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The smeared-horizon observer of a black hole

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Abstract A class of observers is introduced that interpolate smoothly between the Schwarzschild observer, stable at spatial infinity, and the Kerr–Schild observer, who falls into a black hole. For these observers, the passing of the event and inner horizon takes a finite time, which diverges logarithmically when the interpolation parameter σ goes to zero. In the field theoretic approach to gravitation, the behavior at the horizons becomes regular, making the mass of the metric well defined.

1 Introduction

The most amazing property of black holes is their event horizon, which even led Einstein initially to be skeptical about the Schwarzschild metric. An observer at spatial "infinity" will never live the day to observe something falling into a black hole. The related infinite redshift at the event horizon is the ultimate limit of the redshift known from sirens on cars that move away from us.

On the other hand, there are the Painlevé–Gullstrand and Kerr–Schild metrics, which do not exhibit a horizon for the observers related to them. It is the aim of the present paper to introduce a class of intermediate observers, the "smeared horizon observers", which interpolate between the Schwarzschild and Kerr-Schild metrics by a free parameter σ. Some related properties are discussed.

In Sect. 2, we introduce the metric of the smeared-horizon observers and consider properties like its eigenvectors and their behaviors in the black hole interior, and analyze outgoing and ingoing spherical mass shells. In Sect. 3, we connect to recent exact solutions for the black hole interior and its role for the smeared-horizon observer. In Sect. 4, we connect to a recent class of exact solutions for smooth, cored black hole interiors. In Sect. 5, we show that the field theoretic approach to gravitation, connected to the Landau–Lifshitz pseudo tensor for the gravitational field, becomes well defined at the would-be horizons and hence everywhere, so that allows to properly define the mass of the metric.

Work dedicated to Igor V. Volovich.

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2 Generalized Schwarzschild and Reissner–Nordström metric

For smooth functions $N(r)$ and $S(r)$, we consider the generalization of the Schwarzschild metric in spherical coordinates $r^\mu = (t, r, \theta, \phi)$, $\mu = 0, 1, 2, 3$,

$$\mathrm{d}s^2 = -N^2\bar{S}\mathrm{d}t_{\mathrm{S}}^2 + \frac{1}{\bar{S}}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2 = g_{\mu\nu}\mathrm{d}r^\mu\mathrm{d}r^\nu, \quad \bar{S} = S - 1, \tag{1}$$

with $\mathrm{d}\Omega = (\mathrm{d}\theta, \sin\theta\mathrm{d}\phi)$. The Schwarzschild metric [1] is described by $N(r) = 1$, $S(r) = 2GM/r$; the Reissner Nordström metric [2, 3] by $N = 1$, $S = G(2M/r - Q^2/r^2)$. The latter has an event (e) horizon and an inner (i) horizon, where $S(R_{e,i}) = 1$ $(\bar{S} = 0)$. They are located at

$$R_e = G\Big(M + \sqrt{M^2 - \mathcal{Q}^2}\Big), \quad R_i = G\Big(M - \sqrt{M^2 - \mathcal{Q}^2}\Big), \qquad \mathcal{Q} = m_P Q. \tag{2}$$

In our units $\hbar = c = 1$ and $\mu_0 = 4\pi$, the Planck mass is $m_P = 1/\sqrt{G}$.

It holds that $0 < S < 1$ in the outer space, so that $\bar{S} < 0$ there. Inside a core bounded by the inner horizon R_i, one has $S < 1$. In the Schwarzschild metric, the inner horizon coincides with the origin, but this is a special case. In the Reissner Nordström metric, $S \to -\infty$ for $r \to 0$.

We have recently proposed a class of exact solutions where S is regular with $S \sim r^2$ for $r \to 0$. The latter property implies the presence of a finite core bounded by an inner horizon R_i. As in the Reissner Nordström metric, the region between the inner and event horizons, termed the *mantle*, is a standard vacuum, described by the Reissner Nordström metric with $S > 1$. In these models one has $N(r) = 1$ for $r > R_i$ and $0 < N \leq 1$ for $r < R_i$.

2.1 The Painlevé–Gullstrand observer

De één heeft meer in zijn mars dan de ander[1]

Dutch expression

One of the mysteries of black holes is that in its interior, from the event horizon to the inner horizon, the roles of r and t are interchanged. The reversed role of r and t in its interior is counterintuitive, and so is the infinite redshift for signals from the horizon to a stationary observer at spatial infinity.

The first step to investigate the issue was made by Painlevé in 1921 [4], and Gullstrand in 1922 [5], actually intended to question the Schwarzschild solution. For completeness, we recall their approach. In the Schwarzschild metric, they introduce a new time coordinate by setting $\mathrm{d}t_S = \mathrm{d}t_{\mathrm{pg}} - \sqrt{(2GM/r)}\,\mathrm{d}r/(1 - 2GM/r)$, so as to obtain

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{r}\right)\mathrm{d}t_{\mathrm{pg}}^2 - 2\sqrt{\frac{2GM}{r}}\mathrm{d}t_{\mathrm{pg}}\mathrm{d}r - \mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \quad (3)$$

This metric is regular except for $r \to 0$, so that the observer does not notice an event horizon. In integral form one has $t_S = t_{\mathrm{pg}} - 2GM\{2y - \log[(y+1)/(y-1)]\}$, where $y = \sqrt{r/2GM}$.

2.2 The ingoing Kerr–Schild observer

In this work, we follow the approach of Kerr–Schild [6] to generate new metrics. We start by allowing in the Kerr–Schild metric a dilated time $\mathrm{d}t \to N(r)\mathrm{d}t_{\mathrm{ks}}$,

$$\mathrm{d}s^2 = N^2(r)\mathrm{d}t_{\mathrm{ks}}^2 - \mathrm{d}r^2 - r^2\mathrm{d}\Omega^2 - S(r)\mathrm{d}k^2(r). \quad (4)$$

The one-form $\mathrm{d}k = k_\mu \mathrm{d}r^\mu$ involves a null vector k_μ, viz. $g_{\mathrm{ks}}^{\mu\nu} k_\mu k_\nu = 0$. The standard cases and their connection to t_S are

$$\mathrm{d}k = k_\mu \mathrm{d}r^\mu = N\mathrm{d}t_{\mathrm{ks}} + \mathrm{d}r,$$
$$\mathrm{d}t_{\mathrm{S}} = \mathrm{d}t_{\mathrm{ks}} + \frac{S}{N\bar{S}}\mathrm{d}r, \qquad (iks), \quad (5)$$

$$\mathrm{d}k = k_\mu \mathrm{d}r^\mu = N\mathrm{d}t_{\mathrm{ks}} - \mathrm{d}r,$$
$$\mathrm{d}t_{\mathrm{S}} = \mathrm{d}t_{\mathrm{ks}} - \frac{S}{N\bar{S}}\mathrm{d}r, \qquad (oks) \quad (6)$$

where *iks* stands for an ingoing Kerr–Schild observer and *oks* for an outgoing one. Unlike (1), this is regular also at $S = 1$, $\bar{S} = 0$. For the *iks* case, it reads

$$\mathrm{d}s^2 = -N^2\bar{S}\mathrm{d}t_{\mathrm{ks}}^2 - 2NS\mathrm{d}t_{\mathrm{ks}}\mathrm{d}r - (S+1)\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \quad (7)$$

The inverse *iks* metric is coded in

$$g^{\mu\nu}\partial_\mu\partial_\nu = \frac{S+1}{\bar{S}N^2}\partial_{t_{\mathrm{ks}}}^2 - 2\frac{S}{N}\partial_{t_{\mathrm{ks}}}\partial_r + \bar{S}\partial_r^2 - \frac{1}{r^2}\partial_\Omega^2,$$
$$\partial_\Omega = \left(\partial_\theta, \frac{1}{s_\theta}\partial_\phi\right). \quad (8)$$

Indeed, k_μ is a null vector, $k^\mu k_\mu = N^2 g^{00} + 2Ng^{01} + g^{11} = 0$.

The situation for the *oks* is obtained by setting $\mathrm{d}r \to -\mathrm{d}r$ and $\partial_r \to -\partial_r$.

2.3 The ingoing smeared-horizon observer (sho)

The present work proposes a new class of observers, to be called "smeared-horizon observers" (*sho*); the ingoing ones are for some $\sigma \geq 0$ defined by

$$\mathrm{d}t_{\mathrm{ks}} = \mathrm{d}t - \frac{S\bar{S}}{N(\sigma^2 + \bar{S}^2)}\mathrm{d}r,$$
$$\mathrm{d}t_{\mathrm{S}} = \mathrm{d}t + \frac{\sigma^2}{\sigma^2 + \bar{S}^2}\frac{S}{N\bar{S}}\mathrm{d}r. \quad (9)$$

The latter form interpolates between Schwarzschild's stationary observer at infinity ($\sigma = 0$) and the ingoing PG observer ($\sigma \to \infty$). This observer still falls in, but slower than the latter.

We shall take σ constant, though it can actually be a function of r.

The line element takes the form

$$\mathrm{d}s^2 = -N^2\bar{S}\mathrm{d}t^2 - 2\frac{\sigma^2 NS}{\sigma^2 + \bar{S}^2}\mathrm{d}t\mathrm{d}r + (\bar{S} - \sigma^2)\frac{\sigma^2(S+1) + \bar{S}^2}{\left(\sigma^2 + \bar{S}^2\right)^2}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2, \quad (10)$$

with $g = \det(g_{\mu\nu}) = -r^4 N^2 s_\theta^2$. The inverse metric is coded in

$$g^{\mu\nu}\partial_\mu\partial_\nu = \frac{\sigma^2 - \bar{S}}{N^2}\frac{\sigma^2(1+S) + \bar{S}^2}{\left(\sigma^2 + \bar{S}^2\right)^2}\partial_t^2 - \frac{2}{N}\frac{\sigma^2 S}{\sigma^2 + \bar{S}^2}\partial_t\partial_r + \bar{S}\partial_r^2 - \frac{1}{r^2}\partial_\Omega^2. \quad (11)$$

At any finite σ, $g_{\mu\nu}$ and $g^{\mu\nu}$ are regular, notably at the would-be horizon(s) where $\bar{S} = 0$. Taking σ small exposes Schwarzschild's event horizon at an arbitrary precision.

[1] The one is more capable than the other.

2.4 Eigenbasis of the ingoing smeared-horizon metric

The eigenvalues of $g^{ish}_{\mu\nu}$ in (10) are, next to $\lambda_2 = -r^2$ and $\lambda_3 = -r^2 s_\theta^2$,

$$\lambda_0 = N e^{-2\mathcal{L}}, \qquad \lambda_1 = -N e^{2\mathcal{L}}, \tag{12}$$

with $\mathcal{L}$ defined by

$$\sinh 2\mathcal{L} = \frac{(S+1)\sigma^4 - 2\sigma^2\bar{S} - \bar{S}^3}{2N\big(\sigma^2+\bar{S}^2\big)^2} + \frac{N\bar{S}}{2}. \tag{13}$$

Outside the BH core, one has $N = 1$ and $S = 2GM/r - GQ^2/r^2$, so that $\mathcal{L} \to GM/r$ at large r. In the limit $\sigma \to 0$ one has $\sinh 2\mathcal{L} \to \frac{1}{2}(N^2\bar{S}^2 - 1)/N\bar{S}$, which diverges at $\bar{S} = 0$ and changes sign there; at small σ, these divergences are rounded. In that case, $\mathcal{L}$ changes sign at $\bar{S} = \sigma^2 + \mathcal{O}(\sigma^4)$, which codes two locations in the mantle, one beyond R_i and the other below R_e.

The eigenvectors of the (0,1) sector of $g_{\mu\nu}$ are mixed,

$$e_0 = \frac{\big(e^{\mathcal{E}}, -e^{-\mathcal{E}}, 0, 0\big)}{\sqrt{e^{2\mathcal{E}} + e^{-2\mathcal{E}}}}, \qquad e_1 = \frac{\big(e^{-\mathcal{E}}, \ e^{\mathcal{E}}, 0, 0\big)}{\sqrt{e^{2\mathcal{E}} + e^{-2\mathcal{E}}}}. \tag{14}$$

while $e_2 = (0, 0, 1, 0)$ and $e_3 = (0, 0, 0, 1)$. The parameter $\mathcal{E}$ is defined by

$$\begin{aligned}\sinh 2\mathcal{E} &= \frac{\sigma^2+\bar{S}^2}{\sigma^2 S}\big(\sinh 2\mathcal{L} - N\bar{S}\big)\\ &= \frac{\sigma^2+\bar{S}^2}{\sigma^2 S}\left(\frac{(S+1)\sigma^4 - 2\sigma^2\bar{S} - \bar{S}^3}{2N(\sigma^2+\bar{S}^2)^2} - \frac{N\bar{S}}{2}\right).\end{aligned} \tag{15}$$

These identities show that $\mathcal{E} = \mathcal{L} = \frac{1}{2}\log(\sqrt{2}+1)$ at the horizons where $N = 1$, $S = 1$. Eliminating S in favor of $\mathcal{E}$, the line element (10) takes the form

$$\begin{aligned}\mathrm{d}s^2 = {}& N e^{-2\mathcal{L}} \frac{\big(e^{\mathcal{E}}\mathrm{d}t - e^{-\mathcal{E}}\mathrm{d}r\big)^2}{e^{2\mathcal{E}} + e^{-2\mathcal{E}}}\\ & - N e^{2\mathcal{L}} \frac{\big(e^{\mathcal{E}}\mathrm{d}r + e^{-\mathcal{E}}\mathrm{d}t\big)^2}{e^{2\mathcal{E}} + e^{-2\mathcal{E}}} - r^2\mathrm{d}\Omega^2.\end{aligned} \tag{16}$$

Identifying this with the local Minkowski line element $\mathrm{d}\xi^{02} - \mathrm{d}\xi^{12} - \mathrm{d}\xi^{22} - \mathrm{d}\xi^{32}$, one reads off $\mathrm{d}\xi^2 = r\mathrm{d}\theta$, $\mathrm{d}\xi^3 = r s_\theta \mathrm{d}\phi$ and the more interesting ones,

$$\begin{aligned}\mathrm{d}\xi^0 &= \sqrt{N}\, e^{-\mathcal{L}} \frac{e^{\mathcal{E}}\mathrm{d}t - e^{-\mathcal{E}}\mathrm{d}r}{\sqrt{e^{2\mathcal{E}} + e^{-2\mathcal{E}}}},\\ \mathrm{d}\xi^1 &= \sqrt{N}\, e^{\mathcal{L}} \frac{e^{\mathcal{E}}\mathrm{d}r + e^{-\mathcal{E}}\mathrm{d}t}{\sqrt{e^{2\mathcal{E}} + e^{-2\mathcal{E}}}}.\end{aligned} \tag{17}$$

The signs are such that in the exterior, where $\mathcal{E} \gg 1$, $\mathrm{d}\xi^0 \sim \mathrm{d}t$ and $\mathrm{d}\xi^1 \sim \mathrm{d}r$. For $\sigma \to \infty$ one describes the Kerr-Schild observer; with $N \le 1$ and $S \ge 0$, $\mathcal{L}$ and $\mathcal{E}$ remain positive, making t act as timelike coordinate also in the interior.

For small σ, there is a transition region which regularizes the singularity at the event horizon. The regime $\bar{S} < 0$ applies to the outer space and also to the BH core; in both cases, the prefactor $1/\sigma^2$ in $\sinh\mathcal{E}$ makes $\mathcal{E} \gg 1$ and hence $\mathrm{d}\xi^0 \sim \mathrm{d}t$ and $\mathrm{d}\xi^1 \sim \mathrm{d}r$ as usual. In the mantle, the opposite case $\bar{S} > 1$ involves $\sinh\mathcal{E} < 0$ and $\mathcal{E} \ll -1$, so that there is the switched connection between t and r, $\mathrm{d}\xi^0 \sim -\mathrm{d}r$ and $\mathrm{d}\xi^1 \sim \mathrm{d}t$, known from the interior of the Schwarzschild BH. The width of the transition region is $\Delta\bar{S} \sim \sigma^2$. For small σ, one can analyze the transition by coding r in a parameter λ such that

$$\bar{S} = (1 - N\sinh\lambda)\sigma^2 + \mathcal{O}\big(\sigma^4\big), \tag{18}$$

This yields $\sinh 2\mathcal{E} = \sinh 2\mathcal{L} + \mathcal{O}(\sigma^2) = \sinh\lambda + \mathcal{O}(\sigma^2)$. In the absence of surface layers, $N = 1$ at the event and inner horizon, whence r reads

$$r(\lambda) = G(M+w)\left[1 + \frac{M+w}{2w}(\sinh\lambda - 1)\sigma^2 + \mathcal{O}(\sigma^4)\right], \tag{19}$$

where $w = \pm\sqrt{M^2 - \mathcal{Q}^2}$, with the + (−) sign at the event (inner) horizon. Starting inside the mantle near the event horizon and going outwards, λ increases from negative to positive values, with the event horizon $\bar{S} = 0$ located at $\lambda_c = \ln(\sqrt{2}+1)$. Related behavior occurs around the inner horizon.

2.5 Outgoing shells in the frame of the ingoing sho

An outgoing spherical shell for a massless field involves $\mathrm{d}\xi^0 = \mathrm{d}\xi^1$, that is,

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\sinh(\mathcal{E} - \mathcal{L})}{\cosh(\mathcal{E} + \mathcal{L})}. \tag{20}$$

This vanishes at the inner and event horizons where $\mathcal{E} = \mathcal{L}$. With $N = 1$ outside the core, the large r behaviors $\mathcal{L} \to GM/2r$ and $\sinh 2\mathcal{E} \to (1+\sigma^2)r/2GM\sigma^2$, imply $\mathrm{d}r/\mathrm{d}t \to 1 - 2GM\sigma^2/(1+\sigma^2)r$, an outgoing motion.

Equation (15) shows that the crossing $\mathcal{E} = \mathcal{L}$ occurs at the would-be horizons $R_{i,e}$ where $S = 1$. Setting $t_e = -1/S'(R_e)$, the relation

$$\begin{aligned}\mathrm{d}t &= \frac{\cosh(\mathcal{E}+\mathcal{L})}{\sinh(\mathcal{E}-\mathcal{L})}\frac{\mathcal{E}-\mathcal{L}}{\mathcal{E}'-\mathcal{L}'}\\ &\quad \mathrm{d}\log|\mathcal{E}-\mathcal{L}| \approx 2t_e\, \mathrm{d}\log\frac{|r-R_e|}{\sqrt{2}t_e},\end{aligned} \tag{21}$$

leads for $r > R_e$ to $r - R_e \sim \exp(t/2t_e)$ and for $r < R_e$ to $R_e - r \sim \exp(t/2t_e)$. The passing of the event horizon occurs for $t \to -\infty$. Near the inner horizon, we set $t_i = 1/S'(R_i)$ and obtain likewise $r - R_i \sim \exp(-t/2t_i)$

and for $r < R_i$ to $R_i - r \sim \exp(-t/2t_i)$; the passing occurs for $t \to +\infty$. In the mantle, t decreases when r increases.

Let us apply this to the Schwarzschild metric ($S = 2GM/r$ and $N = 1$), employ units $2GM \to 1$, so that $t_e \to 1$, and define $\bar{r} = r - 1$. One has

$$e^t = (r-1)^2\, e^{f(r)}$$
$$f(r) = r + \frac{\sigma}{\sigma^2+1} \arctan \frac{\sigma^2 r + \bar{r}}{\sigma} - \frac{\log\left(\sigma^2 r^2 + \bar{r}^2\right)}{2(\sigma^2+1)}. \tag{22}$$

It keeps the Schwarzschild singularity $t \sim |\log \bar{r}|$ for the time to go from a point $0 < r - 1 \ll 1$ just outside of the event horizon to a location well away.

Including a charge Q, the adimensional units express $S = 2GM/r - GQ^2/r^2$ as $S = 1/r - q^2/4r^2$ with $q = m_P Q/M$. It yields

$$t = r + \Re\left(\frac{1-\frac{1}{2}q^2}{\sqrt{1-q^2}} \log \frac{r-r_e}{r-r_i} + \log[(r-r_e)(r-r_i)]\right) + \Delta t,$$
$$\Delta t = \Re \sum_{j=1}^{4} \frac{q^4 - 8q^2 r_j + 4(4+q^2) r_j^2 - 16 r_j^3}{-q^2 + 2(2+q^2) r_j - 12 r_j^2 + 8(1+\sigma^2) r_j^3} \frac{\log(r-r_j)}{8} \tag{23}$$

where $r_{e,\,i} = \frac{1}{2}(1 \pm \sqrt{1-q^2})$ denote the event and inner horizons, respectively, and the r_j are the 4 complex roots of $\bar{S}^2 + \sigma^2 = 0$. Since they arise from $\bar{S} = \pm i\sigma$, they take the explicit forms

$$r_{1,\,2} = \frac{1 \pm \sqrt{1 - q^2(1+i\sigma)}}{2(1+i\sigma)},$$
$$r_{3,\,4} = \frac{1 \pm \sqrt{1 - q^2(1-i\sigma)}}{2(1-i\sigma)}. \tag{24}$$

Since $r_{3,\,4} = r^*_{1,\,2}$, it follows that $\Delta t = 2\Re(\Delta t_1 + \Delta t_2)$ with

$$\Delta t_{1,\,2} = \frac{\mp\left[1 \pm \sqrt{1-q^2(1+i\sigma)}\,\right]^2}{8(1+i\sigma)\sqrt{1-q^2(1+i\sigma)}} \log\left(r - \frac{1 \pm \sqrt{1-q^2(1+i\sigma)}}{2(1+i\sigma)}\right). \tag{25}$$

It is seen that t keeps logarithmic divergences for signals emitted close to $r_{e,\,i}$. As above for $q = 0$, the effect of a finite σ is to double their prefactor. In other words, for $\sigma \to 0$, Δt absorbs half of the logarithms of the first line in (23). But at finite σ, Δt itself is regular for all real r.

In conclusion, in the description by the ingoing *sho*, it takes infinite time for the shell to emerge from the BH. This stems with the popular statement "nothing can escape from a black hole".

2.6 Ingoing shells described by the ingoing sho

An ingoing spherical shell for a massless field involves $\mathrm{d}\xi^0 = -\mathrm{d}\xi^1$, that is,

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{\cosh(\mathcal{E}-\mathcal{L})}{\sinh(\mathcal{E}+\mathcal{L})}, \tag{26}$$

which is finite for $\mathcal{E} = \mathcal{L}$, that is, at the horizons. Integrating this for the Schwarzschild metric yields

$$t = -r + \frac{\sigma}{\sigma^2+1} \arctan \frac{\sigma^2 r + \bar{r}}{\sigma} - \frac{\log\left(\sigma^2 r^2 + \bar{r}^2\right)}{2(\sigma^2+1)}. \tag{27}$$

This remains finite as $t \sim \log 1/\sigma$ for $r \to 1$, so the ingoing massless shells are observed to go quickly through the event horizon. For the charged case, it involves Δt of Eq. (23),

$$t = -r + \Delta t. \tag{28}$$

Compared to the outgoing case (23), next to $r \to -r$, the explicit logarithms disappear, while the root sum is maintained.

In conclusion, for the incoming *sho*, incoming shells only need a finite time to pass the event horizon: for that process, no event horizon is noticed.

3 The outgoing smeared-horizon observer

So far, we considered the ingoing PG observer. The outgoing PG observer relates formally to the Schwarzschild observer by r-reversal in going from (5) to (6), so that

$$\mathrm{d}t_{\mathrm{S}} = \mathrm{d}t_{\mathrm{ks}} - \frac{S}{N\bar{S}}\mathrm{d}r \tag{3.29}$$

Correspondingly, there is a sign change in the shift term in (9).

$$\mathrm{d}t_{\mathrm{ks}} = \mathrm{d}t + \frac{S\bar{S}}{N(\sigma^2+\bar{S}^2)}\mathrm{d}r,$$
$$\mathrm{d}t_{\mathrm{S}} = \mathrm{d}t - \frac{\sigma^2}{\sigma^2+\bar{S}^2}\frac{S}{N\bar{S}}\mathrm{d}r. \tag{3.30}$$

The latter form interpolates between Schwarzschild's stationary observer at infinity ($\sigma = 0$) and the outgoing KS observer ($\sigma \to \infty$).

The reversed role of ingoing and outgoing motion corresponds to time-reversal, and turns the black hole into a so-called white hole. In the exterior, $\mathrm{d}r/\mathrm{d}t > 0$ leads to time delay $\mathrm{d}t_S/\mathrm{d}t > 1$. In the Schwarzschild interior, increasing time-like coordinate r corresponds to increasing time.

With $\mathrm{d}r \to -\mathrm{d}r$ and fixing the overall sign such that $e_0 \to (1,\,0,\,0,\,0)$ and $e_1 \to (0,\,1,\,0,\,0)$ for large r, the eigenvectors (14) become

$$e_0 = \frac{\left(e^{\mathcal{E}},\,e^{-\mathcal{E}},\,0,\,0\right)}{\sqrt{e^{2\mathcal{E}}+e^{-2\mathcal{E}}}},\qquad e_1 = \frac{\left(-e^{-\mathcal{E}},\,e^{\mathcal{E}},\,0,\,0\right)}{\sqrt{e^{2\mathcal{E}}+e^{-2\mathcal{E}}}},\tag{3.31}$$

with $\mathcal{E}$ again defined by (15). This leads to

$$\mathrm{d}\xi^0 = \sqrt{N}\,e^{-\mathcal{L}}\,\frac{e^{\mathcal{E}}\mathrm{d}t + e^{-\mathcal{E}}\mathrm{d}r}{\sqrt{e^{2\mathcal{E}}+e^{-2\mathcal{E}}}},\qquad \mathrm{d}\xi^1 = \sqrt{N}\,e^{\mathcal{L}}\,\frac{-e^{-\mathcal{E}}\mathrm{d}t + e^{\mathcal{E}}\mathrm{d}r}{\sqrt{e^{2\mathcal{E}}+e^{-2\mathcal{E}}}}.\tag{3.32}$$

In the exterior, one has $N=1$ and $\mathcal{E} \gg 1$ for small σ, so that $\mathrm{d}\xi_0 = e^{-\mathcal{L}}\mathrm{d}t$ and $\mathrm{d}\xi_1 = e^{\mathcal{L}}\mathrm{d}r$, as usual; in the mantle $\mathcal{E} \ll -1$ so that $\mathrm{d}\xi_0 = e^{-\mathcal{L}}\mathrm{d}r$ and $\mathrm{d}\xi_1 = -e^{\mathcal{L}}\mathrm{d}t$; and in the core $\mathrm{d}\xi_0 = \sqrt{N}e^{-\mathcal{L}}\mathrm{d}t$ and $\mathrm{d}\xi_1 = \sqrt{N}e^{\mathcal{L}}\mathrm{d}r$.

Outgoing shells are described by $\mathrm{d}\xi_1 = \mathrm{d}\xi_0$, so that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\sinh(\mathcal{L}+\mathcal{E})}{\cosh(\mathcal{L}-\mathcal{E})}\tag{3.33}$$

and ingoing ones by $\mathrm{d}\xi_1 = -\mathrm{d}\xi_0$, so that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\sinh(\mathcal{L}-\mathcal{E})}{\cosh(\mathcal{L}+\mathcal{E})}.\tag{3.34}$$

Apart from an overall sign, this coincides with (20). Hence ingoing shells, described by the outgoing *sho*, take infinite time to pass the horizon. Outgoing shells, on the other hand, only take a finite time, since (3.33) is regular.

4 Exact solutions for the black hole interior

A class of exact solutions for the BH metric, which is regular everywhere in the core, was proposed recently.

4.1 The stress energy tensor for the sho

For general N, S and σ, the Einstein tensor has the nontrivial elements

$$\begin{aligned}
G^0_{\ 0} &= \frac{S + rS'}{r^2},\\
G^1_{\ 1} &= \frac{S + rS'}{r^2} + \frac{2N'\bar{S}}{rN},\\
G^0_{\ 1} &= \frac{-2\sigma^2 N' S}{rN^2(\sigma^2+\bar{S}^2)},\\
G^2_{\ 2} = G^3_{\ 3} &= \frac{2S' + rS''}{2r} + \frac{N'}{N}\frac{2\bar{S}+3rS'}{2r} + \frac{N''}{N}\bar{S},
\end{aligned}\tag{4.35}$$

with $G^2_{\ 2} = G^3_{\ 3}$ due to spherical symmetry. In the Schwarzschild case $\sigma = 0$, $G^\mu_{\ \nu}$ is diagonal; a value $\sigma > 0$ does not modify these elements, but creates the $G^0_{\ 1}$ element provided $N' \neq 0$. This represents a radial energy current for the smeared-horizon observer falling in onto the static energy distribution.

We express the stress energy tensor in terms of a local cosmological constant ρ_λ, an electrostatic energy density ρ_E and thermal matter with velocity vector $U^\mu = \delta^\mu_0/N\sqrt{-\bar{S}}$ and stress energy tensor $T^\mu_{\vartheta\,\nu} = (\rho_\vartheta + p_\vartheta)U^\mu U_\nu - p_\vartheta \delta^\mu_{\ \nu}$ involving an energy density ρ_ϑ and isotropic pressure p_ϑ. The full stress energy tensor reads

$$\begin{aligned}
T^\mu_{\ \nu} &= (\rho_\lambda - p_\vartheta)\delta^\mu_{\ \nu} + \rho_E \mathcal{C}^\mu_{\ \nu} + (\rho_\vartheta + p_\vartheta)U^\mu U_\nu,\\
\mathcal{C}^\mu_{\ \nu} &= \mathrm{diag}(1,\,1,\,-1,\,-1).
\end{aligned}\tag{4.36}$$

The Einstein equations $G^\mu_{\ \nu} = 8\pi G T^\mu_{\ \nu}$ express the coefficients of (4.36) in the functions N and S,

$$\bar{\rho}_\lambda = \frac{2S + 4rS' + r^2 S''}{32\pi G r^2} + \frac{N'}{N}\frac{4\bar{S} + 3rS'}{32\pi G r} + \frac{N''}{N}\frac{\bar{S}}{16\pi G},\tag{4.37}$$

$$\bar{\rho}_E = \frac{2S - r^2 S''}{32\pi G r^2} - \frac{N'}{N}\frac{3S'}{32\pi G} - \frac{N''}{N}\frac{\bar{S}}{16\pi G},\tag{4.38}$$

$$\bar{\rho}_\vartheta = -\frac{N'\bar{S}}{4\pi G r N},\tag{4.39}$$

where

$$\bar{\rho}_\lambda \equiv \rho_\lambda + \frac{\rho_\vartheta - 3p_\vartheta}{4},\qquad \bar{\rho}_E \equiv \rho_E + \frac{\rho_\vartheta + p_\vartheta}{4},\qquad \bar{\rho}_\vartheta \equiv \rho_\vartheta + p_\vartheta.\tag{4.40}$$

They combine as

$$\rho_{\mathrm{tot}} \equiv \rho_\lambda + \rho_E + \rho_\vartheta = \frac{S + rS'}{8\pi G r^2}.\tag{4.41}$$

While $U_0 = 1/U^0 = N\sqrt{-\bar{S}}$ for any σ, $U_1 = -\sigma^2 S/(\sigma^2 + \bar{S}^2)\sqrt{-\bar{S}}$ connects the radial energy current $T^0_{\ 1} = G^0_{\ 1}/8\pi G$ from (4.35) to the thermal matter by $T^0_{\ 1} = \bar{\rho}_\vartheta U^0 U_1$. Indeed, the ρ_λ and ρ_E terms of (4.38) are proportional to the unit matrix in their (0,1) sectors, evidently for all σ. Hence the energy current relates to the real, thermal particles; not to vacuum or electrostatic energy.

The electrostatic potential reads $A_\mu = \delta^0_\mu A_0(r)$. Conservation of the Maxwell tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ leads to

$$F^{\nu\mu}_{\ \ \ ;\nu} = \left[A_0'\left(\frac{N'}{N^3} - \frac{2}{rN^2}\right) - \frac{A_0''}{N^2}\right]\delta^\mu_0 = \mu_0 J^\mu.\tag{4.42}$$

With electric field $E(r) = -A_0'/N$ and source $J^\mu = \delta_0^\mu \rho_q/N$ this implies

$$E' + \frac{2}{r}E = \mu_0 \rho_q. \quad (4.43)$$

Taking $\mu_0 = 4\pi$, as usual in gravitation, this has the solution

$$E(r) = \frac{Q(r)}{r^2}, \qquad Q(r) = 4\pi \int_0^r \mathrm{d}u\, u^2 \rho_q(u). \quad (4.44)$$

So $Q(r)$ is the enclosed charge. The related energy density is

$$\rho_E = \frac{E^2}{2\mu_0} = \frac{Q^2(r)}{8\pi r^4}. \quad (4.45)$$

Notice that the connections (4.44) and (4.45) are just as in special relativity.

Assuming that ρ_λ, ρ_q, ρ_ϑ and p_ϑ vanish in the mantle, our task is to provide their physical meaning in the core $r \le R_i$, for suitable functions N, S.

4.2 A class of exact solutions

We recently presented an exact solution for a charged black hole core $r \le R_i$ [7]. Here, we recall it and then consider it for the smeared horizon observer.

The basic motivation is that in the stellar collapse, electrons are more easily ejected than protons, so that the black hole is positively charged. Since the Coulomb force is much stronger than the Newton force, the fraction of surplus charge needs only be of order $m_N/m_P \sim 10^{-19}$. The binding energy of the nucleons is released when their density is large enough. The rest mass of the up and down quarks and the electrons makes up only 1% of the energy; when it is neglected, the problem allows an exact solution with $N = 1$, corresponding to a vanishing matter temperature and neglect of rest masses.

Consider a core charge $Q_c$be distributed as

$$Q(r) = q_i F_q\left(\frac{r}{R_i}\right) = m_P R_i\, q_i\, F_q(x),$$
$$q_i = \frac{\mathcal{Q}_i}{m_P R_i}, \qquad x = \frac{r}{R_i}, \quad (4.46)$$

with $F_q = 1$for $x \ge 1$. This generates an electrostatic energy density

$$\rho_E = \frac{Q^2(r)}{8\pi r^4} = \frac{m_P^2}{8\pi R_i^2}\frac{q_i^2 F_q^2(x)}{x^4}. \quad (4.47)$$

The solution for $S(r)$which goes from 0 at $r = 0$to 1 at R_i, is given by

$$\frac{S(xR_i)}{x^2} = 1 + \frac{4}{3}q_i^2[J(1) - J(x)],$$

$$J(x) = \int_0^x \mathrm{d}y \left(\frac{1}{y^3} - \frac{1}{x^3}\right)\frac{F_q^2(y)}{y^2}, \quad (4.48)$$

with $0 \le x \le 1$. The integrals are well behaved when the charge density ρ_q is finite at $r = 0$, so that $F_q(y) \sim y^3$ for $y \to 0$.

The solution rests on the insight that the zero point energy density of the quantum vacuum can act as a zero point battery or zero point storage, and locally absorb the energy density

$$\rho_\lambda = \frac{2S + 4rS' + r^2 S''}{32\pi G r^2}, \quad q_i^2 = \frac{3}{1 + 4I_q},$$
$$I_q = \int_0^1 \frac{\mathrm{d}x}{x^2} F_q^2(x). \quad (4.49)$$

For continuity with the vacuum, this must vanish at R_i, which fixes q_i. It holds that $s_i^- = R_i S'(R_i^-) = q_i^2 - 1$, which should be non-negative at this first crossing of $S = 1$ starting from $S = 0$ at $r = 0$.

Continuity $s_i^- = s_i^+ = R_i S'(R_i^+) = 2(M_c R_e/q_i^2 - 1)$ sets

$$\frac{\mathcal{Q}_c}{M_c} = \frac{2q_i}{1 + q_i^2}, \quad R_i = \frac{2GM_c}{1 + q_i^2}. \quad (4.50)$$

With s_i between 0 and 2, $\mathcal{Q}_c/M_c$ ranges from $\frac{1}{2}\sqrt{3}$ to 1, that is to say, from quite charged to maximally charged.

It has been put forward that surface charge layers may be present on the outer side of the inner and event horizons [7].

The same idea of a nonuniform vacuum energy combined with electric fields has been applied to dark matter [8].

4.3 The exact solutions for the smeared-horizon observer

The connections (9), (3.30) lead to the transformation from r_{S}^μ to r_{sh}^μ given by $\partial r_{\mathrm{S}}^\mu/\partial r_{sh}^\nu = \delta_\nu^\mu + \alpha\delta_0^\mu\delta_\nu^1$, with the inverse $\partial r_{sh}^\mu/\partial r_{\mathrm{S}}^\nu = \delta_\nu^\mu - \alpha\delta_0^\mu\delta_\nu^1$, where $\alpha = \pm\sigma^2 S/(\sigma^2 + \bar{S}^2)N\bar{S}$. The Einstein tensor transforms as $G_{sh\,\nu}^{\mu} = (\partial r_{sh}^\mu/\partial r_{\mathrm{S}}^{\dot\mu})G^{\dot\mu}_{\dot\nu}(\partial r_{\mathrm{S}}^{\dot\nu}/\partial r_{sh}^\nu)$, which coincides with the diagonal G^μ_ν tensor of the Schwarzschild case, and contains an extra term $G^0_{sh\,1} = \alpha(G^0_{\ 0} - G^1_{\ 1})$, consistent with (4.35).

The class of exact solutions of Sect. 4.2 involves $N(r) = 1$ and hence $G^0_{\ 0} = G^1_{\ 1}$, so that the $G^0_{\ 1}$ term does not show up. But when ρ_ϑ and p_ϑ are nontrivial, so is $G^0_{\ 1}$. This relates already to the situation where thermal matter still involves a temperature $T = 0$, but the rest masses of the up and down quarks and electrons are accounted for. They bring deviations at the percent level, as shown in a numerical approach [7].

5 Einstein gravity as a field in flat space time

The aim of the present section is to consider the mass of the regularized metrics of previous section. When the stress energy tensor $T^{\mu\nu}$ is integrated over space, the conservation $T^{\mu\nu}{}_{;\nu}=0$ does not lead to a conserved quantity. This led Landau and Lifshitz to derive their pseudo tensor $\tau^{\mu\nu}$ representing the energy momentum tensor of the gravitational field itself [9]. However, it does not transform as a tensor. It has been pointed out that it becomes a proper object provided it is evaluated in Cartesian coordinates and transformed from there [10, 11]. This approach is based on an underlying Minkowski space time, in which Noether's theorem assures a properly stress energy tensor.

5.1 General approach

Gravitation can be described as a field (a "pudding") in a standard Minkowski space, so that fits naturally with the matter fields in the standard model of particle physics. We consider Cartesian and spherical coordinates,

$$\begin{aligned} &\mathrm{d}\sigma^2=\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu=\gamma_{\mu\nu}\mathrm{d}r^\mu\mathrm{d}r^\nu, \quad x^\mu=(t,\,\mathrm{x},\,\mathrm{y},\,\mathrm{z}),\\ &r^\mu=(t,\,r,\,\theta,\,\phi);\\ &\eta_{\mu\nu}=\mathrm{diag}(1,\,-1,\,-1,\,-1),\\ &\gamma_{\mu\nu}=\left(1,\,-1,\,-r^2,\,-r^2s_\theta^2\right), \end{aligned} \tag{5.51}$$

and, for some metric $g_{\mu\nu}$, the Riemann metric

$$\mathrm{d}s^2=g_{\mu\nu}\mathrm{d}r^\mu\mathrm{d}r^\nu. \tag{5.52}$$

With $g=\det(g_{\mu\nu})$ and $\gamma=\det(\gamma_{\mu\nu})$, the combinations

$$k^{\mu\nu}=\sqrt{\frac{g}{\gamma}}g^{\mu\nu}, \qquad k_{\mu\nu}=\sqrt{\frac{\gamma}{g}}g_{\mu\nu}, \tag{5.53}$$

act as tensor fields in flat space time. This also applies to the mantle of the BH, which for the Schwarzschild metric is the full interior, even though the role of physical time is played there by the radial parameter r.

One can define the "acceleration tensor" [11]

$$A^{\mu\nu}=\frac{1}{2}\left(k^{\mu\nu}k^{\alpha\beta}-k^{\mu\alpha}k^{\nu\beta}\right)_{:\alpha\beta}, \tag{5.54}$$

where the column denotes covariant differentiation in flat space with its Christoffel coefficients $\gamma^\mu{}_{\nu\rho}$ vanishing for Cartesian coordinates. The Einstein equations take the form

$$A^{\mu\nu}=8\pi G\Theta^{\mu\nu}, \quad \Theta^{\mu\nu}=\frac{g}{\gamma}T^{\mu\nu}+\theta^{\mu\nu}=\frac{g}{\gamma}(T^{\mu\nu}+t^{\mu\nu}). \tag{5.55}$$

$\Theta^{\mu\nu}$ is conserved in Minkowski space, $\Theta^{\mu\nu}{}_{:\nu}=0$; this condition coincides with $T^{\mu\nu}{}_{;\nu}=0$, the conservation of $T^{\mu\nu}$ in Riemann space. By eliminating $T^{\mu\nu}$ with use of the Einstein equations $G^{\mu\nu}=8\pi GT^{\mu\nu}$, one gets

$$8\pi G\theta^{\mu\nu}=A^{\mu\nu}-\frac{g}{\gamma}G^{\mu\nu}, \qquad 8\pi Gt^{\mu\nu}=\frac{\gamma}{g}A^{\mu\nu}-G^{\mu\nu}. \tag{5.56}$$

This expresses $t^{\mu\nu}$ in terms of the metric alone. It can be verified that the second order derivatives drop out, together with certain first-order derivatives, so that there remains only a bilinear expression in first order derivatives,

$$\begin{aligned} t^{\mu\nu}=\frac{1}{8\pi G}\Big(&\frac{1}{2}k^{\mu\nu}{}_{:\lambda}k^{\lambda\rho}{}_{:\rho}-\frac{1}{2}k^{\mu\lambda}{}_{:\lambda}k^{\nu\rho}{}_{:\rho}\\ &+\frac{1}{2}k_{\lambda\dot\lambda}k^{\rho\dot\rho}k^{\mu\lambda}{}_{:\rho}k^{\nu\dot\lambda}{}_{:\dot\rho}-\frac{1}{2}k_{\lambda\dot\lambda}k^{\mu\dot\mu}k^{\lambda\rho}{}_{:\dot\mu}k^{\nu\dot\lambda}{}_{:\rho}\\ &-\frac{1}{2}k_{\lambda\dot\lambda}k^{\nu\dot\nu}k^{\mu\lambda}{}_{:\rho}k^{\dot\lambda\rho}{}_{:\dot\nu}+\frac{1}{4}k_{\lambda\dot\lambda}k_{\rho\dot\rho}k^{\mu\dot\mu}k^{\nu\dot\nu}k^{\lambda\rho}{}_{:\dot\mu}k^{\dot\lambda\dot\rho}{}_{:\dot\nu}\\ &-\frac{1}{8}k_{\lambda\dot\lambda}k_{\rho\dot\rho}k^{\mu\dot\mu}k^{\nu\dot\nu}k^{\lambda\dot\lambda}{}_{:\dot\mu}k^{\rho\dot\rho}{}_{:\dot\nu}\\ &+\frac{1}{4}k^{\mu\nu}k_{\lambda\dot\lambda}k^{\lambda\rho}{}_{:\alpha}k^{\dot\lambda\alpha}{}_{:\rho}-\frac{1}{8}k^{\mu\nu}k_{\lambda\dot\lambda}k_{\rho\dot\rho}k^{\alpha\dot\alpha}k^{\lambda\rho}{}_{:\alpha}k^{\dot\lambda\dot\rho}{}_{:\dot\alpha}\\ &+\frac{1}{16}k^{\mu\nu}k_{\lambda\dot\lambda}k_{\rho\dot\rho}k^{\alpha\dot\alpha}k^{\lambda\dot\lambda}{}_{:\alpha}k^{\rho\dot\rho}{}_{:\dot\alpha}\Big). \end{aligned} \tag{5.57}$$

The fact that all second order derivatives could be collected in $A_{\mu\nu}$ arrives from absorbing the square-root factors in $k^{\mu\nu}=\sqrt{g/\gamma}\,g^{\mu\nu}$.

In this field, theoretic approach in Minkowski space, it is natural to identify $t^{\mu\nu}$ as the stress energy tensor of the gravitational field. In Cartesian coordinates it coincides with the Landau Lifshitz pseudo tensor. This clarifies its role: the Landau Lifshitz approach is correct in Cartesian coordinates; from them one can transform the results to any other coordinate systems. In the above approach, this is guaranteed by the covariant derivatives in flat space. The material stress energy tensor in Minkowski space is $(g/\gamma)T^{\mu\nu}$.

In the above cases, the determinants of the metrics are $\gamma=-r^4s_\theta^2$ and $g=-N^2r^4s_\theta^2$, which implies $k^{\mu\nu}=Ng^{\mu\nu}$. The mass (energy) of the metric is

$$\begin{aligned} &E=\int_{R^3}\mathrm{d}^3r\sqrt{-\gamma}\Theta^{00}=\int_{R^3}\mathrm{d}^3r\,r^2\sin\theta\,\frac{A^{00}}{8\pi G},\\ &\mathrm{d}^3r=\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi. \end{aligned} \tag{5.58}$$

For the Friedman metric in cosmology, it follows [11] that $\Theta^{00}=0$, so that, loosely speaking, "it costs no energy to create a universe". This is the ultimate free lunch, more ultimate than the effect of the cosmological constant (dark energy, inflation) alone, for which the energy cost is known to be compensated by the gain of work. Indeed, $\Theta^{00}=0$ holds also in the radiation and matter phases.

5.2 The mass experienced by the smeared-horizon observer

General relativity allows various identifications of mass, in particular the far-field mass experienced by a Newtonian observer. But a complete theory must deal with the near field and behaviour in the interior, and show that all is well there—if it is. As we demonstrate now, this goal is reached for our class of smooth exact solutions observed by the smeared horizon observer.

For the metric (10), Θ^{00} is equal for the ish and osh, reading

$$\Theta^{00} = \frac{\sigma^6(S+rS')+\sigma^4[S(S-3)\bar{S}-rS'(3S^2+2S-3)]}{8\pi Gr^2(\sigma^2+\bar{S}^2)^3} - \frac{3\sigma^2\bar{S}^2(S\bar{S}-rS')+\bar{S}^4(S\bar{S}-rS')}{8\pi Gr^2(\sigma^2+\bar{S}^2)^3}, \quad (5.59)$$

independent of N. It is seen that σ regularizes the $1/\bar{S}^2$ term, that is, the poles at the horizons. The other nontrivial elements of $\Theta^{\mu}{}_{\nu}$, which is diagonal, are independent of σ,

$$\Theta^1{}_1 = N\frac{N(rS'+S)+2rN'\bar{S}}{8\pi Gr^2},$$
$$\Theta^2{}_2 = \Theta^3{}_3 = \frac{N^2(rS''+2S')+2rN'^2\bar{S}+2N(rN''\bar{S}+N'(1+2\bar{S}+2rS'))}{6\pi Gr}. \quad (5.60)$$

The Einstein equations yield for the material (non-gravitational) energy density

$$T^{00} = \frac{\sigma^2+\bar{S}^2+\sigma^2 S}{8\pi Gr^2N^2(\sigma^2+\bar{S}^2)^2}(\sigma^2-\bar{S})(S+rS'). \quad (5.61)$$

The gravitational energy density $t^{00}=\Theta^{00}/N^2-T^{00}$ is proportional to $1/N^2$and reads

$$t^{00} = -\frac{\bar{S}^5SS'}{2\pi GrN^2(\bar{S}^2+\sigma^2)^3} - \frac{S(\bar{S}^2+S)+r(\bar{S}^2-7S)S'}{8\pi Gr^2N^2(\bar{S}^2+\sigma^2)^2}\bar{S}^3 + \frac{S(\bar{S}^2+S)+r(\bar{S}^2-S)S'}{4\pi Gr^2N^2(\bar{S}^2+\sigma^2)}\bar{S} - \frac{1+S^2+rSS'}{8\pi Gr^2N^2}. \quad (5.62)$$

The definition of $\Theta^{\mu\nu}$ imposes that Θ^{00} is a total derivative. Indeed,

$$\Theta^{00} = \frac{1}{4\pi r^2}\frac{\mathrm{d}}{\mathrm{d}r}\frac{\sigma^4-2\sigma^2\bar{S}-\bar{S}^3}{(\sigma^2+\bar{S}^2)^2}\frac{rS}{2G}. \quad (5.63)$$

It leads to the integral

$$P^0(r) = \int_0^r \mathrm{d}r\, 4\pi r^2\,\Theta^{00} = \frac{\sigma^4-2\sigma^2\bar{S}-\bar{S}^3}{(\sigma^2+\bar{S}^2)^2}\frac{rS}{2G}. \quad (5.64)$$

The origin $r=0$did not contribute, since $S\sim r^2$ in our regularized approach. Actually, the $r\to 0$ value of (5.64) vanishes even in the Schwarzschild case $S=2GM/r$and the RN case $S=2GM/r-GQ^2/r^2$.

The important finding is that the quadratic divergencies of (5.63) in the $\sigma=0$ case, at the $\bar{S}=0$ locations of the inner and event horizons, are regulated by any finite σ, so that there is no longer a nasty “integration across the poles”.

With regularized $S(r)\to 0$ at $r\to 0$, the mass is $E=P^0(\infty)$; since $S=2GM/r-GQ^2/r^2$ for $r>R_e$, this yields

$$E = M \text{ for all } \sigma. \quad (5.65)$$

For the Schwarzschild metric, the mass M is in the field theoretic approach determined by the gravitational field alone, since $T^{00}=0$ implies that its energy density equals $t^{00}=\Theta^{00}$, while the singularity at $r\to 0$ is put under the rug. In terms of $\tilde{r}=r/2GM$ it reads when regularized by σ

$$t^{00} = \frac{2\tilde{r}-1-3(2+\sigma^2)\tilde{r}^2+2(2+3\sigma^2+2\sigma^4)\tilde{r}^3-(1+3\sigma^2+2\sigma^4)\tilde{r}^4}{(8\pi Gr^2)\,[(\tilde{r}-1)^2+\sigma^2\tilde{r}^2]^3} = \frac{1}{4\pi r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left\{\frac{(\bar{r}-1)^2+2\sigma\bar{r}^2}{[(\bar{r}-1)^2+\sigma^2\bar{r}^2]^2}(\bar{r}-1)M\right\} \quad (5.66)$$

That the integral is regular and finite, yielding M as it should be, cannot hide that Θ^{00} and t^{00} have an $1/r^2$ divergence at the origin, pointing at singular behavior of the Schwarzschild metric.

Equations (5.63), (5.61), (5.62) and (5.66) show that the smeared horizon observer encounters for small σ a smeared, non-sharp horizon, and only a true one when the limit $\sigma\to 0$ is taken first. The mass is thus well defined and takes the far field value M for any finite value of σ.

6 Outlook

The class of smeared horizon observers which are introduced in this papers allow for a complete and singularity-free description of black holes. On one hand, the exact solutions of Ref. [7] have been carried over without effort to these new observers; on the other hand, their singularities at the inner and event horizons in the field theoretic description of gravitation, get smeared, so that the energy density is finite everywhere and the mass of the black hole well defined.

These smeared horizon observers keep some peculiarities: for the ingoing observer, infalling shells cross the horizon in a finite time, but outgoing shells need infinite time. For the outgoing observer, outgoing shells emerge in finite time, but infalling ones need an infinite time.

This puts forward the possibility to see matter falling into the core of a black hole in a finite time by an ingoing smeared horizon observer; when this shell is next repelled by exerting some force on it, and made to go outwards, and, likewise, the observer is modified into an outgoing smeared horizon observer, he is capable to see the shell emerging in a finite time. A perhaps simpler setup is to investigate, for this type of observation, the geodesic motion of a point particle which enters the black hole mantle and next the core, turns around the origin, and goes out again.

Another open question is the form of Hawking radiation for smeared-horizon observers.

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