f'(\varphi(w))g(w)dw$, on $H(\mathbb{D})$. The boundedness and compactness of the operator between two weighted Bloch-type spaces are investigated in this paper.

1. Introduction

First, we introduce some basic notation which is used in this paper. Throughout the entire paper, the unit disk in the finite complex plane $\mathbb{C}$ will be denoted by $\mathbb{D}$. $H(\mathbb{D})$ will denote the space of all analytic functions on $\mathbb{D}$. Every analytic self-map φ of the unit disk $\mathbb{D}$ induces through composition a linear composition operator C_φ from $H(\mathbb{D})$ to itself. It is a well-known consequence of Littlewood's subordination principle [1] that the formula $C_\varphi(f) = f \circ \varphi$ defines a bounded linear operator on the classical Hardy and Bergman spaces. That is, $C_\varphi : H^p \to H^p$ and $C_\varphi : A^p \to A^p$ are bounded operators. A problem that has received much attention recently is to relate function theoretic properties of φ to operator theoretic properties of the restriction of C_φ to various Banach spaces of analytic functions. Some characterizations of the boundedness and compactness of the composition operator between various Banach spaces of analytic functions can be found in [2–6]. Recently, Yoneda in [7] gave some necessary and sufficient conditions for a composition operator C_φ to be bounded and compact on the logarithmic Bloch space defined as follows:

$$\mathcal{B}_{\log} = \left\{ f \in H(\mathbb{D}) : \|f\| = \sup_{z\in\mathbb{D}} (1-|z|^2)\left(\log\frac{2}{1-|z|^2}\right)\left|f'(z)\right| < \infty \right\}. \tag{1.1}$$

The space $\mathcal{B}_{\log}$ is a Banach space under the norm $\|f\|_{B_{\log}} = |f(0)| + \|f\|$. Ye in [8] characterized the boundedness and compactness of the weighted composition operator uC_φ between the logarithmic Bloch space $\mathcal{B}_{\log}$ and the α-Bloch space $\mathcal{B}^\alpha$ on the unit disk as well as the boundedness and compactness of the weighted composition operator uC_φ between the little logarithmic Bloch space $\mathcal{B}_{\log}^0$ and the little α-Bloch space $\mathcal{B}_0^\alpha$ on the unit disk. A function $f \in H(\mathbb{D})$ is said to belong to the Bloch-type space (or α-Bloch space), denoted by $\mathcal{B}^\alpha = \mathcal{B}^\alpha(\mathbb{D})$, if

$$B_\alpha(f) = \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right)^\alpha \left|f'(z)\right| < \infty. \tag{1.2}$$

The space $\mathcal{B}^\alpha$ becomes a Banach space with the norm $\|f\|_\alpha = |f(0)| + B_\alpha(f)$. Let $\mathcal{B}_0^\alpha$ denote the subspace of $\mathcal{B}^\alpha$ consisting of those $f \in \mathcal{B}^\alpha$ such that

$$\lim_{|z|\to 1} \left(1 - |z|^2\right)^\alpha \left|f'(z)\right| = 0. \tag{1.3}$$

This space is called the little Bloch-type space. For $\alpha = 1$, we obtain the well-known classical Bloch space and the little Bloch space, simply denoted by $\mathcal{B}$ and $\mathcal{B}_0$. Let $\mathcal{B}_{\log}^0$ denote the subspace of $\mathcal{B}_{\log}$ consisting of those $f \in \mathcal{B}_{\log}$ such that

$$\lim_{|z|\to 1} (1 - |z|^2) \left(\log \frac{2}{1 - |z|^2}\right) \left|f'(z)\right| = 0. \tag{1.4}$$

Ye in [9] proved that $\mathcal{B}_{\log}^0$ is a closed subspace of $\mathcal{B}_{\log}$. Galanopoulos in [10] characterized the boundedness and compactness of the composition operator $C_\varphi : \mathcal{B}_{\log} \to Q_{\log}^p$ and the boundedness and compactness of the weighted composition operator $uC_\varphi : \mathcal{B}_{\log} \to \mathcal{B}_{\log}$. Some characterizations of the weighted composition operator between various Bloch-type spaces can be found in [11–16]. Li and Stević in [17] studied the boundedness and compactness of the following two Volterra-type integral operators:

$$J_g f(z) = \int_0^z f(\xi) g'(\xi) \mathrm{d}\xi, \qquad I_g f(z) = \int_0^z f'(\xi) g(\xi) \mathrm{d}\xi, \tag{1.5}$$

on the Zygmund space, for any $g \in H(\mathbb{D})$. Li and Stević in [18], for $f, g \in H(\mathbb{D})$, defined a linear operator as follows:

$$C_\varphi^g f(z) = \int_0^z f'(\varphi(w)) g(w) \mathrm{d}w. \tag{1.6}$$

They called the operator the generalized composition operator and studied the boundedness and compactness of the operator on the Zygmund space, the Bloch-type space $\mathcal{B}^\alpha$, and the little Bloch-type space $\mathcal{B}_0^\alpha$. They also studied the weak compactness of the operator on the little

Bloch-type space. When $g(w) = \varphi'(w)$, we see that the operator $C_\varphi^{\varphi'}$-C_φ is a point evaluation operator. The operator C_φ^g is closely related with some integral operators in papers [19–26]. Our goal here is to characterize the boundedness and compactness of the Li-Stević integral-type operator between different weighted Bloch-type spaces.

Throughout this paper, the letter C denotes a positive constant which may vary at each occurrence but it is independent of the essential variables.

2. Preliminary material

In order to prove the main results, we need the following lemmas.

Lemma 2.1. *There exist two functions $f_1, f_2 \in \mathcal{B}_{\log}$ such that*

$$\frac{1}{(1-|z|^2)\log(2/(1-|z|^2))} \le C\left(\left|f_1'(z)\right| + \left|f_2'(z)\right|\right), \quad z \in \mathbb{D}. \tag{2.1}$$

Proof. For each $z \in \mathbb{D}$, it is easy to see that $1 \le 1+|z| < 2$ and $0 < 1-|z|^2 \le 1$. So

$$\begin{aligned}(1-|z|)\log\frac{2}{1-|z|} &\le (1-|z|^2)\log\frac{2}{1-|z|} \\ &\le (1-|z|^2)\log\frac{4}{1-|z|^2} \\ &\le (1-|z|^2)\log\frac{4}{(1-|z|^2)^2} \\ &= 2(1-|z|^2)\log\frac{2}{1-|z|^2}.\end{aligned} \tag{2.2}$$

According to [10, Lemma 3.1], there exist two functions $f_1, f_2 \in \mathcal{B}_{\log}$ such that

$$\frac{1}{(1-|z|)\log(2/(1-|z|))} \le C\left(\left|f_1'(z)\right| + \left|f_2'(z)\right|\right), \quad z \in \mathbb{D}. \tag{2.3}$$

From (2.2) and (2.3), the lemma follows. □

Lemma 2.2 (see [8]). *Let $f(z) = (1-|z|)\log(2/(1-|z|))/(|1-z|\log(4/|1-z|))$, $z \in \mathbb{D}$, then $|f(z)| < 2$.*

Lemma 2.3 (see [8]). *Let $f \in \mathcal{B}_{\log}$, then $\|f_t\|_{\mathcal{B}_{\log}} \le C\|f\|_{\mathcal{B}_{\log}}$, where $f_t(z) = f(tz)$, $0 < t < 1$.*

Lemma 2.4 (see [8]). *Suppose $0 \le t \le 1$. Let $f(z,t) = (1-|z|)\log(2/(1-|z|))/((1-|tz|)\log(2/(1-|tz|)))$, $z \in \mathbb{D}$, then $|f(z,t)| < 2$.*

3. The boundedness of $C_\varphi^g : \mathcal{B}_{\log}$ (or $\mathcal{B}_{\log}^0$) $\to \mathcal{B}^\alpha$ (or $\mathcal{B}_0^\alpha$)

In this section, we study the boundedness of $C_\varphi^g : \mathcal{B}_{\log}$ (or $\mathcal{B}_{\log}^0$) $\to \mathcal{B}^\alpha$ (or $\mathcal{B}_0^\alpha$).

Theorem 3.1. *Suppose $0 < \alpha < \infty$, φ is an analytic self-map of $\mathbb{D}$, and $g \in H(\mathbb{D})$. Then $C_\varphi^g : \mathcal{B}_{\log} \to \mathcal{B}^\alpha$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)\log(2/(1-|\varphi(z)|^2))} < \infty. \tag{3.1}$$

Proof. We first prove that the condition is sufficient. Suppose

$$M = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)\log(2/(1-|\varphi(z)|^2))} < \infty. \tag{3.2}$$

Then, for $z \in \mathbb{D}$ and $f \in \mathcal{B}_{\log}$, we have

$$\begin{aligned} \left|(1-|z|^2)^\alpha (C_\varphi^g f)'(z)\right| &= (1-|z|^2)^\alpha |f'(\varphi(z))||g(z)| \\ &\le C\|f\| \frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)\log(2/(1-|\varphi(z)|^2))} \\ &\le CM\|f\|_{B_{\log}} < \infty, \end{aligned} \tag{3.3}$$

thus

$$\|C_\varphi^g f\|_\alpha = |C_\varphi^g f(0)| + B_\alpha(C_\varphi^g f) \le CM\|f\|_{B_{\log}}, \tag{3.4}$$

so $C_\varphi^g : \mathcal{B}_{\log} \to \mathcal{B}^\alpha$ is bounded.

Conversely, using Lemma 2.1, there exist two functions $f_1, f_2 \in \mathcal{B}_{\log}$, satisfying

$$\frac{1}{(1-|z|^2)\log(2/(1-|z|^2))} \le C(|f_1'(z)| + |f_2'(z)|). \tag{3.5}$$

Setting $z = \varphi(w)$ in the above inequality, we obtain

$$\frac{1}{(1-|\varphi(w)|^2)\log(2/(1-|\varphi(w)|^2))} \le C(|f_1'(\varphi(w))| + |f_2'(\varphi(w))|). \tag{3.6}$$

Hence,

$$\begin{aligned} \frac{(1-|w|^2)^\alpha |g(w)|}{(1-|\varphi(w)|^2)\log(2/(1-|\varphi(w)|^2))} &\le C(1-|w|^2)^\alpha |g(w)|(|f_1'(\varphi(w))| + |f_2'(\varphi(w))|) \\ &\le C(\|C_\varphi^g f_1\|_\alpha + \|C_\varphi^g f_2\|_\alpha). \end{aligned} \tag{3.7}$$

Since $f_1, f_2 \in \mathcal{B}_{\log}$, and $C_\varphi^g : \mathcal{B}_{\log} \to \mathcal{B}^\alpha$ is bounded, then $C_\varphi^g f_1$ and $C_\varphi^g f_2$ are in $\mathcal{B}^\alpha$. So the supremum over $w \in \mathbb{D}$ in (3.7) is finite, which implies that

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)\log(2/(1-|\varphi(z)|^2))} < \infty, \tag{3.8}$$

as desired. □

Theorem 3.2. *Suppose* $0 < \alpha < \infty$, φ *is an analytic self-map of* $\mathbb{D}$, *and* $g \in H(\mathbb{D})$. *Then* $C_\varphi^g : \mathcal{B}_{\log}^0 \to \mathcal{B}^\alpha$ *is bounded if and only if* (3.1) *holds.*

Proof. If (3.1) holds, from Theorem 3.1, $C_\varphi^g : \mathcal{B}_{\log} \to \mathcal{B}^\alpha$ is bounded, which along with the fact $\mathcal{B}_{\log}^0 \subset \mathcal{B}_{\log}$ implies $C_\varphi^g : \mathcal{B}_{\log}^0 \to \mathcal{B}^\alpha$ is bounded.

Conversely, assume that $C_\varphi^g : \mathcal{B}_{\log}^0 \to \mathcal{B}^\alpha$ is bounded. For $w \in \mathbb{D}$, put

$$f_w(z) = \int_0^z (1-\overline{w}\zeta)^{-1}\left(\log\frac{4}{1-\overline{w}\zeta}\right)^{-1} d\zeta. \tag{3.9}$$

By the inequality $1/(1+|z|) < 2/(1-|z|)$, $z \in \mathbb{D}$, Lemmas 2.2 and 2.4, we get

$$\begin{aligned} &\sup_{z\in\mathbb{D}}(1-|z|^2)\left(\log\frac{2}{1-|z|^2}\right)|f_w'(z)| \\ &\quad= \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)\log(2/(1-|z|^2))}{|1-\overline{w}z|\,|\log(4/(1-\overline{w}z))|} \\ &\quad\le \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)\log(2/(1-|z|^2))}{(1-|\overline{w}z|)\log(2/(1-|\overline{w}z|))} \times \sup_{z\in\mathbb{D}}\frac{(1-|\overline{w}z|)\log(2/(1-|\overline{w}z|))}{|1-\overline{w}z|\,|\log(4/(1-\overline{w}z))|} \\ &\quad\le 4\sup_{z\in\mathbb{D}}\frac{(1-|z|)\log(2/(1-|z|))}{(1-|\overline{w}z|)\log(2/(1-|\overline{w}z|))} \times \sup_{u\in\mathbb{D}}\frac{(1-|u|)\log(2/(1-|u|))}{|1-u|\log(4/|1-u|)} \\ &\quad\le 16, \end{aligned} \tag{3.10}$$

so $\|f_w\|_{B_{\log}} \le 16$. Since

$$\begin{aligned} \lim_{|z|\to1}(1-|z|^2)\left(\log\frac{2}{1-|z|^2}\right)|f_w'(z)| &= \lim_{|z|\to1}\frac{(1-|z|^2)\log(2/(1-|z|^2))}{|1-\overline{w}z|\,|\log(4/(1-\overline{w}z))|} \\ &\le \lim_{|z|\to1}\frac{(1-|z|^2)\log(2/(1-|z|^2))}{(1-|w|)\log 2} \\ &= 0, \end{aligned} \tag{3.11}$$

we see that $f_w \in \mathcal{B}_{\log}^0$. Thus, for $w \in \mathbb{D}$,

$$\begin{aligned}\frac{(1-|w|^2)^\alpha |g(w)|}{(1-|\varphi(w)|^2)\log(2/(1-|\varphi(w)|^2))} &\le C\frac{(1-|w|^2)^\alpha |g(w)|}{(1-|\varphi(w)|^2)\log(4/(1-|\varphi(w)|^2))} \\ &= C(1-|w|^2)^\alpha |f'_{\varphi(w)}(\varphi(w))||g(w)| \\ &\le C\|C_\varphi^g f_{\varphi(w)}\|_\alpha \le C\|C_\varphi^g\|\|f_{\varphi(w)}\|_{B_{\log}} \\ &\le C\|C_\varphi^g\| < \infty,\end{aligned} \tag{3.12}$$

which gives

$$\sup_{w\in\mathbb{D}}\frac{(1-|w|^2)^\alpha |g(w)|}{(1-|\varphi(w)|^2)\log(2/(1-|\varphi(w)|^2))} \le C\|C_\varphi^g\| < \infty, \tag{3.13}$$

finishing the proof of the theorem. □

Theorem 3.3. *Suppose $0 < \alpha < \infty$, φ is an analytic self-map of $\mathbb{D}$, and $g \in H(\mathbb{D})$. Then $C_\varphi^g : \mathcal{B}_{\log} \to \mathcal{B}_0^\alpha$ is bounded if and only if*

$$\lim_{|z|\to 1}\frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)\log(2/(1-|\varphi(z)|^2))} = 0. \tag{3.14}$$

Proof (Necessity). If $C_\varphi^g : \mathcal{B}_{\log} \to \mathcal{B}_0^\alpha$ is bounded, we use the fact that for each function $f \in \mathcal{B}_{\log}$, the analytic function $C_\varphi^g f \in \mathcal{B}_0^\alpha$. Then using the functions of Lemma 2.1, we get the following:

$$\begin{aligned}\frac{(1-|w|^2)^\alpha |g(w)|}{(1-|\varphi(w)|^2)\log(2/(1-|\varphi(w)|^2))} &\le C(1-|w|^2)^\alpha |g(w)|(|f_1'(\varphi(w))| + |f_2'(\varphi(w))|) \\ &\le C(1-|w|^2)^\alpha (|(C_\varphi^g f_1)'(w)| + |(C_\varphi^g f_2)'(w)|) \\ &\longrightarrow 0 \quad (\text{as } |w| \longrightarrow 1),\end{aligned} \tag{3.15}$$

hence, (3.14) holds.

Sufficiency. For $f \in \mathcal{B}_{\log}$, we have

$$\begin{aligned}|(1-|z|^2)^\alpha (C_\varphi^g f)'(z)| &= (1-|z|^2)^\alpha |f'(\varphi(z))||g(z)| \\ &\le C\|f\|_{B_{\log}}\frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)\log(2/(1-|\varphi(z)|^2))} \\ &\longrightarrow 0 \quad (\text{as } |z| \longrightarrow 1),\end{aligned} \tag{3.16}$$

thus, $C_\varphi^g f \in \mathcal{B}_0^\alpha$. Since (3.14) implies (3.1), by Theorem 3.1, $C_\varphi^g : \mathcal{B}_{\log} \to \mathcal{B}^\alpha$ is bounded. Using these two facts, we obtain that $C_\varphi^g : \mathcal{B}_{\log} \to \mathcal{B}_0^\alpha$ is bounded. The proof is complete. □

Theorem 3.4. *Let $0 < \alpha < \infty$, $g \in H(\mathbb{D})$, and φ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi^g : \mathcal{B}_{\log}^0 \to \mathcal{B}_0^\alpha$ is bounded if and only if $C_\varphi^g : \mathcal{B}_{\log} \to \mathcal{B}^\alpha$ is bounded and $\lim_{|z|\to 1}(1-|z|^2)^\alpha |g(z)| = 0$.*

Proof. If $C_\varphi^g : \mathcal{B}_{\log}^0 \to \mathcal{B}_0^\alpha$ is bounded, then for $f \in \mathcal{B}_{\log}$, $f_t \in \mathcal{B}_{\log}^0$ $(0 < t < 1)$. From Lemma 2.3, we have

$$\left\| C_\varphi^g f_t \right\|_\alpha \le \left\| C_\varphi^g \right\| \left\| f_t \right\|_{B_{\log}} \le C \left\| C_\varphi^g \right\| \| f \|_{B_{\log}}, \tag{3.17}$$

from this, it follows that

$$\left(1-|z|^2\right)^\alpha t \left| f'(t\varphi(z)) \right| \left| g(z) \right| \le C \left\| C_\varphi^g \right\| \| f \|_{B_{\log}}, \quad z \in \mathbb{D}. \tag{3.18}$$

Letting $t \to 1$ and taking the supremum in the above inequality over $z \in \mathbb{D}$, we obtain that

$$\left\| C_\varphi^g f \right\|_\alpha \le C \left\| C_\varphi^g \right\| \| f \|_{B_{\log}}, \tag{3.19}$$

thus $C_\varphi^g : \mathcal{B}_{\log} \to \mathcal{B}^\alpha$ is bounded. Since $f(z) = z$ is in $\mathcal{B}_{\log}^0$, the boundedness of $C_\varphi^g : \mathcal{B}_{\log}^0 \to \mathcal{B}_0^\alpha$ implies that

$$\lim_{|z|\to 1} \left(1-|z|^2\right)^\alpha \left| g(z) \right| = \lim_{|z|\to 1} \left(1-|z|^2\right)^\alpha \left| \left(C_\varphi^g f\right)'(z) \right| = 0. \tag{3.20}$$

For the converse, by Theorem 3.1,

$$M_1 = \sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)^\alpha \left| g(z) \right|}{\left(1-\left|\varphi(z)\right|^2\right) \log\left(2/\left(1-\left|\varphi(z)\right|^2\right)\right)} < \infty. \tag{3.21}$$

For any $f \in \mathcal{B}_{\log}^0$, we have

$$\lim_{|z|\to 1} \left(1-|z|^2\right) \left(\log \frac{2}{1-|z|^2} \right) \left| f'(z) \right| = 0, \tag{3.22}$$

so that for any $\epsilon > 0$, there exists a $\delta_1 \in (0,1)$, such that when $\delta_1 < |z| < 1$,

$$\left(1-|z|^2\right) \left(\log \frac{2}{1-|z|^2} \right) \left| f'(z) \right| < \frac{\epsilon}{2M_1}. \tag{3.23}$$

Hence, when $|\varphi(z)| > \delta_1$, we get

$$\begin{aligned}\left|(1-|z|^2)^{\alpha}(C_{\varphi}^{g}f)'(z)\right| &= (1-|z|^2)^{\alpha}\left|f'(\varphi(z))\right|\left|g(z)\right| \\ &< \frac{\epsilon}{2M_1}\frac{(1-|z|^2)^{\alpha}\left|g(z)\right|}{(1-\left|\varphi(z)\right|^2)\log\left(2/(1-\left|\varphi(z)\right|^2)\right)} \\ &< \frac{\epsilon}{2}.\end{aligned} \tag{3.24}$$

We know that there exists a constant M_2 such that $|f'(\varphi(z))| \le M_2$, for any z belonging to the set $\{z \in \mathbb{D} : |\varphi(z)| \le \delta_1\}$. Since $\lim_{|z|\to 1}(1-|z|^2)^{\alpha}|g(z)| = 0$, there exists a $\delta \in (0,1)$, such that when $\delta < |z| < 1$, $(1-|z|^2)^{\alpha}|g(z)| < \epsilon/(2M_2)$, then

$$\begin{aligned}\left|(1-|z|^2)^{\alpha}(C_{\varphi}^{g}f)'(z)\right| &= (1-|z|^2)^{\alpha}\left|f'(\varphi(z))\right|\left|g(z)\right| \\ &\le M_2(1-|z|^2)^{\alpha}\left|g(z)\right| \\ &< \frac{\epsilon}{2},\end{aligned} \tag{3.25}$$

thus, we get that for $z \in \mathbb{D}_1 = \{z \in \mathbb{D} : \delta < |z| < 1\}$,

$$\begin{aligned}\sup_{z\in\mathbb{D}_1}\left|(1-|z|^2)^{\alpha}(C_{\varphi}^{g}f)'(z)\right| &= \sup_{\{z\in\mathbb{D}_1:|\varphi(z)|>\delta_1\}}(1-|z|^2)^{\alpha}\left|(C_{\varphi}^{g}f)'(z)\right| \\ &\quad + \sup_{\{z\in\mathbb{D}_1:|\varphi(z)|\le\delta_1\}}(1-|z|^2)^{\alpha}\left|(C_{\varphi}^{g}f)'(z)\right| \\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon,\end{aligned} \tag{3.26}$$

so that $C_{\varphi}^{g}f \in \mathcal{B}_0^{\alpha}$, that is, $C_{\varphi}^{g} : \mathcal{B}_{\log}^{0} \to \mathcal{B}_0^{\alpha}$ is bounded. The proof of Theorem 3.4 is complete. □

4. The compactness of $C_{\varphi}^{g} : \mathcal{B}_{\log}$ (or $\mathcal{B}_{\log}^{0}$) $\to \mathcal{B}^{\alpha}$ (or $\mathcal{B}_0^{\alpha}$)

In this section, we characterize the compactness of $C_{\varphi}^{g} : \mathcal{B}_{\log}$ (or $\mathcal{B}_{\log}^{0}$) $\to \mathcal{B}^{\alpha}$ (or $\mathcal{B}_0^{\alpha}$). For this purpose, we start this section by stating some useful lemmas. By standard arguments (see, e.g., [2]), the following lemmas follow.

Lemma 4.1. *Suppose $0 < \alpha < \infty$, φ is an analytic self-map of $\mathbb{D}$, and $g \in H(\mathbb{D})$. Let $X = \mathcal{B}_{\log}$ or $\mathcal{B}_{\log}^{0}$, $Y = \mathcal{B}^{\alpha}$ or $\mathcal{B}_0^{\alpha}$. Then $C_{\varphi}^{g} : X \to Y$ is compact if and only if $C_{\varphi}^{g} : X \to Y$ is bounded and for any bounded sequence $\{f_n\}$ in X which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$, one has $\|C_{\varphi}^{g}f_n\|_Y \to 0$ as $n \to \infty$.*

Lemma 4.2. *Let $0 < \alpha < \infty$. A closed set K in $\mathcal{B}_0^{\alpha}$ is compact if and only if K is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in K}(1-|z|^2)^{\alpha}\left|f'(z)\right| = 0. \tag{4.1}$$

The proof of Lemma 4.2 is similar to [27, Lemma 2.1] (see, also [28]) and is omitted.

We begin with the following necessary and sufficient condition for the compactness of $C_\varphi^g : \mathcal{B}_{\log}^0 \to \mathcal{B}_0^\alpha$.

Theorem 4.3. *Suppose* $0 < \alpha < \infty$, φ *is an analytic self-map of* $\mathbb{D}$, *and* $g \in H(\mathbb{D})$. *Then the following statements are equivalent:*

(1) $C_\varphi^g : \mathcal{B}_{\log} \to \mathcal{B}_0^\alpha$ *is compact;*

(2) $C_\varphi^g : \mathcal{B}_{\log}^0 \to \mathcal{B}_0^\alpha$ *is compact;*

(3)

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)\log(2/(1-|\varphi(z)|^2))} = 0. \tag{4.2}$$

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) Since $C_\varphi^g : \mathcal{B}_{\log}^0 \to \mathcal{B}_0^\alpha$ is compact, we obtain by Lemma 4.2

$$\lim_{|z|\to 1} \sup_{\|f\|_{B_{\log}}\le 1} (1-|z|^2)^\alpha \left|(C_\varphi^g f)'(z)\right| = 0. \tag{4.3}$$

Thus, for any $\epsilon > 0$, there exists a $\delta \in (0,1)$, such that when $\delta < |z| < 1$,

$$\sup_{\|f\|_{B_{\log}}\le 1} (1-|z|^2)^\alpha \left|(C_\varphi^g f)'(z)\right| < \frac{\epsilon}{C}. \tag{4.4}$$

Let f_w be defined by (3.9). It is easy to see that

$$\frac{1}{C} \le \frac{1}{\|f_w\|_{B_{\log}}}. \tag{4.5}$$

Set $h_w = f_w/\|f_w\|$, then for $\delta < |z| < 1$, $w = \varphi(z)$,

$$\begin{aligned} \frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)\log(2/(1-|\varphi(z)|^2))} &\le C(1-|z|^2)^\alpha |h_w'(\varphi(z))||g(z)| \\ &= C(1-|z|^2)^\alpha \left|(C_\varphi^g h_w)'(z)\right| \\ &\le C \sup_{\|f\|_{B_{\log}}\le 1} (1-|z|^2)^\alpha \left|(C_\varphi^g f)'(z)\right| < \epsilon, \end{aligned} \tag{4.6}$$

which gives that

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)\log(2/(1-|\varphi(z)|^2))} = 0. \tag{4.7}$$

(3) $\Rightarrow$ (1) For any bounded sequence $\{f_n\}$ in $\mathcal{B}_{\log}$ with $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, we must prove that by Lemma 4.1

$$\left\|C_\varphi^g f_n\right\|_\alpha \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{4.8}$$

We assume that $\|f_n\|_{B_{\log}} \le 1$. From (4.2), given $\epsilon > 0$, there exists a $\delta \in (0,1)$, when $\delta < |z| < 1$,

$$\frac{\left(1-|z|^2\right)^\alpha |g(z)|}{\left(1-|\varphi(z)|^2\right)\log\left(2/\left(1-|\varphi(z)|^2\right)\right)} < \frac{\epsilon}{2}, \tag{4.9}$$

then using (4.9), we get for $\delta < |z| < 1$,

$$\left|\left(1-|z|^2\right)^\alpha \left(C_\varphi^g f_n\right)'(z)\right| = \left(1-|z|^2\right)^\alpha \left|f_n'(\varphi(z))\right| |g(z)| < \frac{\epsilon}{2}. \tag{4.10}$$

Since $\{f_n'\}$ converges uniformly to 0 on a compact subset $\{\varphi(z) : |z| \le \delta\}$ of $\mathbb{D}$ and there exists a constant M_3 such that $\sup_{|z|\le\delta}\left(1-|z|^2\right)^\alpha |g(z)| \le M_3$, we see that there exists an $N > 0$, such that for all $n \ge N$,

$$\sup_{|z|\le\delta}\left|f_n'(\varphi(z))\right| < \frac{\epsilon}{2M_3}. \tag{4.11}$$

Therefore, for all $n \ge N$, $|z| \le \delta$,

$$\left|\left(1-|z|^2\right)^\alpha \left(C_\varphi^g f_n\right)'(z)\right| = \left(1-|z|^2\right)^\alpha \left|f_n'(\varphi(z))\right| |g(z)| < \frac{\varepsilon}{2}. \tag{4.12}$$

Note that $|C_\varphi^g f_n(0)| = 0$. Combining (4.10) and (4.12), we obtain

$$\left\|C_\varphi^g f_n\right\|_\alpha \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{4.13}$$

The proof is complete. □

By Theorems 3.3 and 4.3, we get the following corollary.

Corollary 4.4. *Suppose $0 < \alpha < \infty$, φ is an analytic self-map of $\mathbb{D}$, and $g \in H(\mathbb{D})$. Then $C_\varphi^g : \mathcal{B}_{\log} \to \mathcal{B}_0^\alpha$ is bounded if and only if $C_\varphi^g : \mathcal{B}_{\log} \to \mathcal{B}_0^\alpha$ is compact.*

Theorem 4.5. *Suppose $0 < \alpha < \infty$, φ is an analytic self-map of $\mathbb{D}$, and $g \in H(\mathbb{D})$. Then $C_\varphi^g : \mathcal{B}_{\log} \to \mathcal{B}^\alpha$ is compact if and only if $C_\varphi^g : \mathcal{B}_{\log} \to \mathcal{B}^\alpha$ is bounded and*

$$\lim_{|\varphi(z)|\to 1} \frac{\left(1-|z|^2\right)^\alpha |g(z)|}{\left(1-|\varphi(z)|^2\right)\log\left(2/\left(1-|\varphi(z)|^2\right)\right)} = 0. \tag{4.14}$$

Proof. Suppose that (4.14) is true. For any sequence $\{f_n\}$ in $\mathcal{B}_{\log}$ such that $\|f_n\|_{\mathcal{B}_{\log}} \leq 1$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, it is required to show that by Lemma 4.1,

$$\left\|C_\varphi^g f_n\right\|_\alpha \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{4.15}$$

From (4.14), we have that for every $\epsilon > 0$, there exists a $\delta \in (0,1)$, such that $\delta < |\varphi(z)| < 1$ implies

$$\frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)\log(2/(1-|\varphi(z)|^2))} < \frac{\epsilon}{2}, \tag{4.16}$$

then using (4.16), we get for $\delta < |\varphi(z)| < 1$,

$$\begin{aligned} \left|(1-|z|^2)^\alpha (C_\varphi^g f_n)'(z)\right| &= (1-|z|^2)^\alpha \left|f_n'(\varphi(z))\right|\left|g(z)\right| \\ &\leq \frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)\log(2/(1-|\varphi(z)|^2))} \\ &< \frac{\epsilon}{2}. \end{aligned} \tag{4.17}$$

Since $C_\varphi^g : \mathcal{B}_{\log} \to \mathcal{B}^\alpha$ is bounded, taking $f(z) = z$, we see that $L = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |g(z)| < \infty$. Let $U = \{w \in \mathbb{D} : |w| \leq \delta\}$, since $\{f_n'\}$ converges uniformly to 0 on a compact subset U of $\mathbb{D}$, then there exists an $N > 0$, such that for all $n \geq N$,

$$\sup_{w\in U}\left|f_n'(w)\right| < \frac{\epsilon}{2L}. \tag{4.18}$$

Therefore, for all $n \geq N$,

$$\begin{aligned} \sup_{\{|\varphi(z)|\leq\delta\}} (1-|z|^2)^\alpha (C_\varphi^g f_n)'(z)\Big| &= \sup_{\{|\varphi(z)|\leq\delta\}} (1-|z|^2)^\alpha \left|f_n'(\varphi(z))\right|\left|g(z)\right| \\ &\leq L \sup_{w\in U}\left|f_n'(w)\right| \\ &< \frac{\epsilon}{2}. \end{aligned} \tag{4.19}$$

Note that $|C_\varphi^g f_n(0)| = 0$. Combining (4.17) and (4.19), we obtain

$$\left\|C_\varphi^g f_n\right\|_\alpha \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{4.20}$$

Conversely, suppose that $C_{\varphi}^{g}:\mathcal{B}_{\log}\to\mathcal{B}^{\alpha}$ is compact, then $C_{\varphi}^{g}:\mathcal{B}_{\log}\to\mathcal{B}^{\alpha}$ is bounded. Hence, we only need to prove that (4.14) holds. Assume to the contrary that there is a positive number ϵ_0 and a sequence $\{z_n\}$ in $\mathbb{D}$ such that $\lim_{n\to\infty}|\varphi(z_n)|=1$, and

$$\frac{\left(1-|z_n|^2\right)^{\alpha}|g(z_n)|}{\left(1-|\varphi(z_n)|^2\right)\log\left(2/\left(1-|\varphi(z_n)|^2\right)\right)}\geq\epsilon_0, \tag{4.21}$$

for all n. For each n, writing $\varphi(z_n)=r_ne^{i\theta_n}$, we choose the test functions f_n defined by

$$f_n(z)=\int_0^z\left(\frac{r_n}{1-e^{-i\theta_n}r_nw}-\frac{r_n^2}{1-e^{-i\theta_n}r_n^2w}\right)\left(\log\frac{4}{1-e^{-i\theta_n}r_n^2w}\right)^{-1}dw, \tag{4.22}$$

then

$$f_n'(z)=\left(\frac{r_n}{1-e^{-i\theta_n}r_nz}-\frac{r_n^2}{1-e^{-i\theta_n}r_n^2z}\right)\left(\log\frac{4}{1-e^{-i\theta_n}r_n^2z}\right)^{-1}, \tag{4.23}$$

thus, $|f_n'(z)|\leq((1-r_n)/(1-|z|)^2)(\log(4/(1-|z|)))^{-1}$, we see that f_n converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n\to\infty$. Using Lemmas 2.2 and 2.4, we have $\|f_n\|_{B_{\log}}\leq C$. In view of Lemma 4.1, it follows that

$$\left\|C_{\varphi}^{g}f_n\right\|_{\alpha}\longrightarrow 0\quad\text{as } n\longrightarrow\infty. \tag{4.24}$$

Since

$$\begin{aligned}\left\|C_{\varphi}^{g}f_n\right\|_{\alpha}&\geq\left(1-|z_n|^2\right)^{\alpha}\left|f_n'(\varphi(z_n))\right||g(z_n)|\\&=\left(1-|z_n|^2\right)^{\alpha}|g(z_n)|\left(\frac{r_n}{1-r_n^2}-\frac{r_n^2}{1-r_n^3}\right)\left(\log\frac{4}{1-r_n^3}\right)^{-1}\\&\geq C\frac{\left(1-|z_n|^2\right)^{\alpha}|\varphi(z_n)||g(z_n)|}{\left(1-|\varphi(z_n)|^2\right)\log\left(4/\left(1-|\varphi(z_n)|^3\right)\right)}\\&\geq C\frac{\left(1-|z_n|^2\right)^{\alpha}|\varphi(z_n)||g(z_n)|}{\left(1-|\varphi(z_n)|^2\right)\log\left(2/\left(1-|\varphi(z_n)|^2\right)\right)},\end{aligned} \tag{4.25}$$

and $|\varphi(z_n)|\to 1$ as $n\to\infty$, we obtain $\lim_{n\to\infty}(1-|z_n|^2)^{\alpha}|g(z_n)|/((1-|\varphi(z_n)|^2)\log(2/(1-|\varphi(z_n)|^2)))=0$, which is a contradiction with (4.21). Hence, we are done.

Similarly, we can obtain the following result. The proof of the following theorem will be omitted. □

Theorem 4.6. *Suppose $0<\alpha<\infty$, φ is an analytic self-map of $\mathbb{D}$, $g\in H(\mathbb{D})$, and $C_{\varphi}^{g}:\mathcal{B}_{\log}\to\mathcal{B}^{\alpha}$ is bounded. Then $C_{\varphi}^{g}:\mathcal{B}_{\log}^{0}\to\mathcal{B}^{\alpha}$ is compact if and only if* (4.14) *holds.*

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