

# Quartet condensation in Fermi-systems. The Hoyle state in the $^{12}\text{C}$ nucleus and life on earth

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**Abstract.** The possibility that a four component Fermi gas with attractive interaction can form a quartet condensate at low density is pointed out. It is discussed that for quartets only the Bose-Einstein Condensatio (BEC) phase exists and that the analogue to the weak coupling long coherence BCS phase of pairing is absent. Precursor phenomena in finite nuclei are presented. For instance, the present understanding of the structure of the Hoyle state in  $^{12}\text{C}$  being the gateway for Carbon production in the universe is reviewed. It is pointed out that a crucial test of any theory is the good reproduction of the experimental results for the inelastic form factor from ground to the Hoyle state. The performances of the so-called THSR wavve function are outlined confirming the  $\alpha$  particle condensation hypothesis proposed 15 years back in [1].

## 1 Introduction

One of the most amazing phenomena in quantum many-particle systems is the formation of quantum condensates. At present, the formation of condensates is of particular interest in strongly coupled fermion systems in which the crossover from Bardeen-Cooper-Schrieffer (BCS) pairing to Bose-Einstein condensation (BEC) may be investigated. Among very different quantum systems, nuclear matter is especially well suited for the study of correlation effects in a quantum liquid. In [2], the possibility of  $\alpha$  particle (quartet) condensation in infinite matter was investigated. It was found that quartetting is possible at low densities, below about a fifth of saturation density. At higher densities, around the point where the chemical potential  $\mu$  turns from negative (binding) to positive, the condensation breaks down. This is contrary to ordinary pairing which can exist for considerably positive  $\mu$  values, depending only on the range of the pairing force. The reason for this strong qualitative difference between the two cases is explained in [3].

The question then arises whether in analogy to pairing, also for quartetting exist nuclei where this phenomenon is born out. In [1], we found that such a possibility very likely exists in lighter self-conjugate nuclei for excitation energies around the  $\alpha$

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disintegration threshold. In this contribution, we want to discuss the successes and eventual failures of this idea which was proposed 15 years back. We will specially deal with the Hoyle state which is the first  $0^+$  state at 7.65 MeV in  $^{12}\text{C}$ , since this state is responsible for the massive Carbon production in the universe and, thus, for the existence of life on earth. It was predicted on those grounds by the astrophysicist Fred Hoyle in 1952 [4] and discovered by Fowler a couple of years later [5]. The Hoyle state couples resonantly to the so-called triple  $\alpha$  reaction present in stars and, thus, accelerates very much the  $^{12}\text{C}$  synthesis.

The paper is organised as follows. In Section 2, we describe quartet condensation in infinite nuclear matter. In Section 3, we discuss precursor phenomena of quartet condensation in finite nuclei. For instance the famous Hoyle state at 7.65 MeV in  $^{12}\text{C}$  will be described. In Section 4, we outline shortly the situation in  $^{16}\text{O}$  and in Section 5, we present our conclusions and give further discussions.

## 2 Quartet BEC with applications to nuclear systems: alpha condensation in infinite nuclear matter

The possibility of quartet, i.e.,  $\alpha$  particle condensation in nuclear systems has only come to the forefront in recent years. First, this may be due to the fact that quartet condensation, i.e., the one of four tightly correlated fermions, is a technically by far more difficult problem than is pairing. Second, as we will see, the BEC-BCS transition for quartets is very different from the pair case. As a matter of fact the weak coupling BCS like, long coherence length regime does not exist for quartets. Rather, at higher densities the quartets dissolve and go over into two Cooper pairs.

Quartets are of course present in nuclear systems. In other fields of physics they are much rarer. One knows that two excitons in semiconductors can form a bound state and the question has been asked in the past whether bi-excitons can condense [6]. In future cold atom devices, one may trap four different species of fermions which, with the help of Feshbach resonances, could form quartets (please note that four different fermions are quite necessary to form quartets for Pauli principle and, thus, energetic reasons). Theoretical models have already been treated and a quartet phase predicted in [7].

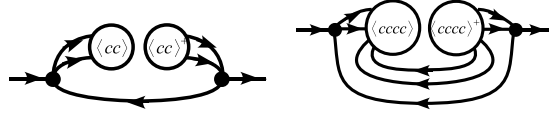
Let us start the theoretical description. For this it is convenient to recapitulate what is done in standard S-wave pairing. On the one hand, we have the equation for the order parameter  $\kappa(\mathbf{p}_1, \mathbf{p}_2) = \langle c_{\mathbf{p}_1} c_{\mathbf{p}_2} \rangle$  (we suppress the spin dependence;  $c^+$ ,  $c$  are fermion creation and annihilation operators)

$$\kappa(\mathbf{p}_1, \mathbf{p}_2) = \frac{1 - n(\mathbf{p}_1) - n(\mathbf{p}_2)}{e_{\mathbf{p}_1} + e_{\mathbf{p}_2} - 2\mu} \sum_{\mathbf{p}'_1, \mathbf{p}'_2} \langle \mathbf{p}_1 \mathbf{p}_2 | v | \mathbf{p}'_1 \mathbf{p}'_2 \rangle \kappa(\mathbf{p}'_1, \mathbf{p}'_2) \quad (1)$$

with  $e_k$  kinetic energy, eventually with a HF shift, and  $\langle \mathbf{p}_1 \mathbf{p}_2 | v | \mathbf{p}'_1 \mathbf{p}'_2 \rangle = \delta(\mathbf{K} - \mathbf{K}') v(\mathbf{q} - \mathbf{q}')$  the matrix element of the force with  $\mathbf{K}, \mathbf{q}$  c.o.m. and relative momenta, one recognises an atrophiated two particle Bethe-Salpeter equation at  $T = 0$ , taken at the eigenvalue  $E = 2\mu$  where  $\mu$  is the chemical potential. Inserting the standard BCS expression for the occupation numbers

$$n(p) = \frac{1}{2} \left( 1 - \frac{e_p - \mu}{2\sqrt{(e_p - \mu)^2 + \Delta^2}} \right) \quad (2)$$

leads for pairs at rest, that is  $\mathbf{K} = \mathbf{p}_1 + \mathbf{p}_2 = 0$ , to the gap equation (5) below. We want to proceed in an analogous way with the quartets. In obvious short hand notation, the in medium four fermion Bethe-Salpeter equation for the quartet order



**Fig. 1.** Single particle mass operator in case of pairing (left) and quartetting (right).

parameter  $K(1234) = \langle c_1 c_2 c_3 c_4 \rangle$  is given by [3]

$$(e_1 + e_2 + e_3 + e_4 - 4\mu)K(1234) = (1 - n_1 - n_2) \sum 1'2' \langle 12|v|1'2' \rangle K(1'2'34) + \text{permutations}, \tag{3}$$

We see that above equation is a rather straight forward extension of the pairing case to the quartet one. The crux lies in the problem how to find the single particle occupation numbers  $n_k$  in the quartet case. Again, we will proceed in analogy to the pairing case. Eliminating there the anomalous Green's function from the  $2 \times 2$  set of Gorkov equations [8] leads to a mass operator in the Dyson equation for the normal Green's function of the form

$$M_{1,1'} = \frac{|\Delta_1|^2}{\omega + e_1 - 2\mu} \delta_{1,1'} \tag{4}$$

with the gap defined by

$$\Delta_1 = \sum_2 \langle 1\bar{1}|v|2\bar{2} \rangle \langle c_2 c_2 \rangle \tag{5}$$

where  $'\bar{1}'$  is the time reversed state of  $'1'$ . Its graphical representation is given in Figure 1 (left). In the case of quartets, the derivation of a single particle mass operator is more involved and we only want to give the final expression here (for detailed derivation, see [3]):

$$M_{1,1}^{\text{quartet}}(\omega) = \sum_{234} \frac{\Delta_{1234} [\bar{f}_2 \bar{f}_3 \bar{f}_4 + f_2 f_3 f_4] \Delta_{1234}^*}{\omega + e_{234}}, \tag{6}$$

where  $\bar{f} = 1 - f$  and  $f_i = \Theta(\mu - e_i)$  is the Fermi step at zero temperature and the quartet gap matrix is given by

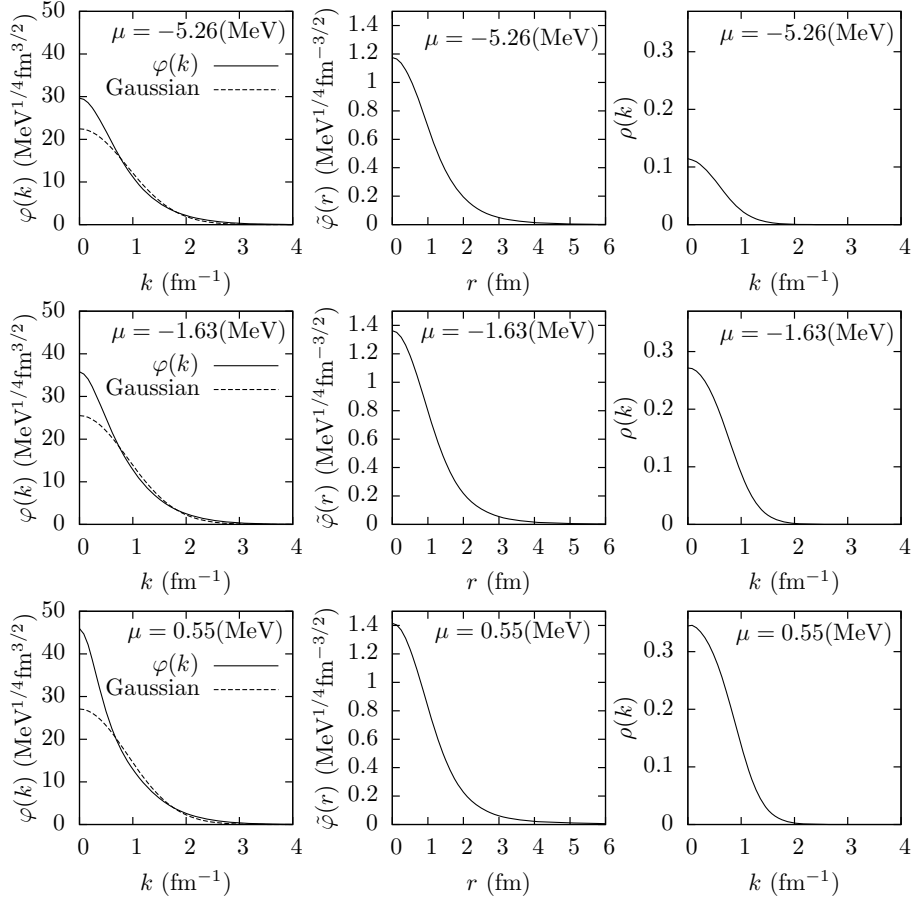
$$\Delta_{1234} = \sum_{1'2'} \langle 12|v|1'2' \rangle \langle c_{1'} c_{2'} c_3 c_4 \rangle. \tag{7}$$

This quartet mass operator is also depicted in Figure 1 (right).

Though, as mentioned, the derivation is slightly intricate, the final result looks plausible. For instance it is seen that the three hole lines seen in Figure 1 give rise to the Fermi occupation factors in the numerator of (1). This makes, as we will see, a strong difference with pairing, since there with only a single fermion line  $\bar{f} + f = 1$  and, thus, no phase space factor appears. Once we have the mass operator, the occupation numbers can be calculated via the standard procedure and the system of equations for the quartet order parameter is closed.

Numerically it is out of question that one solves this complicated nonlinear set of four body equations brute force. Luckily, there exists a very efficient and simplifying approximation. It is known in nuclear physics that, because of its strong binding, a good approximation is to treat the  $\alpha$  particle in mean field as long as it is projected on good total momentum. We, therefore, make the ansatz

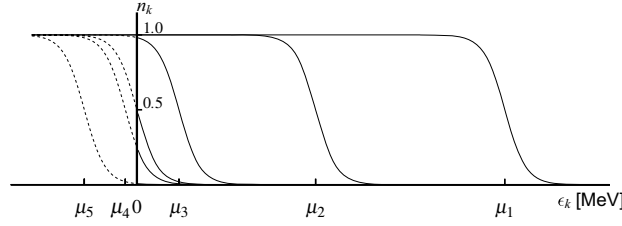
$$\langle c_1 c_2 c_3 c_4 \rangle \rightarrow \varphi(\mathbf{k}_1) \varphi(\mathbf{k}_2) \varphi(\mathbf{k}_3) \varphi(\mathbf{k}_4) \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4), \tag{8}$$



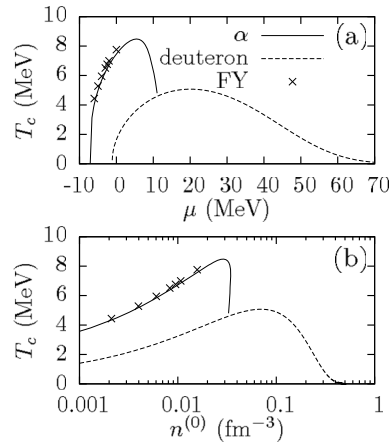
**Fig. 2.** Single particle wave functions  $\varphi$  in momentum and position spaces (two left columns) and s.p. occupation numbers  $\rho(k)$  (right column) [3].

where  $\varphi$  is a  $0S$  single particle wave function in momentum space. With this ansatz which is an eigenstate of the total momentum operator with eigenvalue  $\mathbf{K} = 0$ , the problem is still complicated but reduces to the selfconsistent determination of  $\varphi(\mathbf{k})$  what is a tremendous simplification and renders the problem manageable. Below, we will give an example where the high efficiency of the product ansatz is demonstrated. Of course, with the mean field ansatz we cannot use the bare nucleon-nucleon force. We took a separable one with two parameters (strength and range) which were adjusted to energy and radius of the free  $\alpha$  particle. In Figure 2, we show the evolution with increasing chemical potential  $\mu$  (density) of the single particle wave function in position and momentum space (two left columns). We see that at higher  $\mu$ 's, i.e., densities, the wave function deviates more and more from a Gaussian. At slightly positive  $\mu$  the system seems not to have a solution anymore and selfconsistency cannot be achieved.

Very interesting is the evolution of the occupation numbers  $\rho(k)$  with  $\mu$  (density) also shown in Figure 2 (right column). It is seen that at slightly positive  $\mu$  where the system stops to find a solution, the occupation numbers are still far from saturation. The highest occupation number one obtains lies at around  $n_{k=0} \sim 0.35$ . This is still very far from saturation as it can happen with the BEC-BCS cross-over in the case



**Fig. 3.** Schematic (non-selfconsistent) view of BCS occupation numbers as the chemical potential varies from positive to negative (binding) values.



**Fig. 4.** Critical temperatures for  $\alpha$  particle (binding energy/nucleon  $\sim 7.5$  MeV) and deuteron (binding energy/nucleon  $\sim 1.1$  MeV) condensation as a function of  $\mu$  (upper panel) and as a function of density (lower panel) [10].

of pairing when occupation numbers increase steadily from negative to positive  $\mu$  and finally the occupation numbers saturate at one when  $\mu$  goes well into the positive region, see Figure 3. We, therefore, see that the system is still far from the weak coupling large coherence length regime when it stops to have a solution. One also sees from the extension of the wave functions that the size of the  $\alpha$  particles has barely increased. Before we give an explanation for this behavior, let us study the critical temperature where this breakdown of the solution is seen more clearly.

In order to study the critical temperature for the onset of quartet condensation, we have to linearise the equation for the order parameter (3) in replacing the correlated occupation numbers by the free Fermi-Dirac distributions at finite temperature  $n(p) \rightarrow f(p) = [1 + e^{(e_p - \mu)/T}]^{-1}$ . Determining the temperature  $T$  where the equation is fulfilled gives the critical temperature  $T = T_c^\alpha$ . This is the Thouless criterion for the critical temperature of pairing [9] transposed to the quartet case. In Figure 4, we show the evolution of  $T_c^\alpha$  as a function of the chemical potential (upper panel) and of density (lower panel) [10]. This figure shows very explicitly the excellent performance of our momentum projected mean field ansatz for the quartet order parameter. The crosses correspond to the full solution of equation (3) in the linearised finite temperature regime with the rather realistic Malfliet-Tjohn nucleon-nucleon potential [11] whereas the continuous line corresponds to the projected mean field solution. Both results are literally on top of one another (the full solution is only available for negative chemical potentials). One clearly sees the breakdown of quartetting at small positive  $\mu$  (that the critical  $T$  breakdown occurs at a somewhat larger

positive  $\mu$  may be due to the fact that here we are at finite temperature contrary to the full solution of (3) with (8) which was used at  $T = 0$  in Figure 2) whereas n-p (deuteron) pairing goes on monotonously into the large  $\mu$  region. It is worth mentioning that in the isospin polarised case with more neutrons than protons, n-p pairing is much more affected than quartetting (due to the much stronger binding of the  $\alpha$  particle) and finally loses against  $\alpha$  condensation [12]. So, the fact is that, contrary to the pairing case where there is a smooth cross over from BEC to BCS, in the case of quartetting the transition to the dissolution of the  $\alpha$  particles seems to occur quite abruptly and we have to seek for an explanation of this somewhat surprising difference between pairing and quartetting.

The explanation is in a sense rather trivial. It has to do with the different level densities involved in the two systems. In the pairing case the single particle mass operator only contains a single hole (fermion) line propagator and the level density is given by

$$g_{1h}(\omega) = -\frac{1}{\pi} \text{Im} \sum_p \frac{\bar{f}(p) + f(p)}{\omega + e_p + i\eta} = \sum_p \delta(\omega + e_p). \quad (9)$$

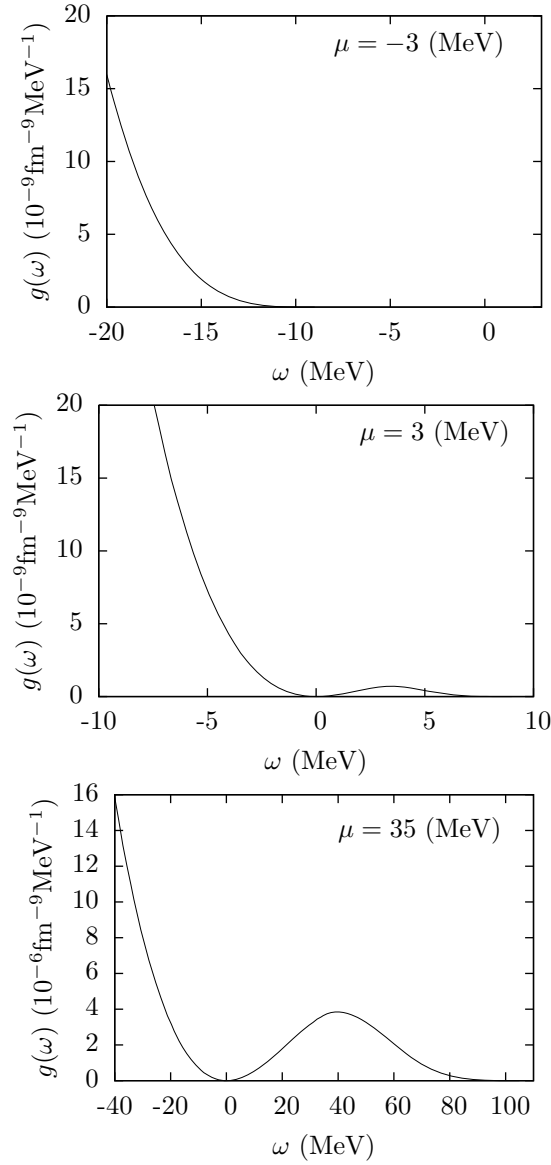
In the case of three holes as is the case for quartetting, we have for the  $3h$  level density (see the equivalent  $3p$  level density in [13])

$$\begin{aligned} g_{3h}(\omega) &= -\frac{1}{\pi} \text{Im} \text{Tr} \frac{\bar{f}(p_1)\bar{f}(p_2)\bar{f}(p_3) + f(p_1)f(p_2)f(p_3)}{\omega + e_1 + e_2 + e_3 + i\eta} \\ &= \text{Tr}[\bar{f}(p_1)\bar{f}(p_2)\bar{f}(p_3) + f(p_1)f(p_2)f(p_3)]\delta(\omega + e_1 + e_2 + e_3). \end{aligned} \quad (10)$$

In Figure 5, we give, for  $T = 0$ , the results for negative and positive  $\mu$ . The interesting case is  $\mu > 0$ . We see that phase space constraint and energy conservation cannot be fulfilled simultaneously at the Fermi energy and level density is zero there. This is just the point where quartetting should build up. With no level density, no quartetting! In the case of pairing there is no phase space restriction and level density is finite at the Fermi energy. For negative  $\mu$ ,  $f(e_k) = 0$  at zero temperature and exponentially small at finite  $T$ . Then there is no fundamental difference between  $1h$  and  $3h$  level densities! This explains the striking difference between pairing and quartetting in the weak coupling regime. The same reasoning holds in considering the in medium four body equation (3). The relevant in medium four fermion level density is also zero at  $4\mu$  for  $\mu > 0$ . Actually the only case for in medium more fermion level densities which remains finite at the Fermi energy is the two particle case when the c.o.m. momentum is zero as one may verify straightforwardly. That is why pairing is such a special case, different from condensation of all higher clusters.

### 3 Finite nuclei and the THSR approach. The Hoyle state in $^{12}\text{C}$

The Hoyle state is the first excited  $0^+$  state in  $^{12}\text{C}$  at 7.65 MeV. This state is one of the most famous states in nuclear physics because without its existence life on earth would be absent in its present form. Indeed, since  $^8\text{Be}$  is unstable, the stellar production of Carbon in the universe would be lower by a huge factor without the existence of the Hoyle state. It is just at the right energy to allow for the so-called triple  $\alpha$  reaction  $\alpha + \alpha + \alpha \rightarrow ^8\text{Be} + \alpha \rightarrow ^{12}\text{C}^*$  to become strongly accelerated. It should be kept in mind that, as mentioned,  $^8\text{Be}$  is unstable. However, with a life time of  $\sim 10^{-17}$  seconds it lives very long on nuclear scales but still decays very fast on absolute scales.



**Fig. 5.**  $3h$  level density for negative and two positive chemical potentials [3].

For the microscopic description of the Hoyle state several approaches have been put forward in the past [14,16,17,19,21–23]. However, only the following, so-called THSR wave function (according to the authors Tohsaki, Horiuchi, Schuck, Roepke) which was proposed in [1] concentrates on the  $\alpha$  particle condensation aspect (the spin-isospin part is not written out)

$$\Psi_{\text{THSR}} \propto \mathcal{A}\psi_1\psi_2\psi_3 \equiv \mathcal{A}|B\rangle \quad (11)$$

with

$$\psi_i = e^{-((\mathbf{R}_i - \mathbf{X}_G)^2)/B^2} \phi_{\alpha_i} \quad (12)$$

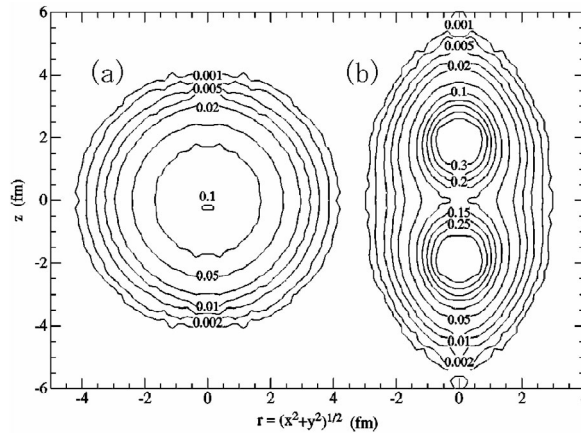
and

$$\phi_{\alpha_i} = e^{-\sum_{k<l}(\mathbf{r}_{i,k}-\mathbf{r}_{i,l})^2/(8b^2)}. \quad (13)$$

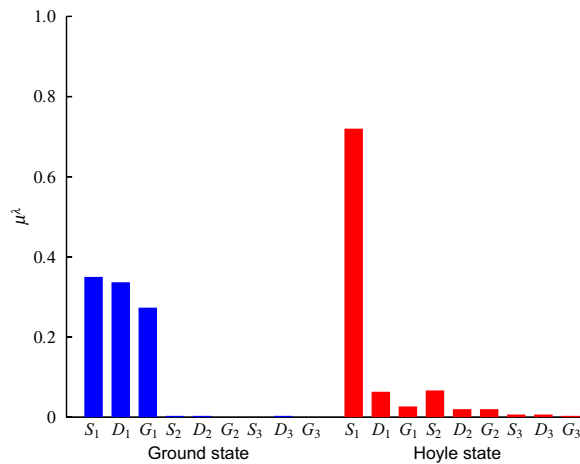
In (11) the  $\mathbf{R}_i$  are the c.o.m. coordinates of  $\alpha$  particle ‘ $i$ ’ and  $\mathbf{X}_G$  is the total c.o.m. coordinate of  $^{12}\text{C}$ .  $\mathcal{A}$  is the antisymmetrizer of the twelve nucleon wave function with  $\phi_{\alpha_i}$  the intrinsic translational invariant wave function of the  $\alpha$ -particle ‘ $i$ ’. The whole 12 nucleon wave function in (11) is, therefore, translationally invariant. The special Gaussian form given in equations (12), (13) was chosen in [1] to ease the variational calculation. The condensate aspect lies in the fact that (11) is a (antisymmetrized) product of three times the same  $\alpha$ -particle wave function and is, thus, analogous to a number projected BCS wave function in the case of pairing. This twelve nucleon wave function has two variational parameters,  $b$  and  $B$ . It possesses the remarkable property that for  $B = b$  it is a pure harmonic oscillator Slater determinant (this aspect of (11) is explained in [24,25]) whereas for  $B \gg b$  the  $\alpha$ ’s are at low density so far apart from one another that the antisymmetrizer can be dropped and, thus, (11) becomes a simple product of three  $\alpha$  particles, all in identical 0S states, that is, a pure condensate state. The minimization of the energy with a Hamiltonian containing a nucleon-nucleon force determined earlier independently [27] allows to obtain a reasonable value for the ground state energy of  $^{12}\text{C}$ . Variation of energy under the condition that (11) is orthogonal to the previously determined ground state allows to calculate the first excited  $0^+$  state, i.e., the Hoyle state. While the size of the individual  $\alpha$  particles remains very close to their free space value ( $b \simeq 1.37$  fm), the variationally determined  $B$  parameter takes on about three times this value. This entails a quite enhanced value of the rms radius of 3.83 fm of the Hoyle state with respect to the one of the ground state (2.4 fm). This gives a volume (density) of the Hoyle state about a factor 3–4 larger (smaller) than for the ground state. In such a large volume the  $\alpha$ ’s have space to develop themselves what is not the case in the ground state where they overlap strongly. The situation for the Hoyle state is then similar to the case of  $^8\text{Be}$  which is the only nucleus which has a pronounced two  $\alpha$  structure in its ground state (we ignore here that  $^8\text{Be}$  is, in fact unstable with a width on the eV level; so on nuclear scales the  $^8\text{Be}$  nucleus can be considered as stable). In Figure 6 we show the result of an exact Monte Carlo calculation based on a realistic nucleon-nucleon force plus a phenomenological three body term [26]. It should be pointed out that this  $\alpha$  cluster structure in the ground state of a nucleus is a singular feature among all nuclei which can be described in first approximation by a dense Fermi gas (Slater determinant). However, expanding the nuclei to densities similar to the one of  $^8\text{Be}$ , that is 3–4 times lower than average ground state densities of ordinary nuclei, the  $\alpha$  structure reappears in excited states.

Still the question may be asked: is the Hoyle state closer to a Slater determinant or to an  $\alpha$  condensate? A precise answer is obtained from the calculation of the bosonic occupation numbers which have been obtained in three different works [23,28,29] with very similar results. The ones of [29] are displayed in Figure 7. We see that the distribution in the ground state is more or less equipartitioned and compatible with the SU3 shell model theory whereas the distribution of the Hoyle state has an overwhelming contribution of over 70% of the  $\alpha$ ’s being in the lowest 0S state, all other contributions are down by a factor of at least ten. We, therefore, can say that the three  $\alpha$  particles in the Hoyle state occupy with their c.o.m. motion to a large fraction the same 0S orbit meaning that, indeed, the Hoyle state can be considered to within good approximation as a condensate of 3  $\alpha$  particles. However, the Pauli principle is still active and antisymmetrisation (plus some residual  $\alpha - \alpha$  interaction) scatters the  $\alpha$ ’s out of the condensate about 30% of the time. It can be mentioned here that this number is very similar concerning good single particle states in odd nuclei where the fermionic occupation numbers also are in the range of 70–80%.



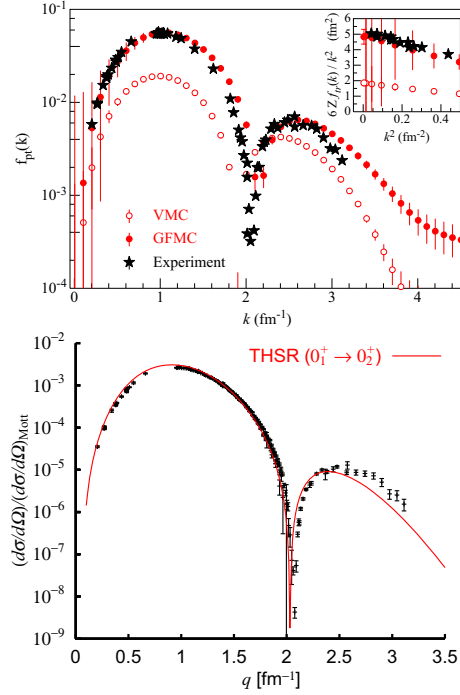


**Fig. 6.** Green's function Monte Carlo results for  $^8\text{Be}$ . Left: laboratory frame; right: intrinsic frame. From [26].



**Fig. 7.**  $\alpha$  particle occupation numbers in the ground state (left) and in the Hoyle state (right) [29].

We mentioned that the Hoyle state has an extended volume being by a factor 3–4 larger than the one of the ground state. How to prove this? It turns out that the inelastic form factor, measured by inelastic electron scattering, is very sensitive to the size of the Hoyle state [30]. Increasing artificially the size of the Hoyle state by about 20% reduces the form factor globally by a factor of two. The fact that the THSR theory reproduces very precisely the experimental values of the inelastic form factor, see Figure 8, without any adjustable parameter can be considered as a great achievement and gives large credit to the picture that in the Hoyle state three  $\alpha$  particles are well born out moving almost independently in their proper mean field. In the same figure we show recent Green's function Monte Carlo (GFMC) results [31] which also reproduce the inelastic form factor very nicely. In the insert of the GFMC-panel, we see that the rather precise experimental transition radius of  $5.29 \pm 0.14 \text{ fm}^2$  given in [21] is better reproduced than with the THSR approach which yields an about 20% too large value. On the other hand, the GFMC approach gives the position of the Hoyle state about 2.5 MeV too high whereas with the THSR wave function the experimental



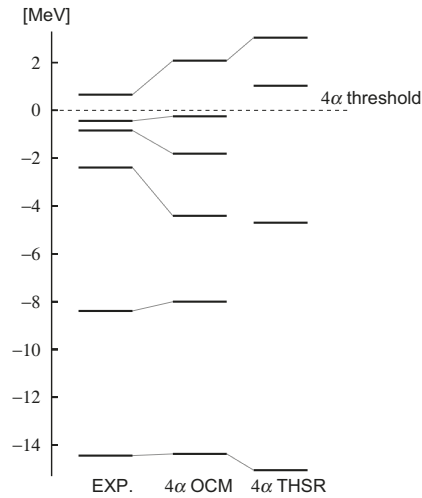
**Fig. 8.** Inelastic form factors from GFMC [31], upper panel, and THSR [30], lower panel. The THSR result cannot be distinguished from the one of [16] on the scale of the figure meaning that also the approach of [16] implicitly contains the  $\alpha$  condensation aspect (this, by the way, is also the case with the approach in [17]).

value of 7.65 MeV is quite well reproduced with no adjustable parameter. There also exist other so-called 'ab-initio lattice Monte Carlo' calculations for the Hoyle state with good reproduction of its energy but the reproduction of the important inelastic form factor is missing so far [32].

#### 4 A brief account of the situation in $^{16}\text{O}$

The situation in  $^{16}\text{O}$  is quite a bit more complicated than in  $^{12}\text{C}$ . The fact is that between the  $4\alpha$  threshold and the ground state, there are several  $0^+$  states which can be interpreted as  $\alpha + ^{12}\text{C}$  cluster configurations. In Figure 9, we show the result of a calculation with the so-called Orthogonal Condition Model (OCM) method [33].

We see that there is a very nice one to one correspondence between the first six calculated  $0^+$  states and experiment. In regard of the complexity of the situation the agreement between both can be considered as very satisfactory. Only the highest state was identified with the  $4\alpha$  condensate state. The four other excited  $0^+$  states are  $\alpha + ^{12}\text{C}$  configurations. For example the 5-th  $0^+$  state is interpreted as an  $\alpha$  orbiting in a higher nodal S-wave around the ground state of  $^{12}\text{C}$ . The 4-th  $0^+$  state contains an  $\alpha$  orbiting in a P-wave around the first  $1^-$  state in  $^{12}\text{C}$ . In the 3-rd  $0^+$  state the  $\alpha$  is in a D-wave coupled to the  $2_1^+$  state of  $^{12}\text{C}$  and in the 2-nd  $0^+$  state the  $\alpha$  is in a 0S-wave and the  $^{12}\text{C}$  in its ground state. The single parameter THSR calculation can only reproduce correctly the ground state and the  $\alpha$  condensate state ( $0_6^+$ ). By construction it cannot describe  $\alpha + ^{12}\text{C}$  configurations. So, the two intermediate states give some sort of average picture of the four  $\alpha$  plus  $^{12}\text{C}$  configurations.



**Fig. 9.** Spectrum of  $0^+$  states in  $^{16}\text{O}$  from the OCM and THSR approaches [1,33].

One would have to employ a more general ansatz like in [34] to cope with the situation. Work in this direction is in progress. The  $0_6^+$  state is theoretically identified as the  $\alpha$ -condensate state from the overlap squared  $|\langle 0_6^+ | \alpha + ^{12}\text{C}(0_2^+) \rangle|^2$  [35].

## 5 Discussion and conclusions

In this contribution, we first discussed the case of quartet condensation in attractive four component Fermi gases. A paradigmatic case of such a quartet is the  $\alpha$  particle in nuclear physics. We showed that  $\alpha$  particle condensation occurs in low density nuclear matter but only in the BEC phase. We explained why for quartets there does not exist an analogue to the weak coupling, long coherence phase which exists for pairing with the BCS description. This is a very important difference between the quartetting and pairing cases. We pointed out that pairing is in fact a very singular situation and that all bosonic clusters formed out of fermions involving more than two fermions do not exhibit the analogue to a long coherence phase what prevails with pairing. We then moved on and discussed the situation in finite nuclei where in some lighter nuclei excited states around the  $\alpha$  particle disintegration threshold can be considered as precursors to quartet condensation in infinite matter. We concentrated on the  $\alpha$  particle condensation aspect of the Hoyle state in  $^{12}\text{C}$  introduced with the THSR wave function 15 years back [1]. This, because the Hoyle state is extremely important for the Carbon production in the universe and, thus, for the existence of life! Our approach reproduces all known experimental results of the Hoyle state without any adjustable parameter and, thus, gives credit to the condensation scenario. This, despite of the fact that its direct experimental verification is difficult. Indeed, while pairing induces clear signs of superfluidity in rotating nuclei, no analogous effects have been detected so far from quartetting. However, several experiments are under way or planned concerning the Hoyle state and analogous states in  $^{16}\text{O}$  or even heavier nuclei, what may shed further light on the situation in the near future [36,37]. A major issue in this respect is the understanding not only of the Hoyle state but of excited states thereof. The  $0_3^+$  and  $0_4^+$  have been identified experimentally recently and have been interpreted as  $\alpha$  gas states with one  $\alpha$  in a higher nodal S-state and a linear chain state, respectively, see [34] and references therein for discussions. Also the structure

of the second  $2^+$  state is strongly debated. It is considered either as a member of a rotational band with the Hoyle state as band head [38] or more as a nodal excitation of one of the  $\alpha$ 's into a D-wave [34]. Further experimental and theoretical studies are necessary to elucidate the situation. With respect to the excited Hoyle states an interesting paper has appeared recently [39] where the authors explain with a single adjustable parameter very well the Hoyle spectrum on grounds that the Hoyle state is a Bose condensate with broken U1 symmetry (particle number). However, also this approach is not well tested and needs further work.

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