

# Dynamical quark mass generation in QCD<sub>3</sub> within the Hamiltonian approach in Coulomb gauge

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**Abstract.** We investigate the equal-time (static) quark propagator in Coulomb gauge within the Hamiltonian approach to QCD in  $d = 2$  spatial dimensions. Although the underlying Clifford algebra is very different from its counterpart in  $d = 3$ , the gap equation for the dynamical mass function has the same form. The additional vector kernel which was introduced in  $d = 3$  to cancel the linear divergence of the gap equation and to preserve multiplicative renormalizability of the quark propagator makes the gap equation free of divergences also in  $d = 2$ .

## 1 Introduction

The two most striking features of low-energy Quantum Chromodynamics (QCD) at ordinary temperature and density are confinement and the spontaneous breaking of chiral symmetry. In recent years the research interest has been shifted to the investigation of thermal properties of QCD and of its phase diagram, where a central challenge is to locate the critical end point. Experimentally, there has been progress at the Relativistic Heavy-Ion Collider and the Large Hadron Collider; searches for the critical end point are on-going at the FAIR and NICA facilities. On the theoretical side, lattice calculations are hindered by the notorious sign problem at finite chemical potential. Furthermore, simulating three or four families of light dynamical quarks involves a high computational cost; an approach to reducing this cost is to reduce the number of physical dimensions.

QCD in  $1 + 1$  dimensions has been widely studied as toy model and in fact displays some relevant properties of real QCD, but fails to be a reliable testing ground for QCD<sub>4</sub>, since gauge symmetries are somewhat trivial in two dimensions (unless a compact manifold is considered). QCD in  $2 + 1$  dimensions is a more interesting alternative, which moreover allows the addition of a topological Chern–Simons term.

In this paper we examine QCD<sub>3</sub> with one massless fermion within the Hamiltonian approach in Coulomb gauge developed previously in  $d = 3$  spatial dimensions [1–3]. Within this approach we will investigate how a mass is dynamically generated by the interaction with the gluons. Our previous work in  $d = 3$  [2–6] has shown that the

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linearly rising colour Coulomb potential is the trigger of chiral symmetry breaking, and that a genuinely non-perturbative Dirac structure in the quark-gluon vertex eliminates the linear divergence of the quark gap equation and makes the latter ultraviolet (UV) finite. In the present paper we investigate whether this cancellation of the UV divergences in the quark gap equation persists also in  $d = 2$ . This is by far not obvious since the algebra of the Dirac matrices is different in  $d = 2$  from the  $d = 3$  case and moreover the degree of divergence is different. Of course, quarks in  $d = 2$  have no chiral symmetry to be broken<sup>1</sup> since there is no counterpart of  $\gamma_5$  in  $d = 2$ . The interesting question is here how a dynamical quark mass, which in  $d = 3$  is a consequence of spontaneous breaking of chiral symmetry, is generated in  $d = 2$  without chiral symmetry breaking. We will show that within the Hamiltonian approach to QCD in Coulomb gauge the dynamical quark mass generation is caused in  $d = 2$  by the confining non-abelian Coulomb interaction of the quarks, like in the  $d = 3$  case. So within this approach the dynamical mass generation seems to be a universal phenomenon which is independent of the number of dimensions and not necessarily linked to the spontaneous breaking of chiral symmetry. We will also show that the cancellation of the leading UV divergence in the quark gap equation found in  $d = 3$  with our Ansatz for the quark vacuum wave functional occurs in *any* dimension.

The structure of the paper is as follows: In Section 2 we review the Hamiltonian approach to QCD with the modifications for  $d = 2$ ; in Section 3 we present our Ansatz for the QCD vacuum wave functional and show that in the bare-vertex approximation, where the full quark-gluon vertex is replaced by the bare one, the quark propagator satisfies the same Dyson–Schwinger equation (DSE) known from  $d = 3$  in *any* number of dimensions; in Section 4 we present the evaluation of the energy density and derive the gap equations for the variational kernels occurring in the Ansatz for the vacuum wave functional. The numerical results are presented in Section 5 and our conclusions are given in Section 6.

## 2 QCD in two space dimensions

The Hamilton operator of QCD in Coulomb gauge  $\nabla \cdot \mathbf{A} = 0$  reads [8]

$$H_{\text{QCD}} = \frac{1}{2} \int d^2x J_A^{-1} \Pi_i^a(\mathbf{x}) J_A \Pi_i^a(\mathbf{x}) + \frac{1}{2} \int d^2x B_i^a(\mathbf{x}) B_i^a(\mathbf{x}) + \int d^2x \psi^\dagger(\mathbf{x}) [-i\boldsymbol{\alpha} \cdot \nabla - g\boldsymbol{\alpha} \cdot \mathbf{A}(\mathbf{x}) + \beta m] \psi(\mathbf{x}) + H_C, \quad (1)$$

where  $\mathbf{A} = \mathbf{A}^a t^a$  are the (transverse) spatial gauge fields with  $t^a$  being the hermitian generators of the  $\mathfrak{su}(N_c)$  algebra,  $\Pi_i^a = -i\delta/\delta A_i^a$  is the canonical momentum,  $B_i^a$  is the chromomagnetic field, and  $J_A = \text{Det } G_A^{-1}$  is the Faddeev–Popov determinant. Furthermore,  $\psi$  and  $\psi^\dagger$  are the quark field operators,  $\alpha_i$  and  $\beta$  are the Dirac matrices (which in  $d = 2$  coincide with Pauli matrices), and  $m$  is the bare current quark mass. The last term in equation (1) is the so-called Coulomb term

$$H_C = \frac{g^2}{2} \int d^2x d^2y J_A^{-1} \rho^a(\mathbf{x}) J_A F_A^{ab}(\mathbf{x}, \mathbf{y}) \rho^b(\mathbf{y}),$$

<sup>1</sup>Although with an even number of fermion fields one can mimic chiral symmetry [7], and the corresponding “chiral symmetry breaking” is in fact a flavour symmetry breaking.

which describes the interaction of the colour charge density

$$\rho^a(\mathbf{x}) = \psi^\dagger(\mathbf{x}) t^a \psi(\mathbf{x}) + f^{abc} A_i^b(\mathbf{x}) \Pi_i^c(\mathbf{x})$$

through the Coulomb kernel

$$F_A^{ab}(\mathbf{x}, \mathbf{y}) = \int d^2z G_A^{ac}(\mathbf{x}, \mathbf{z}) (-\nabla_z^2) G_A^{cb}(\mathbf{z}, \mathbf{y}), \tag{2}$$

where

$$G_A^{-1}(\mathbf{x}, \mathbf{y}) = (-\delta^{ab} \nabla_x^2 - g f^{acb} A_i^c(\mathbf{x}) \partial_i^x) \delta(\mathbf{x} - \mathbf{y})$$

is the Faddeev–Popov operator of Coulomb gauge with  $f^{acb}$  being the structure constants of the  $\mathfrak{su}(N_c)$  algebra.

For the Dirac matrices we choose the “Dirac” representation where  $\beta$  is diagonal

$$\alpha_{i=1,2} = \sigma_{i=1,2}, \quad \beta = \sigma_3.$$

They satisfy the usual Dirac algebra

$$\{\alpha_1, \alpha_j\} = \delta_{ij}, \quad \{\alpha_i, \beta\} = 0, \quad \beta^2 = 1.$$

In two dimensions we have

$$[\alpha_i, \alpha_j] = 2i\varepsilon_{ij}\beta,$$

which leads to

$$\alpha_i \alpha_j = \delta_{ij} + i\varepsilon_{ij}\beta, \quad \beta \alpha_i = i\varepsilon_{ij}\alpha_j.$$

For comparison, in  $d = 3$  we have

$$\alpha_i \alpha_j = \delta_{ij} + i\varepsilon_{ijk} \gamma_5 \alpha_k \quad (d = 3).$$

The crucial difference to  $d = 3$  spatial dimensions is that in  $d = 2$  there is no  $\gamma_5$  and accordingly no chiral symmetry.

### 3 Vacuum wave functional and quark propagator

In the variational approach developed in references [1,6,9–11] one attempts to solve the functional Schrödinger equation

$$H_{\text{QCD}}|\Psi\rangle = E|\Psi\rangle$$

for the QCD vacuum state  $|\Psi\rangle$  by means of the variational principle with suitable trial Ansätze for the vacuum wave functional  $\Psi[A] = \langle A|\Psi\rangle$ . Inspired by the form of the QCD Hamiltonian the vacuum state is assumed of the form

$$|\Psi\rangle = |\Psi_{\text{YM}}\rangle |\Psi_{\text{Q}}\rangle, \tag{3}$$

where  $|\Psi_{\text{YM}}\rangle$  is the vacuum state of the Yang–Mills sector and  $|\Psi_{\text{Q}}\rangle$  is the vacuum state of the quark sector, which includes also the coupling to the gluons. Furthermore it turns out that it is most convenient to use the coordinate representation  $\Psi[A, \xi] = \langle A, \xi | \Psi \rangle$  where  $A$  are classical gauge fields and  $\xi$  are Grassmann variables, the “classical coordinates” of the fermions. In accordance with equation (3) we write the vacuum wave functional of QCD in the form

$$\Psi[A, \xi_+^\dagger, \xi_-] \propto \exp\left\{-\frac{1}{2}S_A[A] - S_f[\xi_+^\dagger, \xi_-, A]\right\}, \tag{4}$$

where  $S_A$  and  $S_f$  define respectively the wave functional of pure Yang–Mills theory and of the quarks interacting with the gluons. For  $S_A$  we could take a Gaussian Ansatz [1] or a more general form involving cubic and quartic couplings [10]. However, in the present paper we do not solve the gluon gap equation but use instead for the gluon propagator a form which is inspired by the IR an UV analysis of the variational equations (which besides the gluon gap equations consist also of Dyson–Schwinger-type of equations, see Ref. [10]) and which fits the lattice data, see equation (19) below. For  $S_f$  we make the Ansatz used already in  $d = 3$  [3]

$$\begin{aligned} S_f[\xi_+^\dagger, \xi_-, A] &= \int \xi_+^\dagger [\beta s + g(V + \beta W)\boldsymbol{\alpha} \cdot \mathbf{A}] \xi_- \\ &= \int \xi^\dagger \Lambda_+ [\beta s + g(V + \beta W)\boldsymbol{\alpha} \cdot \mathbf{A}] \Lambda_- \xi, \end{aligned} \tag{5}$$

where  $s$ ,  $V$ , and  $W$  are variational kernels which will be determined by the minimization of the energy density. Due to the coupling of the quarks to the gluons contained in  $S_f$  the wave functional equation (5) is necessarily non-Gaussian. The vacuum expectation value of an operator  $O$  is given by the functional integral [6,11]

$$\begin{aligned} \langle \Psi | O[A, \Pi, \psi, \psi^\dagger] | \Psi \rangle &= \int \mathcal{D}\xi \mathcal{D}\xi^\dagger \mathcal{D}A J_A e^{-\mu} \Psi^*[\xi_+^\dagger, \xi_-, A] \\ &\quad \times O\left[A, -i\frac{\delta}{\delta A}, \xi_- + \frac{\delta}{\delta \xi_+^\dagger}, \xi_+^\dagger + \frac{\delta}{\delta \xi_-}\right] \Psi[\xi_+^\dagger, \xi_-, A], \end{aligned} \tag{6}$$

where

$$\mu = \Lambda_+ - \Lambda_-$$

is the integration measure of the coherent fermion states, and

$$\xi_\pm(\mathbf{x}) = \int d^2y \Lambda_\pm(\mathbf{x}, \mathbf{y}) \xi(\mathbf{y})$$

are spinor-valued Grassmann fields, with

$$\Lambda_\pm(\mathbf{x}, \mathbf{y}) = \int \frac{d^2\mathbf{p}}{(2\pi)^2} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \left(\frac{1}{2} \pm \frac{\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m}{2\sqrt{\mathbf{p}^2 + m^2}}\right) \tag{7}$$

being the projectors onto the positive/negative eigenstates of the free Dirac Hamilton operator  $\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m$ .

When the form equation (4) of the wave functional is inserted into equation (6) and the functional derivatives are worked out, the expectation value of an operator

reduces to a quantum average of field functionals reminiscent of a Euclidean field theory with action

$$S = S_A + S_f + S_f^* + \mu.$$

This formal equivalence can be exploited to derive Dyson–Schwinger-like equations [10,11] to express the various Green’s functions in terms of the variational kernels contained in the non-Gaussian “action”  $S$ .

The essential quantity of the quark sector is the two-point correlation function of the Grassmann fields  $\xi$

$$Q(\mathbf{x}, \mathbf{y}) = \langle \xi(\mathbf{x}) \xi^\dagger(\mathbf{y}) \rangle,$$

which can be parametrized in momentum space as

$$Q^{-1}(\mathbf{p}) = A(\mathbf{p}) \boldsymbol{\alpha} \cdot \mathbf{p} + B(\mathbf{p}) \beta.$$

With our Ansatz [Eqs. (4) and (5)] for the vacuum wave functional the dressing functions  $A$  and  $B$  obey the Dyson–Schwinger-like equations [6]

$$A(\mathbf{p}) = 1 - \frac{C_F}{2} \int \frac{d^2q}{(2\pi)^2} \text{tr}[\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} \bar{\Gamma}_{0,i}(\mathbf{p}, -\mathbf{q}) Q(\mathbf{q}) D_{ij}(\mathbf{p} - \mathbf{q}) \bar{\Gamma}_j(\mathbf{q}, -\mathbf{p})], \quad (8a)$$

$$B(\mathbf{p}) = s(\mathbf{p}) - \frac{C_F}{2} \int \frac{d^2q}{(2\pi)^2} \text{tr}[\beta \bar{\Gamma}_{0,i}(\mathbf{p}, -\mathbf{q}) Q(\mathbf{q}) D_{ij}(\mathbf{p} - \mathbf{q}) \bar{\Gamma}_j(\mathbf{q}, -\mathbf{p})], \quad (8b)$$

where  $C_F = (N_c^2 - 1)/(2N_c)$  is the quadratic Casimir in the fundamental representation, and

$$\delta^{ab} D_{ij}(\mathbf{x}, \mathbf{y}) = \langle A_i^a(\mathbf{x}) A_j^b(\mathbf{y}) \rangle, \quad D_{ij}(\mathbf{p}) = \frac{t_{ij}(\mathbf{p})}{2\Omega(\mathbf{p})}, \quad t_{ij}(\mathbf{p}) = \delta_{ij} - \frac{p_i p_j}{\mathbf{p}^2}, \quad (9)$$

is the gluon propagator. Furthermore,  $\bar{\Gamma}$  is the full quark-gluon vertex defined by [11]

$$\langle \xi \xi^\dagger A_i \rangle = -Q \bar{\Gamma}_j Q D_{ji}, \quad (10)$$

while  $\bar{\Gamma}_0$  is the bare quark-gluon vertex defined by our Ansatz equation (5) for the vacuum wave functional

$$\bar{\Gamma}_{0,i}(\mathbf{p}, \mathbf{q}) = \Lambda_+(\mathbf{p}) K_i \Lambda_-(-\mathbf{q}) + \Lambda_-(\mathbf{p}) K_i^\dagger \Lambda_+(-\mathbf{q}), \quad (11)$$

where

$$K_i = g[V(\mathbf{p}, \mathbf{q}) + \beta W(\mathbf{p}, \mathbf{q})] \alpha_i.$$

In the bare-vertex approximation, where the full quark-gluon vertex equation (10) is replaced by the bare one equation (11), equation (8) become in the chiral limit  $m = 0$

$$A_p = 1 - \frac{g^2 C_F}{2} \int \frac{d^2q}{(2\pi)^2} \frac{A_q}{A_q^2 + B_q^2} \frac{X_-(\mathbf{p}, \mathbf{q}) V^2(\mathbf{p}, \mathbf{q}) + X_+(\mathbf{p}, \mathbf{q}) W^2(\mathbf{p}, \mathbf{q})}{\Omega(\mathbf{p} + \mathbf{q})}, \quad (12a)$$

$$B_p = s_p - \frac{g^2 C_F}{2} \int \frac{d^2q}{(2\pi)^2} \frac{B_q}{A_q^2 + B_q^2} \frac{X_-(\mathbf{p}, \mathbf{q}) V^2(\mathbf{p}, \mathbf{q}) - X_+(\mathbf{p}, \mathbf{q}) W^2(\mathbf{p}, \mathbf{q})}{\Omega(\mathbf{p} + \mathbf{q})}, \quad (12b)$$

where we have defined the momentum overlap functions

$$X_{\pm}(\mathbf{p}, \mathbf{q}) = \frac{1 \mp \hat{\mathbf{p}} \cdot \hat{\mathbf{q}}}{2} \pm \frac{[\hat{\mathbf{p}} \cdot (\mathbf{p} + \mathbf{q})][\hat{\mathbf{q}} \cdot (\mathbf{p} + \mathbf{q})]}{(\mathbf{p} + \mathbf{q})^2}. \quad (13)$$

In order to simplify the notation, we have denoted in equation (12) the momentum dependence of the dressing functions by a subscript. As we will show now, these equations hold formally in any dimension. Only the overlap functions  $X_{\pm}$  depend on the number  $d$  of spatial dimensions. Although the Dirac matrices depend on the number of spatial dimensions, when the bare vertices are contracted with the (symmetric) transverse projectors only anti-commutators, which are independent of the number of dimensions, enter the final result. To see this, we write the bare quark-gluon vertex equation (11) in the chiral limit as

$$\bar{\Gamma}_{0,i}(\mathbf{p}, \mathbf{q}) = \frac{1}{4} V(\mathbf{p}, \mathbf{q}) [M_i(\mathbf{p}, \mathbf{q}) + M_i(-\mathbf{p}, -\mathbf{q})] - \frac{1}{4} W(\mathbf{p}, \mathbf{q}) [M_i(\mathbf{p}, -\mathbf{q}) - M_i(-\mathbf{p}, \mathbf{q})] \beta,$$

where

$$M_i(\mathbf{p}, \mathbf{q}) = (1 + \boldsymbol{\alpha} \cdot \hat{\mathbf{p}}) \alpha_i (1 + \boldsymbol{\alpha} \cdot \hat{\mathbf{q}}).$$

This quantity has the properties

$$\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} M_i(\mathbf{p}, \mathbf{q}) = M_i(\mathbf{p}, \mathbf{q}) = M_i(\mathbf{p}, \mathbf{q}) \boldsymbol{\alpha} \cdot \hat{\mathbf{q}}, \quad \beta M_i(\mathbf{p}, \mathbf{q}) = -M_i(-\mathbf{p}, -\mathbf{q}) \beta.$$

Using these relations in the DSEs (8) leads then to terms of the form

$$M_i(\pm \mathbf{p}, \mathbf{q}) M_j(\mathbf{q}, \pm \mathbf{p}).$$

For general indices  $i, j$  this will be in general a complicated expression whose details depend on the number of dimensions. However, in DSEs (8) these expressions are always contracted with a transverse projector  $t_{ij}$  [stemming from the gluon propagator Eq. (9)], resulting in

$$t_{ij}(\mathbf{p} + \mathbf{q}) M_i(\pm \mathbf{p}, \mathbf{q}) M_j(\mathbf{q}, \pm \mathbf{p}) = 8(1 \pm \boldsymbol{\alpha} \cdot \hat{\mathbf{p}}) X_{\mp}^{(d)}(\mathbf{p}, \mathbf{q})$$

with

$$X_{\pm}^{(d)}(\mathbf{p}, \mathbf{q}) = \frac{d-1 \mp (3-d)\hat{\mathbf{p}} \cdot \hat{\mathbf{q}}}{2} \pm \frac{[\hat{\mathbf{p}} \cdot (\mathbf{p} + \mathbf{q})][\hat{\mathbf{q}} \cdot (\mathbf{p} + \mathbf{q})]}{(\mathbf{p} + \mathbf{q})^2}.$$

For  $d = 2$  this expression reproduces the previous result equation (13). Note also that in  $d = 2$  the equations (12) for the dressing functions are finite. (QCD in  $d = 2$  spatial dimensions is super renormalizable).

## 4 Energy density and variational equations

The calculation of the vacuum expectation values of the Hamiltonian  $\langle H \rangle$  proceeds completely analogously to the  $d = 3$  case performed in references [6,11]. For the

energy density  $e = \langle H \rangle / (N_c V)$  one finds in  $d = 2$  the following contributions: The single-particle Hamiltonian [first term in the second line of Eq. (1)] yields

$$e_D = \int \frac{d^2 q}{(2\pi)^2} \text{tr}[(\boldsymbol{\alpha} \cdot \mathbf{q} + \beta m) Q(\mathbf{q})] - g C_F \int \frac{d^2 q}{(2\pi)^2} \frac{d^2 \ell}{(2\pi)^2} D_{ij}(\mathbf{q} + \boldsymbol{\ell}) \text{tr}[\alpha_i Q(\mathbf{q}) \bar{\Gamma}_j(\mathbf{q}, \boldsymbol{\ell}) Q(-\boldsymbol{\ell})], \quad (14)$$

while the kinetic energy of the gluon [the term  $J_A^{-1} \Pi_i^a J_A \Pi_i^a$  in Eq. (1)] gives

$$e_E^q = -\frac{C_F}{8} \int \frac{d^2 q}{(2\pi)^2} \frac{d^2 \ell}{(2\pi)^2} t_{ij}(\mathbf{q} + \boldsymbol{\ell}) \text{tr}\{\bar{\Gamma}_{0,i}(\mathbf{q}, -\boldsymbol{\ell}) Q(\boldsymbol{\ell}) \bar{\Gamma}_j(\boldsymbol{\ell}, -\mathbf{q}) Q(\mathbf{q}) - Q_0(\mathbf{q}) \bar{\Gamma}_{0,i}(\mathbf{q}, -\boldsymbol{\ell}) Q(\boldsymbol{\ell}) Q_0(\boldsymbol{\ell}) \bar{\Gamma}_{0,j}(\boldsymbol{\ell}, -\mathbf{q}) Q(\mathbf{q})\}. \quad (15)$$

From the Coulomb term we find

$$e_C^{qq} \simeq -g^2 \frac{C_F}{2} \int \frac{d^2 q}{(2\pi)^2} \frac{d^2 \ell}{(2\pi)^2} F(\mathbf{q} - \boldsymbol{\ell}) \text{tr}\{[Q(\boldsymbol{\ell}) - \frac{1}{2} Q_0(\boldsymbol{\ell})][Q(\mathbf{q}) - \frac{1}{2} Q_0(\mathbf{q})] - \frac{1}{4}\}, \quad (16)$$

where  $F$  is the expectation value of the Coulomb kernel  $F_A$  [Eq. (2)]. Formally, these are exactly the same expressions as in  $d = 3$  (except for the momentum integration measure). However, the differences arise now when taking the traces of the Dirac matrices. In  $d = 2$  the trace of the unit matrix yields a factor 2 instead of 4, and the trace of one  $\beta$  and an even number of  $\alpha_i$  does not vanish like in  $d = 3$ . In particular, we have

$$\text{tr}[\beta \alpha_i \alpha_j] = 2i \varepsilon_{ij}.$$

Although this last expression could in principle make a difference in the calculation, it turns out that *in the bare vertex-approximation*, where we replace the full quark-gluon vertices  $\bar{\Gamma}$  [Eq. (10)] by the bare one  $\bar{\Gamma}_0$  [Eq. (11)], this does not matter. The reason is that the matrix  $\beta$  in the fermion propagator  $Q$  always occurs between two projectors equation (7) and leads to expressions of the form

$$(1 + \boldsymbol{\alpha} \cdot \hat{\mathbf{p}}) \beta (1 + \boldsymbol{\alpha} \cdot \hat{\mathbf{p}}) = \beta (1 - \boldsymbol{\alpha} \cdot \hat{\mathbf{p}}) (1 + \boldsymbol{\alpha} \cdot \hat{\mathbf{p}}) = 0.$$

Therefore, in the bare-vertex approximation we recover for the energy densities equations (14)–(16) the very same expressions found in reference [6] apart from the dimension of the momentum integrals and an overall factor 1/2. The explicit expressions read in  $d = 2$

$$e_D = -2 \int \frac{d^2 q}{(2\pi)^2} \frac{|\mathbf{q}| A_q}{\Delta_q} + g^2 C_F \int \frac{d^2 q}{(2\pi)^2} \frac{d^2 \ell}{(2\pi)^2} \times \frac{X_-(\mathbf{q}, \boldsymbol{\ell}) V(\mathbf{q}, \boldsymbol{\ell}) (A_q A_\ell + B_q B_\ell) + X_+(\mathbf{q}, \boldsymbol{\ell}) W(\mathbf{q}, \boldsymbol{\ell}) (A_q B_\ell + B_q A_\ell)}{\Delta_q \Delta_\ell \Omega(\mathbf{q} + \boldsymbol{\ell})}, \quad (17a)$$

$$e_E^q = \frac{g^2 C_F}{2} \int \frac{d^2 q}{(2\pi)^2} \frac{d^2 \ell}{(2\pi)^2} \frac{A_q A_\ell}{\Delta_q \Delta_\ell} [X_-(\mathbf{q}, \boldsymbol{\ell}) V^2(\mathbf{q}, \boldsymbol{\ell}) + X_+(\mathbf{q}, \boldsymbol{\ell}) W^2(\mathbf{q}, \boldsymbol{\ell})], \quad (17b)$$

$$e_C^{qq} = -g^2 \frac{C_F}{4} \int \frac{d^2q}{(2\pi)^2} \frac{d^2\ell}{(2\pi)^2} F(\mathbf{q} - \ell) \times \frac{4B_q B_\ell + \hat{\mathbf{q}} \cdot \hat{\ell} [A_q(2 - A_q) - B_q^2] [A_\ell(2 - A_\ell) - B_\ell^2]}{\Delta_q \Delta_\ell}, \tag{17c}$$

where we have introduced the abbreviation

$$\Delta_p = A_p^2 + B_p^2.$$

Since the energy density equation (17) differs from its three-dimensional counterpart only by an overall factor and the DSEs (12) have the same form in  $d = 2$  and  $d = 3$  (except for the explicit expression for  $X_\pm$ ), it is clear that the variational equations differ from their  $d = 3$  counterparts only in the number of dimensions of the momentum integrals, while all numeric factors are exactly the same. Minimization of the vacuum energy density equation (17) with respect to the vector kernels  $V$  and  $W$  yields

$$V(\mathbf{p}, \mathbf{q}) = -\frac{1 + s_p s_q}{\Omega(\mathbf{p} + \mathbf{q}) + |\mathbf{p}| \frac{1 - s_p^2 + 2s_p s_q}{1 + s_p^2} + |\mathbf{q}| \frac{1 - s_q^2 + 2s_p s_q}{1 + s_q^2}},$$

$$W(\mathbf{p}, \mathbf{q}) = -\frac{s_p - s_q}{\Omega(\mathbf{p} + \mathbf{q}) + |\mathbf{p}| \frac{1 - s_p^2 - 2s_p s_q}{1 + s_p^2} + |\mathbf{q}| \frac{1 - s_q^2 - 2s_p s_q}{1 + s_q^2}}.$$

We recall here that the vector kernel  $W$  vanishes when  $s_p = 0$  and is therefore of purely non-perturbative nature, since the scalar kernel  $s_p$  vanishes at any order in perturbation theory for a vanishing current quark mass.

The variation of the energy density equation (17) with respect to the scalar kernel  $s_p$  yields the gap equation

$$\begin{aligned} |\mathbf{p}| s_p &= \frac{g^2 C_F}{2} \int \frac{d^2q}{(2\pi)^2} \frac{F(\mathbf{p} - \mathbf{q})}{1 + s_q^2} [s_q(1 - s_p^2) - \hat{\mathbf{p}} \cdot \hat{\mathbf{q}} s_p(1 - s_q^2)] \\ &+ \frac{g^2 C_F}{2} \int \frac{d^2q}{(2\pi)^2} \frac{s_p}{1 + s_q^2} [X_-(\mathbf{p}, \mathbf{q}) V^2(\mathbf{p}, \mathbf{q}) + X_+(\mathbf{p}, \mathbf{q}) W^2(\mathbf{p}, \mathbf{q})] \\ &- \frac{g^2 C_F}{2} \int \frac{d^2q}{(2\pi)^2} \frac{1}{(1 + s_q^2)\Omega(\mathbf{p} + \mathbf{q})} \\ &\times \left\{ X_-(\mathbf{p}, \mathbf{q}) V(\mathbf{p}, \mathbf{q}) [(1 - s_p^2)s_q - 2s_p] + X_+(\mathbf{p}, \mathbf{q}) W(\mathbf{p}, \mathbf{q}) [1 - s_p^2 - 2s_p s_q] \right. \\ &+ \frac{|\mathbf{p}|}{1 + s_p^2} \left[ X_-(\mathbf{p}, \mathbf{q}) V^2(\mathbf{p}, \mathbf{q}) [s_p(s_p^2 - 3) + s_q(1 - 3s_p^2)] \right. \\ &+ \left. X_+(\mathbf{p}, \mathbf{q}) W^2(\mathbf{p}, \mathbf{q}) [s_p(s_p^2 - 3) - s_q(1 - 3s_p^2)] \right] \\ &+ \frac{|\mathbf{q}|}{1 + s_q^2} \left[ X_-(\mathbf{p}, \mathbf{q}) V^2(\mathbf{p}, \mathbf{q}) [(1 - s_p^2)s_q - s_p(1 - s_q^2)] \right. \\ &\left. \left. - X_+(\mathbf{p}, \mathbf{q}) W^2(\mathbf{p}, \mathbf{q}) [(1 - s_p^2)s_q + s_p(1 - s_q^2)] \right] \right\}. \end{aligned} \tag{18}$$

We stress again that, like the DSEs (12), also this gap equation has the same form as its  $d = 3$  counterpart, however, with  $X_\pm$  now given by equation (13).

**Table 1.** Comparison of the  $d = 3$  and  $d = 2$  UV divergences of the gap equation (18) stemming from the Coulomb term, the kernel  $V$ , and the kernel  $W$ .

	$d = 3$	$d = 2$
Coulomb term	$-\frac{g^2 C_F}{(4\pi)^2} s_p  \mathbf{p}  \frac{8}{3} \ln(\Lambda)$	finite
Terms involving $V$	$\frac{g^2 C_F}{(4\pi)^2} s_p \left[ -2\Lambda +  \mathbf{p}  \ln(\Lambda) \left( -\frac{2}{3} + \frac{4}{1 + s_p^2} \right) \right]$	$-\frac{g^2 C_F}{(4\pi)^2} s_p \ln(\Lambda)$
Terms involving $W$	$\frac{g^2 C_F}{(4\pi)^2} s_p \left[ 2\Lambda +  \mathbf{p}  \ln(\Lambda) \left( \frac{10}{3} - \frac{4}{1 + s_p^2} \right) \right]$	$\frac{g^2 C_F}{(4\pi)^2} s_p \ln(\Lambda)$

In  $d = 3$  we had found [5] that the addition of the vector kernel  $W$  makes the gap equation UV finite; there the Coulomb integral [first integral on the right-hand side of Eq. (18)] is logarithmically divergent, and the integrals involving  $V$  and  $W$  are (separately) both linearly and logarithmically divergent. The linear divergence stemming from  $W$  cancels the one stemming from  $V$ , and the three logarithmic divergences cancel altogether. In  $d = 2$  all integrals have one superficial degree of divergence lower than in  $d = 3$ , and quite remarkably the same cancellation of divergences happens also in this case, although the tensorial structures  $X_{\pm}$  [Eq. (13)] look quite differently. Here the Coulomb term is UV finite, and the integrals involving  $V$  and  $W$  are separately logarithmically divergent but in the gap equation (18) these logarithmic divergences cancel. As in  $d = 3$  we find also in this case that the addition of the vector kernel  $W$  makes the gap equation finite; a summary of the UV divergent contributions is given in Table 1. In fact, we have checked that the leading-order divergence of the gap equation (18) cancels in any number of dimensions once both  $V$  and  $W$  are considered.

In  $d = 3$  the vector kernel  $W$  was crucial also to ensure multiplicative renormalizability of the quark propagator [6,12]; this is not the case here, since the DSEs (12) are UV finite.

## 5 Numerical results

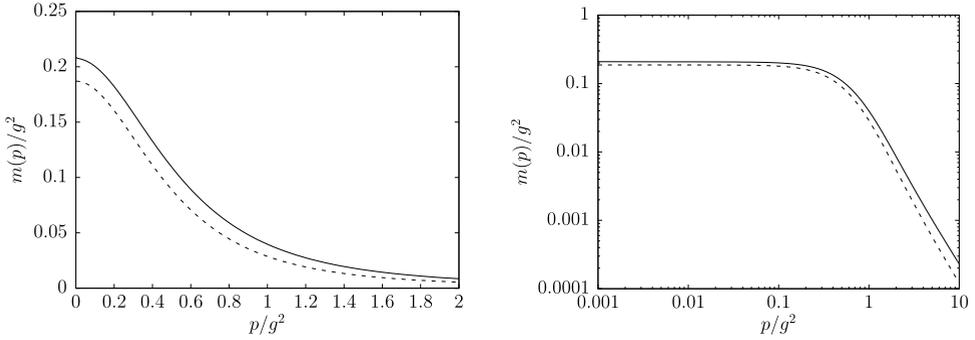
In  $d = 2$  spatial dimensions the squared coupling constant  $g^2$  has the dimension of energy, and we express all dimensionful quantities in terms of  $g^2$ . The colour Coulomb potential can be assumed in the form

$$g^2 F(\mathbf{p}) = \frac{g^2}{\mathbf{p}^2} + \frac{2\pi\sigma_C}{|\mathbf{p}|^3},$$

which consists of the perturbative part ( $\propto 1/\mathbf{p}^2$ ) and the linearly rising, confining part. For the gluon propagator equation (9) we use the Gribov formula [13]

$$\Omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + \frac{m_A^4}{\mathbf{p}^2}}, \quad (19)$$

which excellently fits the lattice data in  $d = 3$  [14]. The infrared analysis of the ghost propagator DSE reveals a relation between the Gribov mass  $m_A$  and the Coulomb



**Fig. 1.** Results (left: linear plot, right: logarithmic plot) for the pseudo-mass function  $m_p$  in units of  $g^2$  with the colour Coulomb potential alone (dashed line) and with the coupling to the transverse gluons included (continuous line).

string tension  $\sigma_C$ . When the angular approximation is used one finds [15]

$$m_A^2 = \frac{5N_c}{12} \sigma_C,$$

while abandoning the angular approximation one obtains [16]

$$m_A^2 = 4N_c \left( \frac{\Gamma(3/4)}{\Gamma(1/4)} \right)^2 \sigma_C.$$

The two values are numerically very close to each other. The Coulomb string tension  $\sigma_C$  is an upper bound for the Wilson string tension  $\sigma$  [17], and in three spatial dimension we have  $\sigma_C \simeq 4\sigma$  [18]. We have no reliable data for the ratio  $\sigma_C/\sigma$  in  $d = 2$ . Since we are interested mostly in a qualitative analysis we choose  $\sigma_C \approx \sigma$ . For the Wilson string tension we take the value [19,20]

$$\sigma = g^4 \frac{N_c^2 - 1}{8\pi}.$$

For numerical stability it is convenient to reformulate the gap equation (18) in terms of the pseudo-mass function

$$m(p) = \frac{2ps_p}{1 - s_p^2}.$$

The resulting gap equation can be found in references [3,5]. The results of the numerical solution of this equation are shown in Figure 1. Like in the three-dimensional case, the main contribution to the dynamical mass generation comes from the colour Coulomb potential [first line in Eq. (18)]. The inclusion of the coupling to the transverse gluons only slightly increases the mass function.

## 6 Conclusions

In this paper we have investigated the dynamical generation of mass in QCD in  $d = 2$  spatial dimensions within the Hamiltonian approach in Coulomb gauge. Somewhat

surprisingly, despite the fundamental differences in the representation of the Lorentz group most results obtained in  $d = 3$  hold also in  $d = 2$ . In particular, the inclusion of the non-perturbative vector kernel  $W$  in the bare quark-gluon vertex  $\bar{\Gamma}_0$  [Eq. (11)] (in addition to the leading kernel  $V$ , which exists also in perturbation theory) makes the gap equation UV finite as in  $d = 3$ . Furthermore, also like in  $d = 3$ , the coupling of the quarks to the spatial gluons only slightly increases the dynamical mass generation. Like in  $d = 3$  this effect is absolutely dominated by the colour Coulomb potential equation (2), which results through the elimination of the temporal gluons  $A_0$  in the Hamiltonian approach and, in fact, represents the instantaneous part of the propagator  $\langle A_0 A_0 \rangle$ .

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