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Editorial

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Fractional calculus, in allowing integrals and derivatives of any positive real order (the term "fractional" is kept only for historical reasons), can be considered a branch of mathematical analysis which deals with integro-differential equations where the integrals are of convolution type and exhibit (weakly singular) kernels of power-law type. It has a history of at least three hundred years because it can be dated back to the letter from G.W. Leibniz to G.A. de L'Hôpital and J. Wallis, dated 30 September 1695, in which the meaning of the one-half order derivative was first discussed and were made some remarks about its possibility. Subsequent mention of fractional derivatives was made, in some context or the other by L. Euler (1730), J.L. Lagrange (1772), P.S. Laplace (1812), S.F. Lacroix (1819), J.B.J. Fourier (1822), N.H. Abel (1823), J. Liouville (1832), B. Riemann (1847), H.L. Greer (1859), H. Holmgren (1865), A.K. Grünwald (1867), A.V. Letnikov (1868), N.Ya. Sonin (1869), H. Laurent (1884), P.A. Nekrassov (1888), A. Krug (1890), O. Heaviside (1892), S. Pincherle (1902), H. Weyl (1919), P. Lévy (1923), A. Marchaud (1927), H.T. Davis (1936), A. Zygmund (1945), M. Riesz (1949), W. Feller (1952), just to cite some relevant contributors up the mid of the last century, see e.g. [1,2]. Recently, a poster illustrating the major contributors during the period 1695-1970 has been published [3].

From its beginning to the 1970's, fractional calculus was studied mainly by mathematicians as an abstract area containing only pure mathematical manipulations of little or no use, except for some applications in rheology (hereditary mechanics). In fact, it is known that one of the earlier applications of fractional calculus is to model viscoelastic bodies as intermediate between fluids and solids. Recently, one of the Guest Editors has provided an historical survey of the contributions on this respect up to those years, see [4], extracted from the book [5].

After 1970 the paradigm began to shift from pure mathematical research to applications in various fields. Up to now, fractional calculus has been found to infiltrate into almost every field of science and engineering. As far as we know, fractional calculus is one of the best tools to characterize long-memory processes and materials, anomalous diffusion, long-range interactions, long-term behaviors, power laws, allometric scaling laws, and so on [6,7]. So the corresponding mathematical models are fractional differential and integral equations. Their evolutions behave in a much more

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complicated way so to study the corresponding dynamics is much more difficult. For example, the solution to an ordinary differential equation of fractional order can not define a dynamic system in the sense of semigroup theory, see [8] which is the early work on fractional dynamics, due to the memory/hereditary property. Although the Lyapunov exponents for the fractional differential system can be similarly defined and their bounds are also obtained [8], not all the classical theory of dynamic systems can be directly applied to the fractional differential equations. Hence, a somewhat theoretical frame needs to be established. On the other hand, new applications such as in viscoelasticity and control theory strongly push the fractional calculus forwards. This special issue, which is the first one on fractional dynamics and control, tries to attract more scientists interested in fractional calculus to focus on this challenging and exciting topic.

Loosely speaking, this issue includes four parts: basic theory, applications, modelling and numerical computation. In the first part, there are three papers in all [9–11]. Often, when we talk about fractional calculus, we must face, although sometime we ignore, a problem. That is, for a given function f, when is it be fractionally integrable and/or fractionally differentiable? In [9], Li and Zhao collect and organize the basic results on fractional integrability and fractional differentiability, and some comments and several persuasive examples are also included. The stability analysis of the fractional differential equations is another aspect of the theory of fractional differential systems. In [10], Li and Zhang summarize recent stability results. In the following paper, Du and Wang study the initialized fractional differential equations in the sense of Riemann-Liouville [11].

In the applications part, there are in total five papers [12–16]. Jiang deals with the time-space fractional Schrödinger-like equation, where the analytical solution is obtained by using the powerful transcendental H function [12]. The paper by Qi and Liu focuses also on finding the exact solution of the duct flow of the fractional Maxwell fluid [13]. The topic of the stochastic response of the physical and/or mechanical system should deserve attention. In [14], Chen, Zhuang and Zhu study the stochastic response of single-degree-of-freedom nonlinear oscillators with fractional damping, where illustrative numerical examples and discussion are included. Since appearance of fractional calculus, the physical and geometrical meaning of fractional integrals and fractional derivatives has always attracted extreme interests although still remaining open. In the following paper, Sheng et al. study the variable-order fractional integrator and fractional differentiator via physical realization which is the early work in this respect [15]. In the last paper of this group, Pagnini discusses for fractional diffusive media the derivation of the evolution equation for the radius of a premixed spherical flame, named also flame ball [16]. In this paper fractional calculus naturally appears on the basis of physical arguments in an strongly engineering application related to combustion science. Understanding premixed combustion in lean conditions plays a key role in product engineering because it is involved in the design of efficient, clean-burning combustion.

Simultaneously, fractional modelling is always an important component in studies of fractional calculus. In this part, five papers [17–21] cover continuous time random walk (CTRW), rheology, anomalous relaxation, heat conduction in rigid bodies, and general systems with memory. In [17], Gorenflo and Mainardi device a threefold pathway leading from CTRW under power law regime to the space-time fractional diffusion equation and the subordination integral form. This pathway offers the method of parametric subordination which is useful for simulating trajectories of diffusing particles. The basic fractional models in rheology are introduced and analyzed by Mainardi and Spada [18]. The models are those that generalize via derivatives of fractional order the classical bodies characterized by two, three and four parameters, referred to as Kelvin-Voigt, Maxwell, Zener, Ant-Zener and Burgers, and have the potential of better characterizing the time dependence of basic transient rheological effects in the Earth. For anomalous relaxation in dielectrics, Capelas de Oliveira, Mainardi and Vaz Jr. [19] use the Mittag-Leffler special functions (in three parameters) to generalize the non-Debye models known in the literature and prove the complete monotonicity of the relaxation functions in order to generate the corresponding relaxation spectra. In this paper theoretical analysis and computer 3D-graphics are displayed to fully understand such general models. Heat conduction is an important phenomenon in physics and thermokinetics, which is well characterized by the classical heat conduction equations. However, such models can not well model the long-range flux. In this situation, the fractional heat conduction can properly solve such problems. In [20], Borino, Di Paola and Zingales propose a fractional model for the long-range flux in rigid bodies. Finally, in the studies of fractional integral and fractional derivatives, their orders also deserve attention. In some physical processes, the order is not a constant but a function of time t and the spatial variable x. Sun, et al. study such a case, where a comparative study of constant-order and variable-order fractional models is presented in characterizing memory properties of systems [21].

Generally speaking, the understanding of the evolution of the solutions to the fractional models largely depends on numerical methods and numerical calculation. In this last part, numerical solutions and approximate analytical solutions to some fractional differential equations are presented in four papers [22–25] in which all the numerical methods turn out to be effective. In [22], Carpinteri, Cornetti and Sapora introduce a fractional calculus approach to the nonlocal elasticity (or fractional elasticity), several numerical examples are displayed as well. In [23], Chen derives an implicit difference method for the percolation model in R^3. Another finite difference scheme based on the nested meshes method for the space fractional differential equation is presented as well by Li and Chen [24]. On the other hand, the approximate solutions to the fractional subdiffusion and drift equations are the goal of Hristov. In [25] an intersting numerical method is developed to generate distributions approaching the analytical solutions expressed through the M-Wright functions.

The Guest Editors hope that the present issue will provide inspiration and encouragement for those working on fractional dynamics and control. The methods and results presented in this special issue show that the exploration of fractional dynamics and control is a fascinating, challenging and applicable promising field of current research.

Last but not the least we thank Prof. Jürgen Kurths for initially encouraging us to follow-up on a successful international workshop entitled “Academic Day of Fractional Dynamics and Control” organized by C.P. Li (one of the Guest Editors of this issue) and W. Chen, held at Shanghai University in May 16-18, 2010, under the chair by Y.Q. Chen [26]. Prof Kurths was so kind to invite us to publish this resulting Special Topics issue, based mostly on selected original contributions from the meeting, as well as including some additionally invited contributions. We further thank him for his careful reading of all submissions and providing pertinent suggestions. We greatly appreciate Mrs Sabine Lehr and Dr. Christian Caron for their sparing no pains to inform us, replying to us and explaining various details regarding this special issue. We specially thank Dr. Christian Caron as well for his careful reading and providing updating suggestions and sparing no effort to carefully edit this volume. We especially thank Miss Isabelle Houlbert for her careful editing and patience. CL thanks his students for actively joining his fractional group. CL also thanks his wife Assoc. Prof. S. Zhao for her unselfish support and his son Mr X. Li for his tolerating less time to give him company. CL acknowledges the financial support of the National Natural Science Foundation of China (10872119) and the Key Disciplines of Shanghai Municipality (S30104). FM acknowledges the financial support of the National Group of Mathematical Physics (GNFM–INDAM) enabling him to attend the interesting workshop in Shanghai University.

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