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*Research Article*
# Sufficient Univalence Conditions for Analytic Functions

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We consider a general integral operator and the class of analytic functions. We extend some univalent conditions of Becker's type for analytic functions using a general integral transform.



## 1. Introduction

Let $\mathcal{U} = \{z \in \mathbb{C},\ |z| < 1\}$ be the unit disk, let $\mathcal{A}$ denote the class of the functions $f$ of the form

$$\{f(z) = z + a_2z^2 + a_3z^3 + \cdots,\ z \in \mathcal{U}\}, \tag{1.1}$$

which are analytic in the open disk, and let $\mathcal{U}$ satisfy the condition $f(0) = f'(0) - 1 = 0$. Consider $\mathcal{S} = \{f \in \mathcal{A} :\ f$ is univalent functions in $\mathcal{U}\}$.

In [1], Pescar needs the following theorem.

Theorem 1.1 [1]. *Let $c$ and $\beta$ be complex numbers with $\mathbf{Re}\,\beta > 0$, $|c| \le 1$, and $c \neq -1$, and let $h(z) = z + a_2z^2 + \cdots$ be a regular function in $\mathcal{U}$.If*

$$\left| c|z|^{2\beta} + (1 - |z|^{2\beta})\frac{zh''(z)}{\beta h'(z)} \right| \le 1 \tag{1.2}$$

*for all the $z \in \mathcal{U}$, then the function*

$$F_\beta(z) = \left[\beta \int_0^z t^{\beta-1}h'(t)dt\right]^{1/\beta} = z + \cdots \tag{1.3}$$

*is regular and univalent in $\mathcal{U}$.*

In [2], Ozaki and Nunokawa give the next result.

Theorem 1.2 [2]. *Let $f \in \mathcal{A}$ satisfy the following condition:*

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \leq 1 \tag{1.4}$$

*for all $z \in \mathcal{U}$, then $f$ is univalent in $\mathcal{U}$.*

Lemma 1.3 (The Schwarz lemma) [3, 4]. *Let the analytic function $f$ be regular in the unit disk and let $f(0) = 0$ . If $|f(z)| \leq 1$, then*

$$|f(z)| \leq |z| \tag{1.5}$$

*for all $z \in \mathcal{U}$, where the equality can hold only if $|f(z)| = Kz$ and $K = 1$.*

In [5], Seenivasagan and Breaz consider, for $f_i \in \mathcal{A}_2$ $(i = 1, 2, \ldots, n)$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$, $\beta \in \mathbb{C}$, the integral operator

$$F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left( \frac{f_i(t)}{t} \right)^{1/\alpha_i} dt \right\}^{1/\beta}. \tag{1.6}$$

When $\alpha_i = \alpha$ for all $i = 1, 2, \ldots, n$, $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z)$ becomes the integral operator $F_{\alpha,\beta}(z)$ considered in [6].

## 2. Main results

Theorem 2.1. *Let $M \geq 1$ and the functions $f_i \in \mathcal{A}$, for $i \in \{1, \ldots, n\}$, satisfy the condition (1.4), and let $\beta$ be a real number, $\beta \geq \sum_{i=1}^{n} (2M+1)/|\alpha_i|$ and $c$ is a complex number.*

*If*

$$|c| \leq 1 - \frac{1}{\beta} \sum_{i=1}^{n} \frac{2M+1}{|\alpha_i|}, \tag{2.1}$$

$$|f_i(z)| \leq M \tag{2.2}$$

*for all $z \in \mathcal{U}$, then the function $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ defined in (1.6) is in the class $\mathcal{S}$.*

*Proof.* Define a function

$$h(z) = \int_0^z \prod_{i=1}^{n} \left( \frac{f_i(t)}{t} \right)^{1/\alpha_i} dt, \tag{2.3}$$

then we have $h(0) = h'(0) - 1 = 0$. Also, a simple computation yields

$$h'(z) = \prod_{i=1}^{n} \left( \frac{f_i(z)}{z} \right)^{1/\alpha_i}, \tag{2.4}$$

$$\frac{z h''(z)}{h'(z)} = \sum_{i=1}^{n} \frac{1}{\alpha_i} \left( \frac{z f_i'(z)}{f_i(z)} - 1 \right). \tag{2.5}$$

From (2.5), we have

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq \sum_{i=1}^{n} \frac{1}{|\alpha_i|}\left(\left|\frac{zf_i'(z)}{f_i(z)}\right| + 1\right) = \sum_{i=1}^{n} \frac{1}{|\alpha_i|}\left(\left|\frac{z^2 f_i'(z)}{(f_i(z))^2}\right|\left|\frac{f_i(z)}{z}\right| + 1\right). \tag{2.6}$$

From the hypothesis, we have $|f_i(z)| \leq M$ $(z \in \mathcal{U},\ i = 1,2,\dots,n)$, then by Lemma 1.3, we obtain that

$$|f_i(z)| \leq M|z| \quad (z \in \mathcal{U},\ i = 1,2,\dots,n). \tag{2.7}$$

We apply this result in inequality (2.6), and we obtain

$$\begin{aligned}\left|\frac{zh''(z)}{h'(z)}\right| &\leq \sum_{i=1}^{n} \frac{1}{|\alpha_i|}\left(\left|\frac{z^2 f_i'(z)}{(f_i(z))^2}\right| M + 1\right)\\ &\leq \sum_{i=1}^{n} \frac{1}{|\alpha_i|}\left(\left|\frac{z^2 f_i'(z)}{(f_i(z))^2} - 1\right| M + M + 1\right)\\ &= \sum_{i=1}^{n} \frac{1}{|\alpha_i|}(M + M + 1) = \sum_{i=1}^{n} \frac{2M+1}{|\alpha_i|}.\end{aligned} \tag{2.8}$$

We have

$$\begin{aligned}\left|c|z|^{2\beta} + \left(1 - |z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right| &= \left|c|z|^{2\beta} + \left(1 - |z|^{2\beta}\right)\frac{1}{\beta}\sum_{i=1}^{n} \frac{1}{|\alpha_i|}\left(\frac{zf_i'(z)}{f_i(z)} - 1\right)\right|\\ &\leq |c| + \frac{1}{\beta}\cdot\sum_{i=1}^{n} \frac{1}{|\alpha_i|}\left(\left|\frac{z^2 f_i'(z)}{f_i^2(z)}\right| \cdot \frac{|f_i(z)|}{|z|} + 1\right).\end{aligned} \tag{2.9}$$

We obtain

$$\left|c|z|^{2\beta} + \left(1 - |z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right| \leq |c| + \frac{1}{\beta}\sum_{i=1}^{n} \frac{2M+1}{|\alpha_i|}. \tag{2.10}$$

So from (2.1), we have

$$\left|c|z|^{2\beta} + \left(1 - |z|^{2\beta}\right)\frac{zh''(z)}{\beta h'(z)}\right| \leq 1. \tag{2.11}$$

Applying Theorem 1.1, we obtain that $F_{\alpha_1,\alpha_2,\dots,\alpha_n,\beta}$ is univalent. □

THEOREM 2.2. *Let $M \geq 1$ and the functions $f_i \in \mathcal{A}$, for $i \in \{1,\dots,n\}$ satisfy the condition (1.4), and let $\beta$ be a real number, $\beta \geq n(2M+1)/|\alpha|$ and $c$ is a complex number.*

*If*

$$\begin{aligned}|c| &\leq 1 - \frac{1}{\beta}\frac{n(2M+1)}{|\alpha|},\\ |f_i(z)| &\leq M\end{aligned} \tag{2.12}$$

*for all $z \in \mathcal{U}$, then the function*

$$F_{\alpha,\beta}(z) = \left\{\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{1/\alpha} dt\right\}^{1/\beta} \tag{2.13}$$

*is in the class $\mathcal{S}$.*

*Proof.* In Theorem 2.1, we consider $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$. □

Corollary 2.3. *Let the functions $f_i \in \mathcal{A}$, for $i \in \{1,\dots,n\}$, satisfy the condition (1.4), and let $\beta$ be a real number, $\beta \geq \sum_{i=1}^{n}(3/|\alpha_i|)$ and $c$ is a complex number.*

*If*

$$\begin{gathered} |c| \leq 1 - \frac{1}{\beta}\sum_{i=1}^{n} \frac{3}{|\alpha_i|}, \\ |f_i(z)| \leq 1 \end{gathered} \tag{2.14}$$

*for all $z \in \mathcal{U}$, then the function $F_{\alpha_1,\alpha_2,\dots,\alpha_n,\beta}$ defined in (1.6) is in the class $\mathcal{S}$.*

*Proof.* In Theorem 2.1, we consider $M = 1$. □

Corollary 2.4. *Let $M \geq 1$ and the function $f \in \mathcal{A}$, satisfy the condition (1.4), and let $\beta$ be a real number, $\beta \geq (2M+1)/|\alpha|$ and $c$ is a complex number.*

*If*

$$\begin{gathered} |c| \leq 1 - \frac{1}{\beta}\frac{2M+1}{|\alpha|}, \\ |f(z)| \leq M \end{gathered} \tag{2.15}$$

*for all $z \in \mathcal{U}$, then the function*

$$G_{\alpha,\beta}(z) = \left\{\beta \int_0^z t^{\beta-1} \left(\frac{f(t)}{t}\right)^{1/\alpha} dt\right\}^{1/\beta} \tag{2.16}$$

*is in the class $\mathcal{S}$.*

*Proof.* In Theorem 2.1, we consider $n = 1$. □

Corollary 2.5. *Let the function $f \in \mathcal{A}$ satisfy the condition (1.4), and let $\beta$ be a real number, $\beta \geq 3/|\alpha|$ and $c$ is a complex number.*

*If*

$$\begin{gathered} |c| \leq 1 - \frac{1}{\beta}\frac{3}{|\alpha|}, \\ |f(z)| \leq 1 \end{gathered} \tag{2.17}$$

*for all* $z \in \mathcal{U}$, *then the function*

$$G_{\alpha,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \left( \frac{f(t)}{t} \right)^{1/\alpha} dt \right\}^{1/\beta} \tag{2.18}$$

*is in the class* $\mathcal{S}$.

*Proof.* In Corollary 2.4, we consider $M = 1$. □

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