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Research Article

Distributed Antenna Channels with Regenerative Relaying: Relay Selection and Asymptotic Capacity

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Multiple-input-multiple-output (MIMO) techniques have been widely proposed as a means to improve capacity and reliability of wireless channels, and have become the most promising technology for next generation networks. However, their practical deployment in current wireless devices is severely affected by antenna correlation, which reduces their impact on performance. One approach to solve this limitation is *relaying diversity*. In relay channels, a set of N wireless nodes aids a source-destination communication by relaying the source data, thus creating a distributed antenna array with uncorrelated path gains. In this paper, we study this multiple relay channel (MRC) following a *decode-and-forward* (D&F) strategy (*i.e.*, regenerative forwarding), and derive its achievable rate under AWGN. A half-duplex constraint on relays is assumed, as well as distributed channel knowledge at both transmitter and receiver sides of the communication. For this channel, we obtain the optimum relay selection algorithm and the optimum power allocation within the network so that the transmission rate is maximized. Likewise, we bound the ergodic performance of the achievable rate and derive its asymptotic behavior in the number of relays. Results show that the achievable rate of regenerative MRC grows as the logarithm of the Lambert W function of the total number of relays, that is, $\mathcal{C} = \log_2(W_0(N))$. Therefore, D&F relaying, cannot achieve the capacity of actual MISO channels.

1. INTRODUCTION

Current wireless applications demand an ever-increasing transmission capacity and highly reliable communications. Voice transmission, video broadcasting, and web browsing require wire-like channel conditions that the wireless medium still cannot support. In particular, channel impairments, namely, path loss and multipath fading do not allow wireless channels to reach the necessary rate and robustness expected for next generation systems. Recently, a wide range of multiple antenna techniques have been proposed to overcome these channel limitations [1–4]; however, the deployment of multiple transmit and/or receive antennas on the wireless nodes is not always possible or worthwhile. For these cases, the most suitable technique to take advantage of spatial diversity is *node cooperation* and *relay channels* [5, 6].

Relay channels consist of single source-destination pairs aided in their communications by a set of wireless relay nodes that creates a distributed antenna array (see Figure 1). The relay nodes can be either infrastructure nodes, placed by the service provider in order to enhance coverage and rate [7], or a set of network users that cooperate with the source, while having own data to transmit [8]. Relay-based architectures have been shown to improve capacity, diversity, and delay of wireless channels when properly allocating network resources, and have become a key technique for the evolution of wireless communications [9].

Background

The use of relays to increase the achievable rate of point-to-point transmissions was initially proposed by Cover and El Gamal in [10]. Motivated by this work, many relaying techniques have been recently studied, which can be classified, based on their forwarding strategy and required processing at the relay nodes, as *regenerative relaying* and *nonregenerative relaying* [5, 11]. The former assumes that relay nodes decode the source information, prior to reencoding and sending it to destination [12, 13]. On the other hand, with the latter, relay nodes transform and retransmit their received signals but do not decode them [14–16].

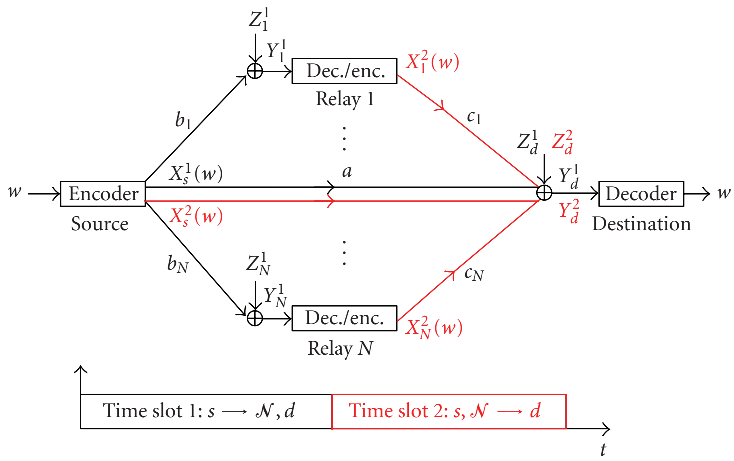

FIGURE 1: Half-duplex regenerative multiple relay channel with N parallel relays.

Regenerative relaying was initially presented in [10, Theorem 1] for a single-relay channel, and consists of relay nodes decoding the source data and transmitting it to destination, ideally without errors. Such signal regeneration allows for cooperative coherent transmissions. Therefore, source and relays can operate as a distributed antenna array and implement multiple-input single-output (MISO) beamforming. We distinguish two techniques: *decode-and-forward* (D&F), presented in [10], and *partial decoding* (PD), analyzed in [17]. D&F requires the relay nodes to fully decode the source message before retransmitting it. Thus, it penalizes the achievable rate when poor source-to-relay channel conditions occur. Nevertheless, for poor source-to-destination channels (e.g., degraded relay channels), it was shown to be the capacity achieving technique [10]. On the other hand, with PD the relay nodes only partially decode the source message. Part of the transmitted message is sent directly to the destination without being relayed [18]. PD is specifically appropriate when the source node can adapt the amount of information transmitted through relays to the network channel conditions; otherwise it does not improve the D&F scheme [19]. The diversity analysis of regenerative multiple relay networks was carried out by Laneman and Wornell in [20], showing that signal regeneration achieves full transmit diversity of the system. However, regenerative relaying has some drawbacks as well: first, decoding errors at the relay nodes generate error propagation; second, synchronization among relays (specifically in the low SNR regime) may complicate its implementation, and finally, the processing capabilities required at the relays increase their cost [5].

The two previously mentioned techniques are well known for the single-relay channel. However, the only significant extensions to the multiple relay setup are found in [6, 21, 22]. In these works, they were applied to physical-layer multihop networks and to the multiple relay channel with orthogonal components, respectively.

Contributions

This paper studies the point-to-point Gaussian channel with N parallel relays that use *decode-and-forward* relaying. On the relays, a half duplex constraint is considered, that is, the relay nodes cannot transmit and receive simultaneously in the same frequency band. The communication is arranged into two consecutive, identical time slots, as shown in Figure 1. The source uses the first time slot to transmit the message to the set of relays and to the destination. Then, during time slot 2, the set of nodes who have successfully decoded the message, and the source, transmit extra parity bits to the destination node, which uses its received signal during the two slots to decode the message. Transmit and receive channel state information (CSI) are available at both transmitter and receiver sides, and channel conditions are assumed not to vary during the two slots of the communication. Additionally, we consider that the source knows all relay-to-destination channels, so that it can implement a relay selection algorithm. Finally, the overall transmitted power during the two time slots is constrained to a constant, and we maximize the achievable rate through power allocation on the two slots of the communication, and on the useful relays.

The contributions of this paper are as follows.

(i) First, the instantaneous achievable rate of the proposed communication is derived in Proposition 1; then the optimum power allocation on the two slots is obtained in Proposition 2. Results show that the achievable rate is maximized through an optimum relay selection algorithm and through power allocation on the two slots, referred to as constrained temporal *waterfilling*.

(ii) Second, we analyze the ergodic performance of the instantaneous achievable rate derived in Proposition 2, assuming independent, identically distributed (i.i.d.) random channel fading and i.i.d. random relay positions. We assume that the source node transmits over several concatenated two-slot transmissions. The channel is invariant during the two slots, and uncorrelated from one two-slot transmission to the next (see Figure 2). Thus, the source transmits with an effective rate equal to the ergodic achievable rate of the link, which is lower- and upper-bounded in this paper.

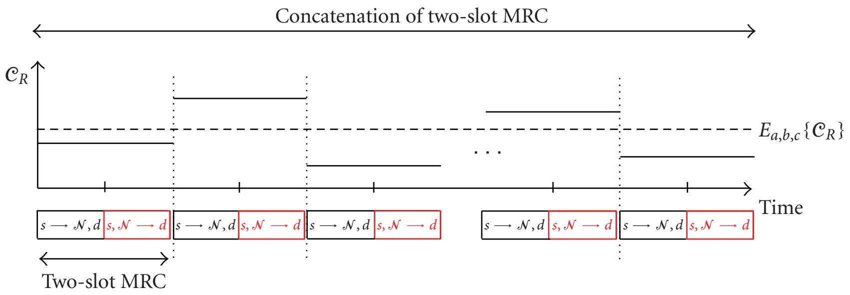

Figure 2: Ergodic capacity: concatenation in time of half-duplex multiple relay channels.

(iii) Finally, we study the asymptotic performance (in the number of relays) of the instantaneous achievable rate, and we show that it grows asymptotically with the logarithm of the branch 0 of the Lambert W function[1] of the total number of relays, that is, $\mathcal{C} = \log_2(W_0(N))$.

The remainder of the paper is organized as follows: in Section 2, we introduce the channel and signal model; in Section 3, the instantaneous achievable of the D&F MRC is derived and the optimum relay selection and power allocation are obtained. In Section 4, the ergodic achievable rate is upper- and lower-bounded, and Section 5 analyzes the asymptotic achievable rate of the channel. Finally, Section 6 contains simulation results and Section 7 summarizes conclusions.

Notation

We define $\mathbf{X}_{1:n}^{(2)} = [X_1^{(2)}, \ldots, X_n^{(2)}]^T$ with $n \in \{1, \ldots, N\}$. Moreover, in the paper, $I(A;B)$ denotes mutual information between random variables A and B, $\mathcal{C}(x) = \log_2(1+x)$, $\mathbf{b}^\dagger$ denotes the conjugate transpose of vector $\mathbf{b}$, and b^* denotes the conjugate of b.

2. CHANNEL MODEL

We consider a wireless multiple-relay channel (MRC) with a source node s, a destination node d, and a set of parallel relays $\mathcal{N} = \{1, \ldots, N\}$ (see Figure 1). Wireless channels among network nodes are *frequency-flat, memoryless*, and modelled with a complex, Gaussian-distributed coefficient; $a \sim \mathcal{CN}(0,1)$ denotes the unitary power, Rayleigh distributed channel between source and destination, and $c_i \sim \mathcal{CN}(0,1)$ the complex channel from relay i to destination. In the system, b_i is modelled as a superposition of path loss (with exponent α) and Rayleigh distributed fading, in order to account for the different transmission distances from the source to relays, d_i, $i = 1, \ldots, N$, and from source to destination d_o (used as reference), that is,

$$b_i \sim \mathcal{CN}\left(0, \left(\frac{d_o}{d_i}\right)^\alpha\right). \tag{1}$$

We assume invariant channels during the two-slot communication.

As mentioned, the communication is arranged in two consecutive time slots of equal duration (see Figure 1). During the first slot, a single-input multiple-output (SIMO) transmission from the source node to the set of relays and destination takes place. The second slot is then used by relays and source to retransmit data to destination via a distributed MISO channel. In both slots, the transmitted signals are received under additive white Gaussian noise (AWGN), and destination attemps to decode making use of the signal received during the two phases. The complex signals transmitted by the source during slot $t = \{1, 2\}$, and by relay i during phase 2, are denoted by $X_s^{(t)}$ and $X_i^{(2)}$, respectively. Therefore, considering memoryless channels, the received signal at the relay nodes during time slot 1 is given by

$$Y_i^{(1)} = b_i \cdot X_s^{(1)} + Z_i^{(1)} \quad \text{for } i \in \mathcal{N}, \tag{2}$$

where $Z_i^{(1)} \sim \mathcal{CN}(0,1)$ is normalized AWGN at relay i. Likewise, considering the channel definition in Figure 1, the received signal at the destination node d during time slots 1 and 2 is written as

$$\begin{aligned} Y_d^{(1)} &= a \cdot X_s^{(1)} + Z_d^{(1)}, \\ Y_d^{(2)} &= a \cdot X_s^{(2)} + \sum_{i=1}^{N} c_i \cdot X_i^{(2)} + Z_d^{(2)}, \end{aligned} \tag{3}$$

where, as previously said, $Z_d^{(t)} \sim \mathcal{CN}(0,1)$ is AWGN. Notice that, due to half-duplex limitations, the relay nodes do not transmit during time slot 1 and do not receive during time slot 2. The overall transmitted power during the two time slots is constrained to $2P$; thus, defining $\gamma_1 = \mathrm{E}\{X_s^{(1)}(X_s^{(1)})^*\}$ and $\gamma_2 = \mathrm{E}\{X_s^{(2)}(X_s^{(2)})^*\} + \sum_{i=1}^{N} \mathrm{E}\{X_i^{(2)}(X_i^{(2)})^*\}$ as the

[1] The branch 0 of the Lambert W function, $W_0(N)$, is defined as the function satisfying $W_0(N)e^{W_0(N)} = N$, with $W_0(N) \in R^+$ [23].

transmitted power[2] during slots 1 and 2, respectively, we enforce the following two-slot power constraint:

$$\gamma_1 + \gamma_2 = 2P. \tag{4}$$

3. ACHIEVABLE RATE IN AWGN

In order to determine the achievable rate of the channel, we consider updated transmitter and receiver channel state information (CSI) at all nodes, and assume symbol and phase synchronization among transmitters. The achievable rate with D&F is given in the following proposition.

Proposition 1. *In a half-duplex multiple-relay channel with decode-and-forward relaying and N parallel relays, the rate*

$$\begin{aligned} \mathcal{C}_{\text{D\&F}} = \max_{1\le n\le N} & \left\{ \max_{p(\mathbf{X}_s,\mathbf{X}_{1:n}^{(2)}):\gamma_1+\gamma_2=2P} \frac{1}{2}\cdot I\left(X_s^{(1)};Y_d^{(1)}\right) \right. \\ & \left. + \frac{1}{2}\cdot I\left(X_s^{(2)},\mathbf{X}_{1:n}^{(2)};Y_d^{(2)}\right)\right\} \\ \text{s.t.}\quad I\left(X_s^{(1)};Y_n^{(1)}\right) & \ge I\left(X_s^{(1)};Y_d^{(1)}\right) + I\left(X_s^{(2)},\mathbf{X}_{1:n}^{(2)};Y_d^{(2)}\right) \end{aligned} \tag{5}$$

is achievable. Source-relay path gains have been ordered as

$$|b_1| \ge \cdots \ge |b_n| \ge \cdots \ge |b_N|. \tag{6}$$

Remark 1. Factor 1/2 comes from time division signalling. Variable n in the maximization represents the number of active relays; hence, the relay selection is carried out through the maximization in (5), considering (6).

Proof. Let the N relays in Figure 1 be ordered as in (6), and assume that only the subset $\mathcal{R}_n = \{1,\ldots,n\} \subseteq \mathcal{N}$ is active, with $n \le N$. The source node selects message $\omega \in [1,\ldots,2^{mR}]$ for transmission (with m the total number of transmitted symbols during the two slots, and R the transmission rate) and maps it into two codebooks $\mathcal{X}_1, \mathcal{X}_2 \in \mathcal{C}^{m/2}$, using two independent encoding functions,[3] $x_1 : \{1,\ldots,2^{mR}\}\to\mathcal{X}_1$ and $x_2 : \{1,\ldots,2^{mR}\}\to\mathcal{X}_2$. The codeword $x_1(\omega)$ is then transmitted by the source during time slot 1, that is, $X_s^{(i)} = x_1(\omega)$. At the end of this slot, all relay nodes belonging to $\mathcal{R}_n$ are able to decode the transmitted message with arbitrarily small error probability if and only if the transmission rate satisfies [24]:

$$\begin{aligned} R &\le \frac{1}{2}\cdot\min_{i\in\mathcal{R}_n}\left\{I\left(X_s^{(1)};Y_i^{(1)}\right)\right\} \\ &= \frac{1}{2}\cdot I\left(X_s^{(1)};Y_n^{(1)}\right), \end{aligned} \tag{7}$$

where equality follows from (6), taking into account that all noises are i.i.d. Later, once decoded ω and knowing the codebook $\mathcal{X}_2$ and its associated encoding function, nodes in $\mathcal{R}_n$ (and also the source) calculate $x_2(\omega)$ and transmit it during phase 2. Hence, considering *memoryless* time-division channels with uncorrelated signalling between the two phases, the destination is able to decode ω if

$$R \le \frac{1}{2}\cdot I\left(X_s^{(1)};Y_d^{(1)}\right) + \frac{1}{2}\cdot I\left(X_s^{(2)},\mathbf{X}_{1:n}^{(2)};Y_d^{(2)}\right). \tag{8}$$

Therefore, the maximum source-to-destination transmission rate for the MRC is given by (8) with equality, subject to (7) being satisfied. Finally, noting that the set of active relay nodes $\mathcal{R}_n$ can be chosen out of $\{\mathcal{R}_1,\ldots,\mathcal{R}_N\}$ concludes the proof. □

As previously mentioned, we consider all receiver nodes under unitary power AWGN. The evaluation of Proposition 1 for faded Gaussian channels is established in Proposition 2. Previously, from an intuitive view of (5), some conclusions can be inferred: first, we note that the relay nodes which have successfully decoded during phase 1 transmit during phase 2 using a distributed MISO channel to destination. Assuming transmit CSI and phase synchronization among them, the performance of such a distributed MISO is equal to that of the actual MISO channel. Therefore, the optimum power allocation on the relays will also be the optimum beamforming [1]. For the power allocation over the two time slots, we also notice the following tradeoff: the higher the power allocated during time slot 1 is, the more the relays belong to the decoding set, but the less power they have during time slot 2 to transmit. Both considerations are discussed in Proposition 2.

Proposition 2. *In a Gaussian, half-duplex, multiple relay channel with decode-and-forward relaying and N parallel relays, the rate*

$$\mathcal{C}_{\text{D\&F}} = \max_{1\le n\le N}\frac{1}{2}\cdot\mathcal{C}(\gamma_{1n}\lambda_1) + \frac{1}{2}\cdot\mathcal{C}(\gamma_{2n}\lambda_{2n}) \tag{9}$$

is achievable, where

$$\lambda_1 = |a|^2, \qquad \lambda_{2n} = |a|^2 + \sum_{i=1}^{n}|c_i|^2 \tag{10}$$

are the beamforming gains during time slots 1 and 2, respectively, and the power allocation is computed from

$$\begin{aligned} \gamma_{1n} &= \max\left\{\left(\frac{1}{\mu_n}-\frac{1}{\lambda_1}\right),\gamma_n^c\right\}, \\ \gamma_{2n} &= \min\left\{\left(\frac{1}{\mu_n}-\frac{1}{\lambda_{2n}}\right),2P-\gamma_n^c\right\} \end{aligned} \tag{11}$$

subject to $(\mu_n^{-1}-\lambda_1^{-1}) + (\mu_n^{-1}-\lambda_{2n}^{-1}) = 2P$, *and*

$$\begin{aligned} \gamma_n^c &= \phi_n + \sqrt{\phi_n^2 + \frac{2P}{\lambda_1}}, \\ \phi_n &= \left(\frac{1}{\mu_n}-\frac{1}{\lambda_1}\right) - \frac{|b_n|^2}{2\lambda_1\lambda_{2n}}. \end{aligned} \tag{12}$$

Source-relay path gains have been ordered as

$$|b_1| \ge \cdots \ge |b_n| \ge \cdots \ge |b_N|, \tag{13}$$

[2] $E\{\cdot\}$ denotes expectation.

[3] Codewords in $\mathcal{X}_1$, $\mathcal{X}_2$ have length $m/2$ since each one is transmitted in one time slot, respectively.

Remark 2. As previously, maximization over n selects the optimum number of relays. The optimum power allocation γ_{1n}, γ_{2n} results in a constrained temporal *water-filling* over the two slots of the communication. Furthermore, γ_n^c is the minimum power allocation during time slot 1 that satisfies simultaneously, for a given set of active relays $\mathcal{R}_n = \{1, \ldots, n\}$, the power constraint (4) and the constraint in (5).

Proof. To derive expression (9), we independently solve the optimization problems in (5):

$$\max_{p(\mathbf{X}_s, \mathbf{X}_{1:n}^{(2)}): \gamma_1 + \gamma_2 = 2P} \frac{1}{2} \cdot I\left(X_s^{(1)}; Y_d^{(1)}\right) + \frac{1}{2} \cdot I\left(X_s^{(2)}, \mathbf{X}_{1:n}^{(2)}; Y_d^{(2)}\right)$$
$$\text{s.t.} \quad I\left(X_s^{(1)}; Y_n^{(1)}\right) \geq I\left(X_s^{(1)}; Y_d^{(1)}\right) + I\left(X_s^{(2)}, \mathbf{X}_{1:n}^{(2)}; Y_d^{(2)}\right) \tag{14}$$

for every $n \in \{1, \ldots, N\}$. First, we notice that for AWGN and *memoryless* channels, the optimum input signal during the two slots is i.i.d. with Gaussian distribution. Hence, the mutual information in (14) are given by

$$\begin{aligned} I\left(X_s^{(1)}; Y_d^{(1)}\right) &= \mathcal{C}(\gamma_1 \lambda_1), \\ I\left(X_s^{(2)}, \mathbf{X}_{1:n}^{(2)}; Y_d^{(2)}\right) &= \mathcal{C}(\gamma_2 \lambda_{2n}), \\ I\left(X_s^{(1)}; Y_n^{(1)}\right) &= \mathcal{C}\left(\gamma_1 |b_n|^2\right), \end{aligned} \tag{15}$$

with λ_1 and λ_{2n} defined in (10), and γ_1 and γ_2 the transmitted powers during time slot 1 and 2, respectively. Then maximization (14) reduces to

$$\begin{aligned} &\max_{\gamma_1, \gamma_2: \gamma_1 + \gamma_2 = 2P} \frac{1}{2} \cdot \mathcal{C}(\gamma_1 \lambda_1) + \frac{1}{2} \cdot \mathcal{C}(\gamma_2 \lambda_{2n}) \\ &\text{s.t.} \quad \mathcal{C}\left(\gamma_1 |b_n|^2\right) \geq \mathcal{C}(\gamma_1 \lambda_1) + \mathcal{C}(\gamma_2 \lambda_{2n}). \end{aligned} \tag{16}$$

The optimization above is solved in Appendix A yielding (9), with γ_{1n} and γ_{2n} the optimum power allocation on each slot for a given value n. Maximization over n results in the optimum relay selection. □

4. ERGODIC ACHIEVABLE RATE

In this section, we analyze the ergodic behavior of the instantaneous achievable rate obtained in Proposition 2. We assume that the source transmits over several, concatenated two-slot multiple relay transmissions, with uncorrelated channel conditions (see Figure 2). Thus, it achieves an effective rate equal to the expectation (on the channel distribution) of the achievable rate defined in Proposition 2, that is, it achieves a rate equal to the ergodic achievable rate. Throughout the paper, we assume random channel fading and random i.i.d. relay positions, invariant during the two-phase transmission but independent between transmissions.

Accordingly, considering the result in (9), we define the ergodic achievable rate[4] of the half-duplex MRC as

$$\begin{aligned} \mathcal{C}_{\text{D\&F}}^e &= \mathrm{E}_{\mathbf{a},\mathbf{b},\mathbf{c}}\{\mathcal{C}_{\text{D\&F}}\} \\ &= \mathrm{E}_{\mathbf{a},\mathbf{b},\mathbf{c}}\left\{\max_{1 \leq n \leq N} \mathcal{C}_n\right\}, \end{aligned} \tag{17}$$

where $\mathbf{a} = |a|^2$ is the source-to-destination channel; $\mathbf{c} = [|c_1|^2, \ldots, |c_N|^2]$ the relay-to-destination channels, and $\mathbf{b} = [|b_1|^2, \ldots, |b_N|^2]$ the source-to-relay channels ordered as (6). Notice that all elements in $\mathbf{c}$ are i.i.d. while, due to ordering, elements in $\mathbf{b}$ are mutually dependent. Finally, $\mathcal{C}_n$ in (17) is defined from Proposition 2 as

$$\mathcal{C}_n = \frac{1}{2} \cdot \mathcal{C}(\gamma_{1n} \lambda_1) + \frac{1}{2} \cdot \mathcal{C}(\gamma_{2n} \lambda_{2n}). \tag{18}$$

There is no closed-form expression for the ergodic capacity of the multiple-relay channel in (17); capacities $\mathcal{C}_1, \ldots, \mathcal{C}_N$ are mutually dependent, therefore closed-form expression for the cumulative density function (cdf) of $\max_{1 \leq n \leq N} \mathcal{C}_n$ cannot be obtained. Hence, we turn our attention to obtaining upper and lower bounds.

4.1. Lower bound

A lower bound can be derived using Jensen's inequality, taking into account the convexity of the pointwise maximum function:

$$\begin{aligned} \mathcal{C}_{\text{D\&F}}^e &= \mathrm{E}_{\mathbf{a},\mathbf{b},\mathbf{c}}\left\{\max_{1 \leq n \leq N} \mathcal{C}_n\right\} \\ &\geq \max_{1 \leq n \leq N} \mathrm{E}_{\mathbf{a},\mathbf{b},\mathbf{c}}\{\mathcal{C}_n\}. \end{aligned} \tag{19}$$

The interpretation of such bound is as follows: the inequality shows that the ergodic capacities achieved assuming a fixed number of active relays are, obviously, always lower than the ergodic capacity achieved with instantaneous optimal relay selection. Analyzing (19) carefully, we notice that $\mathcal{C}_n$ does not depend upon entire vector $\mathbf{b}$ but only upon $|b_n|^2$. Furthermore, we have seen that $\mathcal{C}_n$ depends on fading between source and destination, and between relays and destination just in terms of beamforming gains $\lambda_1 = |a|^2$ and $\lambda_{2n} = |a|^2 + \sum_{i=1}^n |c_i|^2$; therefore, renaming $\delta = |a|^2$ and $\beta_n = \sum_{i=1}^n |c_i|^2$, expression (19) simplifies to

$$\mathcal{C}_{\text{D\&F}}^e \geq \max_{1 \leq n \leq N} \mathrm{E}_{\delta, \beta_n, |b_n|^2}\{\mathcal{C}_n\}, \tag{20}$$

where δ is a unitary-mean, exponential random variable describing the square of the fading coefficient between source and destination. Likewise, β_n describes the relay beamforming gain assuming only the set of relays $\mathcal{R}_n = \{1, \ldots, n\}$ to be active. It is obtained as the sum of n exponentially distributed, unitary mean random variables, and hence it is distributed as a chi-squared random variable with $2n$ degrees

[4] Notice that, due to the power constraint (4), the ergodic achievable rate is directly computed as the expectation of the instantaneous achievable rate of the link.

of freedom. Both variables are described by their probability density functions (pdf) as

$$f_\delta(\delta) = e^{-\delta},\qquad f_{\beta_n}(\beta) = \frac{\beta^{(n-1)}e^{-\beta}}{(n-1)!}. \tag{21}$$

The study of $|b_n|^2$ is more involved; b_n, as defined previously, is the nth better channel from source to relays, following the ordering in (13). As stated earlier, source-to-relay channels in (1) are i.i.d. with complex Gaussian distribution and power $(d_o/d)^\alpha$; d is the random source-to-relay distance, assumed i.i.d. for all relays and with a generic pdf $f_d(d)$, $d \in [0, d^+]$. Hence, defining $\xi \sim \mathcal{CN}(0,(d_o/d)^\alpha)$, we make use of ordered statistics to obtain the pdf of $|b_n|^2$ as [25]

$$\begin{aligned} f_{|b_n|^2}(b) = {} & \frac{N!}{(N-n)!1!(n-1)!} f_{|\xi|^2}(b) P\left[|\xi|^2 \le b\right]^{N-n} \\ & \times P\left[|\xi|^2 \ge b\right]^{n-1}, \end{aligned} \tag{22}$$

where cumulative density function $P[|\xi|^2 \le b]$ may be derived as

$$P\left[|\xi|^2 \le b\right] = 1 - \int_0^{d^+} e^{-b(x/d_o)^\alpha} f_d(x)dx, \tag{23}$$

and probability density function $f_{|\xi|^2}(b)$ is computed as the first derivative of (23) respect to b:

$$f_{|\xi|^2}(b) = \int_0^{d^+} \left(\frac{x}{d_o}\right)^\alpha e^{-b(x/d_o)^\alpha} f_d(x)dx. \tag{24}$$

Therefore, proceeding from (20),

$$\mathcal{C}^e_{\mathrm{D\&F}} \ge \max_{1\le n\le N} \iint_0^\infty \mathrm{E}_{|b_n|^2}\left\{\mathcal{C}_n \mid \delta, \beta_n\right\} f_\delta(\delta) f_{\beta_n}(\beta) db\, d\beta, \tag{25}$$

where $\mathrm{E}_{|b_n|^2}\{\mathcal{C}_n \mid \delta, \beta\}$ is the mean of $\mathcal{C}_n$ over $|b_n|^2$ conditioned on beamforming gains δ and $\beta_n = \beta$. This mean may be readily obtained using the pdf (22) and power allocation defined in (10):

$$\begin{aligned} \mathrm{E}_{|b_n|^2}\left\{\mathcal{C}_n \mid \delta, \beta\right\} = {} & \frac{1}{2}\int_0^\infty \left(\mathcal{C}(\gamma_{1n}\delta) + \mathcal{C}(\gamma_{2n}(\delta+\beta))\right) \\ & \times f_{|b_n|^2}(b) db. \end{aligned} \tag{26}$$

Notice that

$$[\gamma_{1n}, \gamma_{2n}] = \begin{cases} \left[\left(\dfrac{1}{\mu_n} - \dfrac{1}{\delta}\right), \left(\dfrac{1}{\mu_n} - \dfrac{1}{\delta+\beta}\right)\right], & b \ge \psi(\delta,\beta), \\ \left[\gamma^c_n, 2P - \gamma^c_n\right], & b < \psi(\delta,\beta), \end{cases} \tag{27}$$

where

$$\psi(\delta,\beta) = \left[\left(\frac{1}{\mu_n} - \frac{1}{\delta}\right) + \frac{2P}{\delta}\left(\frac{1}{\mu_n} - \frac{1}{\delta}\right)^{-1}\right]\delta(\delta+\beta). \tag{28}$$

4.2. Upper bound

To upper bound the ergodic achievable rate use, once again, Jensen's inequality. Nevertheless, in this case, we focus on the concavity of functions $\mathcal{C}_n$ in (18). As previously mentioned, the capacity $\mathcal{C}_n$ only depends on 3 variables: the random source-to-user channel $|a|^2$, the relays-to-destination beamforming gain $\sum_{i=1}^n |c_i|^2$, and the random path gain $|b_n|^2$. Obviously, it also depends on the power allocation and the power constraint, but notice that power allocation is a direct function of those three variables and that the power constraint is assumed constant.

The concavity of $\mathcal{C}_n$ over the three random variables is shown in Appendix B, and obtained applying properties of the composition of concave functions [26]. This result allows us to conclude that $\mathcal{C}_{\mathrm{D\&F}}$, being defined as the maximum of a set concave functions (9), is also concave over the variables that define $\mathcal{C}_n$. Therefore, the capacity of regenerative MRC is concave over variable **a** and vectors **b** and **c**, and thus we may define the following upper bound:

$$\begin{aligned} \mathcal{C}^e_{\mathrm{D\&F}} & = \mathrm{E}_{\mathbf{a},\mathbf{b},\mathbf{c}}\left\{\max_{1\le n\le N} \mathcal{C}_n\right\} \\ & \le \max_{1\le n\le N} \mathcal{C}_n(\bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}), \end{aligned} \tag{29}$$

where $\bar{\mathbf{a}} = \mathrm{E}_{\mathbf{a}}\{\mathbf{a}\} = 1$, $\bar{\mathbf{c}} = \mathrm{E}_{\mathbf{c}}\{\mathbf{c}\} = [1,\ldots,1]$, and $\bar{\mathbf{b}} = \mathrm{E}_{\mathbf{b}}\{\mathbf{b}\} = [|\overline{b_1}|^2, \ldots, |\overline{b_n}|^2, \ldots, |\overline{b_N}|^2]$ are the mean squared source-to-destination, relay-to-destination, and source-to-relay channels, respectively. Notice that $|\overline{b_n}|^2 = \int_0^\infty b f_{|b_n|^2}(b) db$ is computed by using the pdf in (22). Therefore, considering the capacity derivation in Proposition 2, we obtain

$$\mathcal{C}_n(\bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}) = \frac{1}{2}\log_2(1+\rho_{1n}) + \frac{1}{2}\log_2(1+\rho_{2n}\cdot(n+1)), \tag{30}$$

where

$$\begin{aligned} \rho_{1n} & = \max\left\{\left(\frac{1}{\mu_n} - 1\right), \gamma^c_n\right\} \\ \rho_{2n} & = \min\left\{\left(\frac{1}{\mu_n} - \frac{1}{n+1}\right), (2P - \gamma^c_n)\right\}, \\ \gamma^c_n & = \left(\left(\frac{1}{\mu_n} - 1\right) - \frac{|\overline{b_n}|^2}{2(n+1)}\right) \\ & \quad + \sqrt{\left(\left(\frac{1}{\mu_n} - 1\right) - \frac{|\overline{b_1}|^2}{2(n+1)}\right)^2 + 2P}. \end{aligned} \tag{31}$$

Hence, the upper bound on the ergodic capacity of MRC is

$$\mathcal{C}^e_{\mathrm{D\&F}} \le \max_{1\le n\le N} \frac{1}{2}\log_2(1+\rho_{1n}) + \frac{1}{2}\log_2(1+\rho_{2n}\cdot(n+1)). \tag{32}$$

The interpretation of this upper bound leads to the comparison of faded and nonfaded channels: from (29) we conclude that the capacity of the MRC with nonfaded channels is always higher than the ergodic capacity of the MRC with unitary-mean Rayleigh-faded channels.

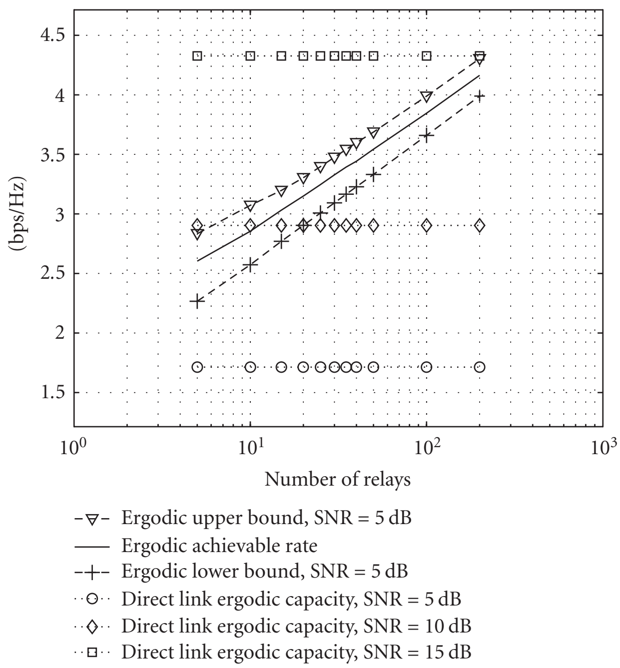

Figure 3: Ergodic achievable rate in [bps/Hz] of a Gaussian multiple relay channel with transmit SNR = 5 dB, under Rayleigh fading. The upper and lower bounds proposed in the paper are shown, and the ergodic capacity of a direct link plotted as reference.

5. ASYMPTOTIC ACHIEVABLE RATE

In previous sections, we analyzed the instantaneous and ergodic achievable rate of multiple-relay channels with full CSI, assuming a finite number of potential relays N. Results suggest (as it can be shown in Figure 3) a growth of the spectral efficiency with the total number relays. Nevertheless, neither the result in Proposition 2 nor the bounds (25) and (32) are tractable enough to infer the asymptotic behavior. In this section, we introduce the necessary approximations to simplify the problem and to analyze the asymptotic achievable rate of the MRC. We show that capacity grows with the logarithm of the branch zero of the Lambert W function of the total number of parallel relays.

Prior to the analysis, in the asymptotic domain ($N \to \infty$), we rename variable n in maximization (9) as $n = \kappa \cdot N$ with $\kappa \in [0, 1]$ (see [25, page 71]), and we introduce four key approximations.

(1) For a large number of network nodes, we consider capacities $\mathcal{C}_n$ in (18) defined only by the second slot mutual information,[5] that is,

$$\mathcal{C}_{\kappa \cdot N} = \frac{1}{2}\mathcal{C}\left(\gamma_{1\kappa \cdot N}\lambda_1\right) + \frac{1}{2}\mathcal{C}\left(\gamma_{2\kappa \cdot N}\lambda_{2\kappa \cdot N}\right) \approx \frac{1}{2}\mathcal{C}\left(\gamma_{2\kappa \cdot N}\lambda_{2\kappa \cdot N}\right). \tag{33}$$

The proposed approximation is justified by the large beamforming gain obtained during time slot 2 when the number of relays grows to ∞ (as shown in approximation 2). As a consequence, $\gamma^c_{\kappa \cdot N}$ computed in Appendix A is recalculated as

$$\gamma^c_{\kappa \cdot N} = 2P \frac{\lambda_{2\kappa \cdot N}}{\left|b_{\kappa \cdot N}\right|^2 + \lambda_{2\kappa \cdot N}}. \tag{34}$$

To derive (34), we recall that $\gamma^c_{\kappa \cdot N}$ is defined in (A.5) as the power allocation during slot 1 that simultaneously satisfies $\sum_{i=1}^{2}\gamma_i = 2P$ and $\mathcal{C}(\gamma_1 |b_{\kappa \cdot N}|^2) = \mathcal{C}(\gamma_1\lambda_1) + \mathcal{C}(\gamma_2\lambda_{2\kappa \cdot N})$ (i.e., $\gamma^c_{\kappa \cdot N} = \{\gamma_1 : \mathcal{C}(\gamma_1 |b_{\kappa \cdot N}|^2) = \mathcal{C}(\gamma_1\lambda_1) + \mathcal{C}((2P - \gamma_1)\lambda_{2\kappa \cdot N})\}$). Hence, neglecting the factor $\mathcal{C}(\gamma_1\lambda_1)$, then (34) is obtained.

(2) From the Law of Large Numbers, $\lambda_{2\kappa \cdot N}$ in (10) is approximated as $\lambda_{2\kappa \cdot N} \approx \kappa \cdot N$.

(3) From [25, pages 255–258], the pdf of the ordered random variable $|b_{\kappa \cdot N}|^2$ asymptotically satisfies $\mathrm{pdf}_{|b_{\kappa \cdot N}|^2} = \mathcal{N}(Q(1-\kappa), \varepsilon \cdot N^{-1})$ as $N \to \infty$ (with ε a fixed constant). $Q(\kappa) : [0, 1] \to R^+$ is the inverse function of the cdf of the squared modulus of the nonordered source-to-relay channel defined in (1), that is , $Q(\Pr\{|b|^2 < \widetilde{b}\}) = \widetilde{b}$ with $b \sim \mathcal{CN}(0, (d_o/d)^\alpha)$ and d the source-to-relay random distance. From the asymptotic pdf, the following convergence in probability holds:

$$\left| b_{\kappa \cdot N} \right|^2 \xrightarrow{P} Q(1-\kappa). \tag{35}$$

(4) We consider high-transmitted power, so that $\mu_{\kappa \cdot N} \approx P^{-1}$ is in the power allocation (11).

Making use of those four approximations, we may apply (9) to define the asymptotic instantaneous capacity as

$$\begin{aligned}
\mathcal{C}^a_{\text{D\&F}} &= \frac{1}{2}\lim_{N\to\infty}\max_{\kappa\in[0,1]} \mathcal{C}_{\kappa \cdot N} \\
&\approx \frac{1}{2}\lim_{N\to\infty}\max_{\kappa\in[0,1]} \mathcal{C}\left(\gamma_{2\kappa \cdot N}\lambda_{2\kappa \cdot N}\right) \\
&= \frac{1}{2}\lim_{N\to\infty}\max_{\kappa\in[0,1]} \min\left\{\mathcal{C}\left(\left(\frac{1}{\mu_{\kappa \cdot N}} - \frac{1}{\kappa \cdot N}\right)\kappa \cdot N\right),\right. \\
&\qquad\qquad \left.\mathcal{C}\left(\left(2P - \gamma^c_{\kappa \cdot N}\right)\kappa \cdot N\right)\right\} \\
&= \frac{1}{2}\lim_{N\to\infty}\max_{\kappa\in[0,1]} \min\left\{\mathcal{C}(P \cdot \kappa \cdot N - 1),\right. \\
&\qquad\qquad \left.\mathcal{C}\left(2P\frac{Q(1-\kappa)\kappa \cdot N}{Q(1-\kappa) + \kappa \cdot N}\right)\right\},
\end{aligned} \tag{36}$$

where first equality follows from Proposition 2, and second equality from approximation 1; third equality comes from the power allocation $\gamma_{2\kappa \cdot N}$ in (11) and considering $\lambda_{2\kappa \cdot N} = 2\kappa \cdot N$ as approximation 2. Finally, forth equality is obtained making use of approximation 4, and introducing the asymptotic convergence of $|b_{\kappa \cdot N}|^2$ in (34).

Let us focus now on the last equality in (36). We notice that (i) $\mathcal{C}(P \cdot \kappa \cdot N - 1)$ is an increasing function in $\kappa \in [0, 1]$,

[5] The proposed approximation is also a lower bound. Thus, the asymptotic performance of the lower bound is valid to lower bound the asymptotic performance of the achievable rate.

(ii) $Q(1-\kappa)$ is a decreasing function in the same interval, (iii) therefore, $\mathcal{C}(2P(Q(1-\kappa)\kappa\cdot N)/(Q(1-\kappa)+\kappa\cdot N))$ is asymptotically a decreasing function in $\kappa \in [0,1]$. Hence, in the limit, the maximum in κ of the minimum of an increasing and a decreasing functions would be given at the intersection of the two curves. As derived in Appendix C, the intersection point[6] $\kappa_o(N)$ satisfies

$$\kappa_o(N) \geq \frac{W_0(\rho N)}{\rho N} \tag{37}$$

with ρ a fixed constant in $(0,1)$, and with equality whenever the relay positions are not random but deterministic. As mentioned earlier, $W_0(N)$ is the branch zero of the Lambert W function evaluated at N [23].

Finally, applying the forth equality in (36), we derive

$$\begin{aligned} \mathcal{C}^{a}_{\text{D\&F}} &= \frac{1}{2}\lim_{N\to\infty} \mathcal{C}\left(P\cdot\kappa_o(N)\cdot N-1\right) \\ &\geq \frac{1}{2}\lim_{N\to\infty} \log_2\left(P\cdot\frac{W_0(\rho N)}{\rho}\right). \end{aligned} \tag{38}$$

This result shows that, for any random distribution of relays, the capacity of MRC with channel knowledge grows asymptotically with the logarithm of the Lambert W function of the total number relays. However, due to approximations 2 and 3, our proof only demonstrates asymptotic performance in probability.

6. NUMERICAL RESULTS

In this section, we evaluate the lower and upper bounds described in (25) and (32), respectively, and compare them with the ergodic achievable rate of the link, obtained through Monte Carlo simulation.

As previously pointed out, we assume i.i.d., unitary mean, Rayleigh-distributed fading from all transmitter nodes to destination, while source-to-relay channels are modelled as a superposition of path loss and unitary mean Rayleigh fading. Likewise, source and destination are fixed nodes, while the position of the N relays is i.i.d. throughout a square, limited at its diagonal by the point-to-point source-to-destination link. As mentioned earlier, the position of relays is invariant during the two-slot communication but variant and uncorrelated from one transmission to the other. To deal with propagation effects, we defined a simplified exponential indoor propagation model with path loss exponent $\alpha = 4$. Finally, we consider normalized distances, defining distance between source and destination equal to 1, and source-relay random distance $d_i \in [0,1]$.

Taking into account the considerations above, we focus the analysis on the number of relay nodes and the transmitted SNR, that is, P/σ_o^2. Figure 3 depicts the ergodic bounds computed for transmit SNR equal to 5 dB for an MRC with the number of relay nodes ranging from 5 to 200. Likewise,

[6] For a fixed number of relays N, a fixed intersection point κ_o is derived. Thus, $\kappa_o = \kappa_o(N)$.

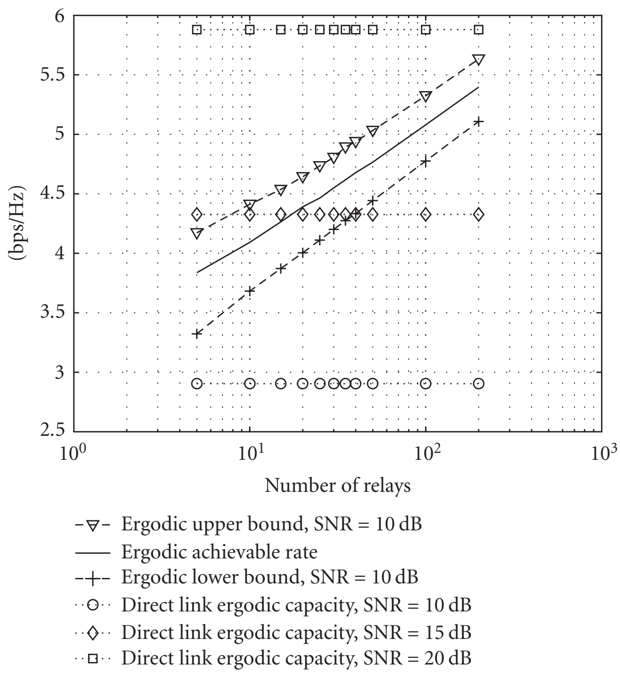

FIGURE 4: Ergodic achievable rate in [bps/Hz] of a Gaussian multiple relay channel with transmit SNR = 10 dB, under Rayleigh fading. The upper and lower bounds proposed in the paper are shown, and the ergodic capacity of a direct link plotted as reference.

Figures 4 and 5 plot results for transmit SNR equal to 10 dB and 20 dB, respectively. Firstly, we clearly note that, for all plots, ergodic bounds and simulated result increase with the number of users, as we have previously demonstrated in the asymptotic capacity section.

Moreover, the comparison of the three plots shows that the advantage of relaying diminishes as the transmitted power increases. In such a way, it can be seen that for transmit SNR = 5 dB only $N = 20$ parallel relay nodes are needed to double the noncooperative capacity, while for SNR = 10 dB more than $N = 200$ nodes would be necessary to obtain twice the spectral efficiency. Furthermore, we may see that for SNR = 5 dB with only 10 relays, it is possible to obtain the same ergodic capacity as a Rayleigh-faded direct link with SNR = 10 dB, while to obtain the same power saving for MRC with SNR = 20 dB, 50 nodes are needed. Finally, plots show that the accuracy of the presented bounds grows as the transmit SNR diminishes, which may be interpreted in terms of the meaning of such bounds: for decreasing transmitted power, the effect of instantaneous relay selection and the effect of Rayleigh fading over the cooperative links lose significance.

Figures 6–8 show results on the mean number of active relays versus the total number of relay nodes. Recall that the optimum number of relay nodes is calculated from maximization over n in Proposition 2. Specifically, Figure 6 depicts results for SNR = 5 dB while Figures 7 and 8 show cooperating nodes for SNR = 10 dB and SNR = 20 dB. In all three, the number of active nodes n that maximizes the lower and upper bounds, (25) and (32), respectively, is

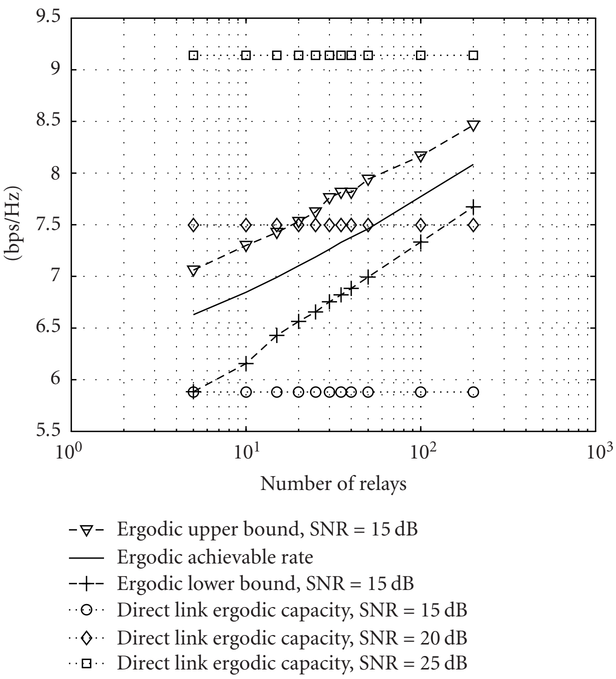

Figure 5: Ergodic achievable rate in [bps/Hz] of a Gaussian multiple relay channel with transmit SNR = 15 dB, under Rayleigh fading. The upper and lower bounds proposed in the paper are shown, and the ergodic capacity of a direct link plotted as reference.

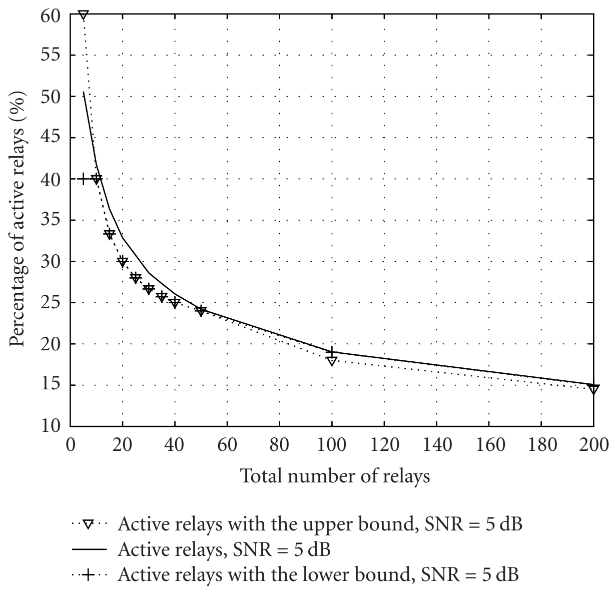

Figure 6: Expected number of active relays (in %) of a multiple relay channel with transmit SNR = 5 dB, under Rayleigh fading. The number of relays that optimizes the upper and lower bounds are shown for comparison.

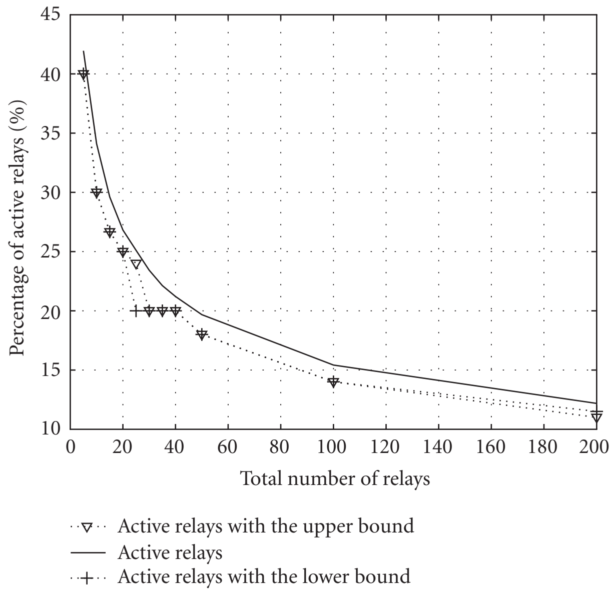

Figure 7: Expected number of active relays (in %) of a multiple relay channel with transmit SNR = 10 dB, under Rayleigh fading. The number of relays that optimizes the upper and lower bounds are shown for comparison.

also plotted; hence, it allows for comparison between the mean number of relays with capacity achieving relaying and the optimum number of relays with no instantaneous relay selection (25) and with no fading channels (32), respectively. Firstly, results show that the simulated mean number of relays is close to the number of relays maximizing the upper and lower bounds, being closer for the low SNR regime. Finally, we notice that, as the transmit SNR increases, the percentage of relays cooperating with the source decreases. Therefore, we conclude that regenerative relaying is, as previously mentioned, more powerful in the low SNR regime.

7. CONCLUSIONS

In this paper, we examined the achievable rate of a decode-and-forward (D&F) multiple-relay channel with half-duplex constraint and transmitter and receiver channel state information. The transmission was arranged in two phases: during the first phase, the source transmits its message to relays and destination. During the second phase, the relays and the source are configured as a distributed antenna array to transmit extra parity bits. The instantaneous achievable rate for the optimum relay selection and power allocation was obtained. Furthermore, we studied and bounded the ergodic performance of the achievable rate for Rayleigh-faded channels. We also found the asymptotic performance of the achievable rate in number of relays. Results show that (i) $\mathcal{C}_{\text{D\&F}} \propto \log(W_0(N))$ as $N \to \infty$; (ii) with regenerative relaying, higher capacity is obtained for low signal-to-noise ratio, (iii) the percentage of active relays (i.e., the number of nodes who can decode the source message) decreases for increasing N, and (iv) this percentage is low, even at low SNR, due to the regenerative constraint.

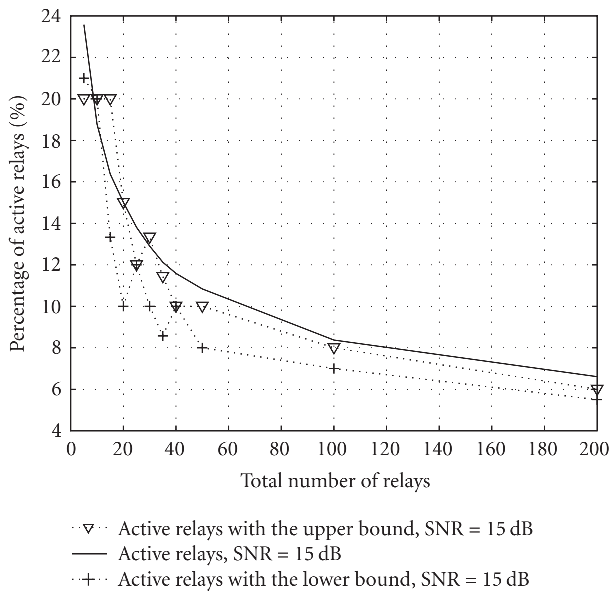

Figure 8: Expected number of active relays (in %) of a multiple relay channel with transmit SNR = 15 dB, under Rayleigh fading. The number of relays that optimizes the upper and lower bounds are shown for comparison.

APPENDICES

A. OPTIMIZATION PROBLEM

For completeness of explanation, in the appendix we solve optimization problem (16), which can be recast as follows:

$$C = \max_{\gamma_1,\gamma_2} \frac{1}{2}\sum_{i=1}^{2} \log_2(1+\gamma_i\lambda_i)$$
$$\text{s.t.}\quad \sum_{i=1}^{2}\gamma_i = 2P, \tag{A.1}$$
$$\gamma_i \geq \frac{\Pi_{i=1}^{2}(1+\gamma_i\lambda_i)-1}{|b_n|^2},$$

which is convex in both $\gamma_1 \in R^+$ and $\gamma_2 \in R^+$. The Lagrange dual function of the problem is

$$L(\gamma_1,\gamma_2,\mu,\nu) = \sum_{i=1}^{2}\log(1+\gamma_i\lambda_i) - \mu\left(\sum_{i=1}^{2}\gamma_i - 2P\right) + \nu\left(\gamma_1 - \frac{\Pi_{i=1}^{2}(1+\gamma_i\lambda_i)-1}{|b_n|^2}\right), \tag{A.2}$$

where μ and ν are the Lagrange multipliers for first and second constraints, respectively. The three KKT conditions (necessary and sufficient for optimality) of the dual problem are

$$\text{(i)}\quad \frac{\lambda_i}{1+\gamma_i\lambda_i} - \mu + \nu\frac{d}{d\gamma_i}\left(\gamma_i - \frac{\Pi_{i=1}^{2}(1+\gamma_i\lambda_i)-1}{|b_n|^2}\right) = 0 \quad \text{for } i \in \{1,2\},$$
$$\text{(ii)}\quad \mu\left(\sum_{i=1}^{2}\gamma_i - 2P\right) = 0,$$
$$\text{(iii)}\quad \nu\left(\gamma_1 - \frac{\Pi_{i=1}^{2}(1+\gamma_i\lambda_i)-1}{|b_n|^2}\right) = 0. \tag{A.3}$$

Notice that the set $(\nu^*,\gamma_1^*,\gamma_2^*,\mu^*)$:

$$\nu^* = 0,\qquad \gamma_i^* = \left(\frac{1}{\mu^*} - \frac{1}{\lambda_i}\right)^+,\qquad \frac{1}{\mu^*} = P + \frac{1}{2}\sum_{i=1}^{2}\frac{1}{\lambda_i}, \tag{A.4}$$

satisfies KKT conditions hence yielding the optimum solution.[7] However, taking into account that optimal primal points must satisfy the two constraints in (A.1), and that

$$\left.\begin{aligned}&\sum_{i=1}^{2}\gamma_i = 2P\\ &\gamma_1 \geq \frac{\Pi_{i=1}^{2}(1+\gamma_i\lambda_i)-1}{|b_n|^2}\end{aligned}\right\} \longrightarrow \gamma_1 \geq \gamma^c = \phi + \sqrt{\phi^2 + \frac{2P}{\lambda_1}} \in R^+ \tag{A.5}$$

with $\phi = (1/\mu^* - 1/\lambda_i) - |b_n|^2/2\lambda_1\lambda_2$. Then, the result in optimum power allocation is

$$\gamma_1^* = \max\left\{\left(\frac{1}{\mu^*} - \frac{1}{\lambda_i}\right), \gamma^c\right\},$$
$$\gamma_2^* = 2P - \gamma_1^*, \tag{A.6}$$
$$\frac{1}{\mu^*} = P + \frac{1}{2}\sum_{i=1}^{2}\frac{1}{\lambda_i}.$$

B. CONCAVITY OF $\mathcal{C}_N$

In the appendix, we prove the concavity of capacity $\mathcal{C}_n$ (defined in (18) based on (9)) over random variables $|a|^2$, $\sum_{i=1}^{n}|c_i|^2$, and $|b_n|^2$. To do so, we first rewrite the function under study as a composition of functions:

$$\mathcal{C}_n = \mathcal{C}\left(\max\left(\Gamma_1(\mathbf{x}),\Gamma_2(\mathbf{x})\right)\right) + \mathcal{C}\left(\min\left(\Psi_1(\mathbf{x}),\Psi_2(\mathbf{x})\right)\right), \tag{B.7}$$

[7] Using standard notation, we define $(A)^+ = \max\{A, 0\}$.

where $\mathbf{x} = [|a|^2, \sum_{i=1}^{n}|c_i|^2, |b_n|^2]$ and

$$\begin{aligned}
\Gamma_1(\mathbf{x}) &= \left(\frac{1}{\mu_n} - \frac{1}{|a|^2}\right)|a|^2, \quad \Gamma_1 : \mathbf{R}^{3+} \longrightarrow \mathbf{R}, \\
\Gamma_2(\mathbf{x}) &= \gamma_n^c(\mathbf{x})|a|^2, \quad \Gamma_2 : \mathbf{R}^{3+} \longrightarrow \mathbf{R}, \\
\Psi_1(\mathbf{x}) &= \left(\frac{1}{\mu_n} - \frac{1}{|a|^2 + \sum_{i=1}^{n}|c_i|^2}\right) \\
&\quad \times \left(|a|^2 + \sum_{i=1}^{n}|c_i|^2\right), \quad \Psi_1 : \mathbf{R}^{3+} \longrightarrow \mathbf{R}, \\
\Psi_2(\mathbf{x}) &= (2P - \gamma_n^c(\mathbf{x}))\left(|a|^2 + \sum_{i=1}^{n}|c_i|^2\right), \quad \Psi_2 : \mathbf{R}^{3+} \longrightarrow \mathbf{R}.
\end{aligned} \tag{B.8}$$

First, we notice that pointwise maximum and pointwise minimum functions are nondecreasing functions with Hessian equal to zero. Next, computing the Hessian of $\Gamma_1(\mathbf{x})$ and $\Gamma_2(\mathbf{x})$ (respect to $\mathbf{x}$), it is shown that both are concave functions. Therefore, from [26, pages 86-87], we derive that $\max(\Gamma_1(\mathbf{x}), \Gamma_2(\mathbf{x}))$ is concave on $\mathbf{x}$. Accordingly, we may show that $\Psi_1(\mathbf{x})$ and $\Psi_2(\mathbf{x})$ are also concave functions, and so is $\min(\Psi_1(\mathbf{x}), \Psi_2(\mathbf{x}))$. Hence, considering that the sum of concave functions is always concave, and that $\mathcal{C}(x)$ is a concave nondecreasing function, we derive that $\mathcal{C}_n$ is concave on $\mathbf{x}$.

C. INTERSECTION OF CAPACITY CURVES

In this appendix, we analyze the intersection point κ_o of curves $f_1(\kappa) = \log_2(P\cdot\kappa N)$ and $f_2(\kappa) = \log_2(1 + 2P(Q(1-\kappa)\kappa\cdot N)/(Q(1-\kappa)+\kappa\cdot N))$ for a given number of relays N. To do so, we set $f_1(\kappa_o) = f_2(\kappa_o)$ to obtain[8]

$$Q(1-\kappa_o) \approx \kappa_o \cdot N. \tag{C.9}$$

From approximation 3 in Section 5, equality above is equivalent to

$$\Pr\{|b|^2 \leq \kappa_o \cdot N\} = 1 - \kappa_o \tag{C.10}$$

with $b \sim \mathcal{CN}(0, (d_o/d)^\alpha)$ and d the source-to-relay random distance. Furthermore, making use of the cdf in (23), we obtain

$$\begin{aligned}
\kappa_o &= 1 - \Pr\{|b|^2 \leq \kappa_o \cdot N\} \\
&= \int_0^{d^+} e^{-(x/d_o)^\alpha \kappa_o \cdot N} f_d(x)dx.
\end{aligned} \tag{C.11}$$

We can now apply Jensen's inequality for convex functions, in order to lower bound the integral as

$$\kappa_o \geq e^{-(E\{x\}/d_o)^\alpha \kappa_o \cdot N} \tag{C.12}$$

with $E\{x\} = \int_0^{d^+} x f_d(x)dx$. Equality is satisfied whenever the relays position are not random but deterministic, that is, $f_d(x) = \delta(x - d_r)$. Next, from [23], we directly solve inequality (C.12) over κ_o as

$$\kappa_o(N) \geq \frac{W_0(\rho N)}{\rho N} \tag{C.13}$$

with $\rho = -(E\{x\}/d_o)^\alpha$ a fixed constant in $(0, 1)$, and $W_0(\cdot)$ the branch zero of the Lambert W function.

This solution is applicable for every possible random distribution of relays.

[8] Approximation (C.9) is obtained neglecting the effect of 1 within the logarithm in $f_2(\kappa)$, assuming sufficiently large transmitted power P.

ACKNOWLEDGMENTS

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