Rotating vector solving method applied for nonlinear oscillator

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Abstract Significant number of procedures for solving of the finite degree-of-freedom forced nonlinear oscillator are developed. For all of them it is common that they are based on the exact solution of the corresponding linear oscillator. For technical reasons, the aim of this paper is to develop a simpler solving procedure. The rotating vector method, developed for the linear oscillator, is adopted for solving of the nonlinear finite degree-of-freedom oscillator. The solution is assumed in the form of trigonometric functions. Assuming that the nonlinearity is small all terms of the series expansion of the function higher than the first are omitted. The rotating vectors for each mass are presented in the complex plane. In the paper, the suggested rotating vector procedure is applied for solving of a three-degree-of-freedom periodically excited oscillator. The influence of the nonlinear stiffness of the flexible elastic beam, excited with a periodical force, on the resonant properties of the system in whole is investigated. It is obtained that the influence of nonlinearity on the amplitude and phase of vibration is more significant for smaller values of the excitation frequency than for higher ones.

1 Introduction

The problem of the many degree-of-freedom (dof) linear oscillators with an external excitation is known for a long time. It is considered in almost all textbooks in theory of oscillation (see for example [1,2]). A system of n masses is connected with linear elastic springs and dashpots with linear damping properties. One mass is harmonically excited. Motion of the system is modeled with coupled second-order differential equations with constant coefficients and the solution is obtained in analytical form. It is found that for the resonance case the particular solutions of the system are harmonic functions with frequency which corresponds to the excitation one, while the amplitude and phase angle of masses differ.

However, the most of oscillators are nonlinear. Then, the mathematical model of many degree-of-freedom oscillator is a system of coupled second-order nonlinear differential equations. To obtain the analytic solution for these equations is not an easy task. Various solving methods for the two mass systems with two degrees of freedom [3–9] are developed (see [10]): the energy balance method [11], the He's frequency–amplitude procedure [12], the He's variation approach [13], the higher order Hamiltonian approach [14] etc. For all of them it is common that the result is unable to give an exact repre-

sentation in the complex plane of the dynamic balance between inertia, stiffness, damping and external forces [15].

To overcome the lack of the previously mentioned methods, in this paper the rotating vector solving method, developed for the one degree-of-freedom oscillator, is extended for solving of multi degree-of-freedom systems.

The paper has four sections. In Sect. 2, the n -mass system with n degrees of freedom are physically and mathematically modeled. In Sect. 3 the rotating vector method based on the dynamic balance of internal and external forces is extended for solving n degrees of freedom systems. The relative amplitude and phase angles for bodies in oscillator are calculated. The procedure is applied on vibration of a system modeled with three coupled differential equations (Sect. 4). The amplitude–frequency and phase–frequency relations are studied. The paper ends with Conclusion.

2 Model of the multi body and multi degree-of-freedom system

In Fig. 1 the multi body oscillator is plotted. Masses $m_1, m_2, \dots, m_n$ are connected with linear springs with rigidity s_l and dashpots with linear viscosity and coefficients of damping k_l where $l = 1, 2, \dots, n$. Connection of mass m_1 to the fixed body is nonlinear. The elastic

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force in the spring is a nonlinear deflection function, while the damping force is a nonlinear velocity function. The nonlinearity may be of any order given with an integer or non-integer not smaller than 1.

On the mass m_1 a periodical force $F(t) = F_0 \sin(\omega_g t)$ acts, where F_0 and ω_g are the amplitude and the frequency of the excitation force, respectively. Using the principle of D'Alembert, the oscillatory system is mathematically modeled as

$$m_1 \ddot{x}_1 + s_1 x_1 + k_1 \dot{x}_1 + s_\alpha x_1 |x_1|^{\alpha-1} + k_\beta \dot{x}_1 |\dot{x}_1|^{\beta-1} + s_2(x_1 - x_2) + k_2(\dot{x}_1 - \dot{x}_2) = F_0 \sin(\omega_g t), \quad (1)$$

$$m_l \ddot{x}_l + s_l(x_l - x_{l-1}) + k_l(\dot{x}_l - \dot{x}_{l-1}) + s_{l+1}(x_l - x_{l+1}) + k_{l+1}(\dot{x}_l - \dot{x}_{l+1}) = 0, \quad l = 2, 3, \dots, n-1 \quad (2)$$

$$m_n \ddot{x}_n + s_n(x_n - x_{n-1}) + k_n(\dot{x}_n - \dot{x}_{n-1}) = 0, \quad (3)$$

where $x_1, x_2, \dots, x_n$ are position functions, $s_1, s_2, \dots, s_n$ are coefficients of linear rigidity, $k_1, k_2, \dots, k_n$ are coefficients of linear damping, s_α is the coefficient of nonlinear rigidity and k_α is the coefficient of nonlinear damping. Due to nonlinearity the closed form analytical solution of the system does not exist.

However, for the linear system when $s_\alpha = 0$ and $k_\alpha = 0$, when the excitation force $F(t) = F_0 \sin(\omega_g t)$ has the amplitude F_0 and the frequency ω_g , the particular solution of (1–3) has the form

$$x_l = A_l \sin(\omega_g t + \varphi_l), \quad l = 1, 2, \dots, n \quad (4)$$

where A_l is the unknown amplitude of vibration, and φ_l is unknown phase angle. For that case, substituting (4) into (1–3) a system of n strong nonlinear algebraic equations are obtained

$$\begin{aligned} & -m_1 A_1 \omega_g^2 \sin(\omega_g t + \varphi_1) + s_1 A_1 \sin(\omega_g t + \varphi_1) \\ & -k_1 A_1 \omega_g \cos(\omega_g t + \varphi_1) \\ & + s_2(A_1 \sin(\omega_g t + \varphi_1) - A_2 \sin(\omega_g t + \varphi_1)) \\ & -k_2 \omega_g (A_1 \cos(\omega_g t + \varphi_1) - A_2 \cos(\omega_g t + \varphi_1)) \\ & = F_0 \sin(\omega_g t), \end{aligned} \quad (5)$$

$$\begin{aligned} & -m_l A_l \omega_g^2 \sin(\omega_g t + \varphi_l) + s_l(A_l \sin(\omega_g t + \varphi_l) \\ & - A_{l-1} \sin(\omega_g t + \varphi_{l-1})) - k_l \omega_g (A_l \cos(\omega_g t + \varphi_l) \\ & - A_{l-1} \cos(\omega_g t + \varphi_{l-1})) + s_{l+1}(A_l \sin(\omega_g t + \varphi_l) \\ & - A_{l+1} \sin(\omega_g t + \varphi_{l+1})) \\ & -k_{l+1} \omega_g (A_l \cos(\omega_g t + \varphi_l) \\ & - A_{l+1} \cos(\omega_g t + \varphi_{l+1})) = 0, \quad l = 2, 3, \dots, n-1 \end{aligned} \quad (6)$$

$$\begin{aligned} & -m_n A_n \sin(\omega_g t + \varphi_n) + s_n(A_n \sin(\omega_g t + \varphi_n) \\ & - A_{n-1} \sin(\omega_g t + \varphi_{n-1})) - k_n \omega_g (A_n \cos \\ & \times (\omega_g t + \varphi_n) - A_{n-1} \cos(\omega_g t + \varphi_{n-1})) = 0. \end{aligned} \quad (7)$$

Separating the terms with the same periodic functions (sine and cosine), Eqs. (5–7) transform into $2n$ algebraic equations with $2n$ unknown coefficients A_l and φ_l where $l = 1, 2, \dots, n$.

Using the exact solutions of (5–7), a solving procedure for the system (1–3) with small nonlinearity is developed.

3 Method of rotating vector

For simplification, let us rewrite the Eqs. (1–3) into

$$\begin{aligned} & m_1 \ddot{x}_1 + s_1 x_1 + k_1 \dot{x}_1 + s_\alpha x_1 |x_1|^{\alpha-1} \\ & + k_\beta \dot{x}_1 |\dot{x}_1|^{\beta-1} + \sum_{i=2}^n m_i \ddot{x}_i \\ & - F_0 \exp(i\omega_g t) = 0, \end{aligned} \quad (8)$$

$$\begin{aligned} & m_l \ddot{x}_l + s_l x_l + k_l \dot{x}_l - s_{l+1} x_{l+1} - k_{l+1} \dot{x}_{l+1} \\ & + \sum_{i=l+1}^n m_i \ddot{x}_i = 0, \quad l = 2, 3, \dots, n-1 \end{aligned} \quad (9)$$

$$\begin{aligned} & m_n \ddot{x}_n + s_n x_n + k_n \dot{x}_n \\ & - s_{n+1} x_{n+1} - k_{n+1} \dot{x}_{n+1} = 0, \end{aligned} \quad (10)$$

where $F_0 \exp(i\Omega t)$ is a periodical excitation force and $i = \sqrt{-1}$ is the imaginary unit. Introducing rotating vectors with constant amplitude A_s and phase θ_s :

$$x_l = A_l \exp(i(\omega_g t + \varphi_l)) \quad (11)$$

and corresponding time derivatives

$$\begin{aligned} \dot{x}_l &= A_l i \omega_g \exp(i(\omega_g t + \varphi_l)) \\ &= A_l \omega_g \exp\left(i\left(\omega_g t + \varphi_l + \frac{\pi}{2}\right)\right), \end{aligned} \quad (12)$$

$$\begin{aligned} \ddot{x}_l &= A_l \omega_g^2 i^2 \exp(i(\omega_g t + \varphi_l)) \\ &= A_l \omega_g^2 \exp(i(\omega_g t + \varphi_l + \pi)) \\ &= -A_l \omega_g^2 \exp(i(\omega_g t + \varphi_l)) \end{aligned} \quad (13)$$

Eqs. (8–10) transform into

$$\begin{aligned} & A_1 \exp(i(\omega_g t + \varphi_1)) \left(-m_1 \omega_g^2 + s_1 + k_1 \omega_g \exp\left(\frac{i\pi}{2}\right) \right) \\ & + s_\alpha A_1 \exp(i(\omega_g t + \varphi_1)) |A_1 \exp(i(\omega_g t + \varphi_1))|^{\alpha-1} \\ & + k_\beta A_1 \omega_g \exp\left(i\left(\omega_g t + \varphi_1 + \frac{\pi}{2}\right)\right) \\ & \left| A_1 \omega_g \exp\left(i\left(\omega_g t + \varphi_1 + \frac{\pi}{2}\right)\right) \right|^{\beta-1} \\ & - \sum_{i=2}^n m_i \omega_g^2 A_i \exp(i(\omega_g t + \varphi_i)) = F_0 \exp(i\omega_g t) \end{aligned} \quad (14)$$

$$\begin{aligned} & A_l \exp(i\varphi_l) \left(-m_l \omega_g^2 + s_l + k_l \omega_g \exp\left(\frac{i\pi}{2}\right) \right) \\ & - A_{l-1} \exp(i\varphi_{l-1}) \left(s_l - k_l \omega_g \exp\left(\frac{i\pi}{2}\right) \right) \\ & - \sum_{i=l+1}^n m_i A_i \omega_g^2 \exp(i\varphi_i) = 0, \quad s = 2, 3, \dots, n-1 \end{aligned} \quad (15)$$

Fig. 1 Model of the multi body oscillator

Fig. 2 Rotating vectors acting on the mass m_n

$$A_n \exp(i\varphi_n) \left(-m_n \omega_g^2 + s_n + k_n \omega_g \exp\left(\frac{i\pi}{2}\right) \right) - \exp(i\varphi_{n-1}) \left(s_n + k_n \omega_g \exp\left(\frac{i\pi}{2}\right) \right) = 0 \quad (16)$$

To solve the system of Eqs. (14–16) we start with the equation (16) which considers the motion of the mass m_n which is the last in the system.

3.1 Mass m_n

In Fig. 2 the rotating vectors for mass m_n in the $x-y$ plane complex plane are plotted. Namely, using the relation for the complex number $z = x + iy$, x and y represent the real and the imaginary value. The plane is assumed to rotate with angular velocity ω_g which corresponds to the excitation frequency.

In the $x-y$ plane, the relation (16) has five terms: one, which represents the inertial force of mass m_n , and two stiffness and two damping terms in the connection between masses m_n and the previous mass m_{n-1} .

For simplification, the last two terms of (16) are summarized and substituted with one force F_{Rn} with angle position ψ_n :

$$F_{Rn} = A_{n-1} \sqrt{s_n^2 + k_n^2 \omega_g^2}, \tan(\psi_n) = \frac{k_n \omega_g}{s_n}. \quad (17)$$

Introducing (17)₁ into (16) and projecting the relation on the displacement and velocity axis d_n and v_n of the mass m_n in the $x-y$ plane, we have

$$s_n A_n - m_n A_n \omega_g^2 - F_{Rn} \cos(\psi_n + \varphi_n) = 0, \quad (18)$$

Fig. 3 Rotating vectors of the mass m_l

$$k_n A_n \omega_g - F_{Rn} \sin(\psi_n + \varphi_n) = 0. \quad (19)$$

Rewriting (18) and (19) we obtain

$$\sqrt{s_n^2 + k_n^2 \omega_g^2} \cos(\psi_n + \varphi_n) = \frac{A_n}{A_{n-1}} K_n, \quad (20)$$

$$\sqrt{s_n^2 + k_n^2 \omega_g^2} \sin(\psi_n + \varphi_n) = \frac{A_n}{A_{n-1}} S_n, \quad (21)$$

where

$$K_n = s_n - m_n \omega_g^2, S_n = k_n \omega_g. \quad (22)$$

After some modification equations the amplitude ratio for masses m_{n-1} and m_n and the phase angle φ_n are obtained

$$\frac{A_n}{A_{n-1}} = q_n = \sqrt{\frac{s_n^2 + k_n^2 \omega_g^2}{K_n^2 + S_n^2}}, \quad \varphi_n = \tan^{-1} \left(\frac{S_n}{K_n} \right) - \tan^{-1} \left(\frac{k_n \omega_g}{s_n} \right) \quad (23)$$

3.2 Mass m_l where $l = 2, 3, \dots, n-1$

On any of mass m_l which is between the first m_1 and the last m_n mass the following forces act: inertial forces of all masses from m_n up to m_l , and the stiffness and damping forces in the connection between the mass m_l and m_{l-1} .

According to (15) the rotating vectors for m_l are plotted (see Fig. 3).

As it was suggested in the previous section, two vectors $A_{l-1} \exp(i\varphi_{l-1}) (-s_l)$ and $A_{l-1} \exp(i\varphi_{l-1} + \frac{i\pi}{2})$

$(-k_l\omega_g)$ in (15) are substituted with only one force. In Fig. 4a, the result of adding of two terms is plotted, i.e.,

$$F_{Rl} = A_{l-1}\sqrt{s_l^2 + k_l^2\omega_g^2}, \tan(\psi_l) = \frac{k_l\omega_g}{s_l}. \quad (24)$$

Using (24) the Eq. (15) is projected on the displacement and velocity axes, d_l and v_l of the mass m_l in the complex plane (see Fig. 3) and we obtain

$$A_l(-m_l\omega_g^2 + s_l) + F_{RL}\cos(\psi_l + \varphi_l) - \sum_{i=l+1}^n m_i A_i \omega_g^2 \cos\left(\sum_{j=l+1}^i \varphi_j\right) = 0, \quad (25)$$

$$A_l k_l \omega_g - F_{RL}\sin(\psi_l + \varphi_l) - \sum_{i=l+1}^n m_i A_i \omega_g^2 \sin\left(\sum_{j=l+1}^i \varphi_j\right) = 0$$

$$l = 2, 3, \dots, n-1 \quad (26)$$

Substituting (24) into (15) and (26) and after some modification it is

$$\sqrt{s_l^2 + k_l^2\omega_g^2}\cos(\psi_l + \varphi_l) = \frac{A_l}{A_{l-1}}K_l, \quad (27)$$

$$\sqrt{s_l^2 + k_l^2\omega_g^2}\sin(\psi_l + \varphi_l) = \frac{A_l}{A_{l-1}}S_l, \quad (28)$$

where

$$K_l = (-m_l\omega_g^2 + s_l) - \sum_{i=l+1}^n m_i \omega_g^2 \left(\prod_{j=l+1}^i q_j\right) \cos\left(\sum_{j=l+1}^i \varphi_j\right), \quad (29)$$

$$S_l = k_l \omega_g - \sum_{i=l+1}^n m_i \omega_g^2 \left(\prod_{j=l+1}^i q_j\right) \sin\left(\sum_{j=l+1}^i \varphi_j\right), \quad (30)$$

$$q_j = \frac{A_j}{A_{j-1}}, \quad (31)$$

Finally, the amplitude of vibration ratio for masses m_l and m_{l+1} and the phase angle φ_l are obtained

$$\frac{A_l}{A_{l-1}} = q_l = \sqrt{\frac{s_l^2 + k_l^2\omega_g^2}{K_l^2 + S_l^2}},$$

$$\varphi_l = \tan^{-1}\left(\frac{S_l}{K_l}\right) - \tan^{-1}\left(\frac{k_l\omega_g}{s_l}\right). \quad (32)$$

3.3 Mass m_1

Finally, the dynamic balance of forces for mass m_1 is given with equation (14). It is seen that the nonlinear terms are with angle position given with trigonometric functions $\cos(\omega_g t + \varphi_1)|\cos(\omega_g t + \varphi_1)|^{\alpha-1}$ and $\sin(\omega_g t + \varphi_1)|\sin(\omega_g t + \varphi_1)|^{\alpha-1}$ and also $\cos(\omega_g t + \varphi_1)|\cos(\omega_g t + \varphi_1)|^{\beta-1}$ and $\sin(\omega_g t + \varphi_1)|\sin(\omega_g t + \varphi_1)|^{\beta-1}$. Let us introduce the Fourier series for $|\cos(\omega_g t + \varphi_1)|^{\alpha-1}$ and $|\sin(\omega_g t + \varphi_1)|^{\alpha-1}$, i.e., $|\cos(\omega_g t + \varphi_1)|^{\beta-1}$ and $|\sin(\omega_g t + \varphi_1)|^{\beta-1}$ using the relation [16]

$$(\cos(\theta))^p = \sum_{i=0}^{\infty} a_i \cos(i\theta),$$

$$(\sin(\theta))^p = \sum_{i=0}^{\infty} b_i \sin(i\theta). \quad (33)$$

where

$$a_i = \left(\frac{4}{\pi}\right) \int_0^{\pi/2} (\cos(\theta))^p \cos(i\theta) d\theta,$$

$$b_i = \left(\frac{4}{\pi}\right) \int_0^{\pi/2} (\sin(\theta))^p \sin(i\theta) d\theta \quad (34)$$

and

$$\int_0^{\pi/2} (\cos(\theta))^p \cos(i\theta) d\theta = \frac{\pi}{2^{p+1}(p+1)B\left(\frac{p+1}{2}, \frac{p-i+1}{2}\right)},$$

where $B(p, q)$ is the beta function [10].

Assuming the first term of the series expansions (33) we have

$$\cos(\theta)|\cos(\theta)|^p = a_0 \cos(\theta) + \dots, \sin(\theta)|\sin(\theta)|^p = b_0 \sin(\theta) + \dots \quad (35)$$

Introducing the first approximation (35) into (14), we obtain the system of vectors rotating with angular velocity ω_g . In Fig. 4 the system after approximation is plotted.

Projecting the Eq. (14) on the axis d_1 and v_1 in complex plane (see Fig. 4) it is

$$A_1(-m_1\omega_g^2 + s_1) + a_0 s_a A_1 - \sum_{i=2}^n m_i \omega_g^2 A_i \cos\left(\sum_{j=2}^i \varphi_j\right) = F_0 \cos(\varphi_1)$$

$$A_1 k_1 \omega_g + b_0 k_b A_1 \omega_g \quad (36)$$

Fig. 4 Rotating vectors of the mass m_1 in complex plane

$$-\sum_{i=2}^n m_i \omega_g^2 A_i \sin \left(\sum_{j=2}^i \varphi_j \right) = F_0 \sin(\varphi_1) \quad (37)$$

i.e.,

$$\begin{aligned} A_1 K_1 &= F_0 \cos(\varphi_1), \\ A_1 S_1 &= F_0 \sin(\varphi_1) \end{aligned} \quad (38)$$

where

$$K_1 = (-m_1 \omega_g^2 + s_1) + a_0 s_\alpha - \sum_{i=2}^n m_i \omega_g^2 \left(\prod_{j=2}^i q_j \right) \cos \left(\sum_{j=2}^i \varphi_j \right), \quad (39)$$

$$S_1 = [k_1 \omega_g + b_0 k_\beta \omega_g - \sum_{i=2}^n m_i \omega_g^2 \left(\prod_{j=2}^i q_j \right) \sin \left(\sum_{j=2}^i \varphi_j \right)] \quad (40)$$

Finally, according to (38) the approximate solution for the amplitude and phase of vibration of the mass 1 is

Using the values (41) and substituting into (32) and (23) amplitudes and phases of vibration of masses $2, 3, \dots, n$ are obtained.

4 Vibration of the three-degree-of freedom system

Let us consider the three mass oscillatory system with three degrees of freedom. The elastic and damping properties of the spring and dashpot connected with the mass m_1 are nonlinear. The order of nonlinearity is assumed to be $3/2$. Connection between masses m_2 and m_3 are linear. On the mass m_1 the excitation force $F_0 \sin(\omega_g t)$ acts, where F_0 and ω_g are amplitude and frequency, respectively. Motion of the three mass system is described with coupled weakly nonlinear differential equations

$$\begin{aligned} m_1 \ddot{x}_1 + s_1 x_1 + k_1 \dot{x}_1 + s_\alpha x_1 |x_1|^{1/2} \\ + k_\beta \dot{x}_1 |\dot{x}_1|^{1/2} + s_2 (x_1 - x_2) + k_2 (\dot{x}_1 - \dot{x}_2) \\ = F_0 \sin(\omega_g t), \end{aligned} \quad (42)$$

$$\begin{aligned} m_2 \ddot{x}_2 + s_2 (x_2 - x_1) + k_2 (\dot{x}_2 - \dot{x}_1) + s_3 (x_2 - x_3) \\ + k_3 (\dot{x}_2 - \dot{x}_3) = 0, \end{aligned} \quad (43)$$

$$\begin{aligned} m_3 \ddot{x}_3 + s_3 (x_3 - x_2) \\ + k_3 (\dot{x}_3 - \dot{x}_2) = 0, \end{aligned} \quad (44)$$

where s_i are coefficients of linear stiffness, $s_\alpha \ll 1$ is the coefficient of small nonlinear stiffness, k_i are coefficients of linear damping, k_β is the coefficient of small nonlinear damping and $i = 1, 2, 3$. For the system with weak nonlinearities, the aforementioned approximate solving method of the problem is applied.

Thus, the relative amplitudes of vibration and the phase angles are due to (21) and (32)

$$\begin{aligned} \frac{A_3}{A_2} &= \sqrt{\frac{s_3^2 + k_3^2 \omega_g^2}{(s_3 - m_3 \omega_g^2)^2 + k_3^2 \omega_g^2}} = q_3, \\ \varphi_3 &= \tan^{-1} \left(\frac{k_3 \omega_g}{s_3 - m_3 \omega_g^2} \right) - \tan^{-1} \left(\frac{k_3 \omega_g}{s_3} \right), \end{aligned} \quad (45)$$

and

$$\begin{aligned} \frac{A_2}{A_1} &= \sqrt{\frac{s_2^2 + k_2^2 \omega_g^2}{(s_2 - m_2 \omega_g^2 - m_3 q_3 \omega_g^2 \cos(\varphi_3))^2 + (k_2 \omega_g + m_3 q_3 \omega_g^2 \sin(\varphi_3))^2}} = q_2, \\ \varphi_2 &= \tan^{-1}(Q_2) - \tan^{-1} \left(\frac{k_2 \omega_g}{s_2} \right), \end{aligned} \quad (46)$$

$$\begin{aligned} A_1 &= \frac{F_0}{\sqrt{K_1^2 + S_1^2}}, \\ \varphi_1 &= \tan^{-1} \left(\frac{S_1}{K_1} \right) \end{aligned} \quad (41)$$

where

$$Q_2 = \tan(\psi_2 + \varphi_2) = \frac{k_2\omega_g + m_3\omega_g^2 q_3 \sin(\varphi_3)}{s_2 - m_2\omega_g^2 - m_3\omega_g^2 q_3 \cos(\varphi_3)}. \quad (47)$$

Introducing the frequency ω_i damping ζ_i and mass μ parameters

$$\omega_i = \sqrt{\frac{s_i}{m_i}}, \quad \zeta_i = \sqrt{\frac{k_i^2}{m_i s_i}}, \quad \mu_{i-1} = \frac{m_i}{m_{i-1}},$$

$$s = \frac{s_\alpha}{s_1}, \quad \zeta = \sqrt{\frac{k_\beta^2}{m_1 s_1}}, \quad (48)$$

where $i = 1, 2, 3$, the amplitude and phase relations (45) and (46) with (47) transform into

$$\frac{A_3}{A_2} = \sqrt{\frac{1 + \zeta_3^2 \left(\frac{\omega_g^2}{\omega_3^2}\right)}{\left(1 - \left(\frac{\omega_g^2}{\omega_3^2}\right)\right)^2 + \zeta_3^2 \left(\frac{\omega_g^2}{\omega_3^2}\right)}}, \quad (49)$$

$$\varphi_3 = \tan^{-1} \left(\frac{\zeta_3 \left(\frac{\omega_g}{\omega_3}\right)}{1 - \left(\frac{\omega_g^2}{\omega_3^2}\right)} \right) - \tan^{-1} \left(\frac{\zeta_3 \omega_g}{\omega_3} \right), \quad (50)$$

and the frequency ratio is smaller than 1. In Fig. 5b it is shown that the phase angle is increasing with the damping coefficient ζ_3 . For the case when damping is omitted and $\zeta_3 = 0$, the phase angle is zero.

In Fig. 6 the ratio of the amplitude of vibration A_2 of the mass m_2 to amplitude of vibration of A_1 of the mass m_1 as the function of the excitation frequency ω_g (51) is plotted. The A_2/A_1 amplitude ratio is shown for the resonant value of the amplitude A_3 when $\omega_g/\omega_3 = 0.995$ and $\zeta_3 = 0.5$. The diagrams are plotted for various damping parameter ζ_2 . For the same parameter values the phase angle φ_2 as functions of ω_g/ω_2 , for various damping parameter ζ_2 (52) is plotted in Fig. 6b. It is obtained that the resonant amplitude decreases with increase of the damping parameter ζ_2 . The resonant amplitude moves toward smaller values of ω_g by increasing of ζ_2 . Even the coefficient ζ_2 is zero, the phase angle exist.

Substituting (35) into (42) it is seen that the mass m_1 is under influence of linear and nonlinear forces. Assuming the first term of Fourier transformation [16] i.e., $\cos(\theta) |\cos(\theta)|^{1/2} = 0.71317 \cos(\theta)$ and $\sin(\theta) |\sin(\theta)|^{1/2} = 0.71317 \cos(\theta)$ and neglecting the other terms as small values, the approximate values for the relations (39) and (40) are obtained

$$K_1 = s_1 - m_1 \omega_g^2 + 0.71317 s_\alpha - m_2 \omega_g^2 q_2 \cos(\varphi_2) - m_3 \omega_g^2 q_2 q_3 \cos(\varphi_2 + \varphi_3), \quad (53)$$

$$\frac{A_2}{A_1} = \sqrt{\frac{1 + \zeta_2^2 \left(\frac{\omega_g^2}{\omega_2^2}\right)}{\left(1 - \left(\frac{\omega_g^2}{\omega_2^2}\right) - \mu_2 q_3 \left(\frac{\omega_g^2}{\omega_2^2}\right) \cos(\varphi_3)\right)^2 + \left(\frac{\omega_g}{\omega_2} \zeta_2 + \mu_2 q_3 \left(\frac{\omega_g^2}{\omega_2^2}\right) \sin(\varphi_3)\right)^2}}, \quad (51)$$

$$\varphi_2 = \tan^{-1}(Q_2) - \tan^{-1} \left(\frac{\omega_g}{\omega_2} \zeta_2 \right),$$

$$Q_2 = \tan(\psi_2 + \varphi_2) = \frac{\frac{\omega_g}{\omega_2} \zeta_2 + \mu_2 \left(\frac{\omega_g^2}{\omega_2^2}\right) q_3 \sin(\varphi_3)}{1 - \left(\frac{\omega_g^2}{\omega_2^2}\right) - \mu_2 \left(\frac{\omega_g^2}{\omega_2^2}\right) q_3 \cos(\varphi_3)} \quad (52)$$

Using (49–52) the amplitude–frequency and phase–frequency diagrams are plotted.

In Fig. 5a the diagram of amplitude ratio A_3/A_2 (49) and in Fig. 5b of the phase angle φ_3 (50) as function of ω_g/ω_3 , for various damping parameter ζ_3 are plotted.

In Fig. 5a the relation between the amplitude of vibration A_3 of the mass m_3 and the amplitude A_2 of the mass m_2 depending on the excitation frequency ω_g is plotted. It is obvious that the resonant frequency and amplitude depend on the damping coefficient ζ_3 : the higher is the damping the resonant amplitude is lower

Using the dimensionless parameters (48) it is

$$K_1 = m_1 \omega_1^2 K, \quad S_1 = m_1 \omega_1^2 S, \quad (55)$$

where

$$K = 1 + 0.71317 s - \frac{\omega_g^2}{\omega_1^2} - \mu_1 \frac{\omega_g^2}{\omega_1^2} q_2 \cos(\varphi_2) - \mu_1 \mu_2 \frac{\omega_g^2}{\omega_1^2} q_2 q_3 \cos(\varphi_2 + \varphi_3), \quad (56)$$

Fig. 5 a $A_3/A_2 - \omega_g/\omega_3$ diagrams for various ζ_3 ; b $\varphi_3 - \omega_g/\omega_3$ diagrams for various ζ_3

Fig. 6 a $A_2/A_1 - \omega_g/\omega_2$ diagrams for various ζ_2 ; b $\varphi_2 - \omega_g/\omega_2$ diagrams for various ζ_2

$$S = \zeta_1 \frac{\omega_g}{\omega_1} + 0.71317 \zeta \frac{\omega_g}{\omega_1} - \mu_1 \frac{\omega_g^2}{\omega_1^2} q_2 \sin(\varphi_2) - \mu_1 \mu_2 \frac{\omega_g^2}{\omega_1^2} q_2 q_3 \sin(\varphi_2 + \varphi_3). \quad (57)$$

Substituting (55) into (41) the amplitude and phase relations are obtained

$$A_1 = \frac{F_0}{m_1 \omega_1^2 Q}, \quad (58)$$

$$\varphi_1 = \tan^{-1} \left(\frac{S}{K} \right), \quad (59)$$

where

$$Q = \sqrt{K^2 + S^2}. \quad (60)$$

In Figs. 7 and 8, the $\frac{F_0}{m_1 \omega_1^2 A_1} = Q$ diagrams, as the function of the frequency ratio $\frac{\omega_g}{\omega_1}$ for various values of nonlinear stiffness s and damping ζ_1 , are plotted. In Fig. 7a the diagrams of ratio of the resonant amplitude of vibration of the flexible beam A_1 and the amplitude

of excitation F_0 i.e. $1/Q$, as the function of the excitation frequency for various damping parameters are plotted. For $\omega_g/\omega_2 = 0.52$ and $\zeta_2 = 0.5$ maximum of the relation $1/Q$ for $\zeta_1 = 0$ is 1 at $\omega_g/\omega_1 = 0.55$ and the maximum increases with ζ_1 . The phase angle has the tendency of decrease with ω_g/ω_1 .

In Fig. 8 the influence on the nonlinear rigidity on the vibration properties of the system is shown. Increasing the coefficient of the nonlinear stiffness the maximal amplitude of vibration decreases and its position tends to $\omega_g/\omega_1 = 1$ (Fig. 8a). In Fig. 8b the phase angle variation according to nonlinear property is shown. Maximum of the phase angle does not depend on s , but its position tends to move from $\omega_g/\omega_1 = 1$ with increasing of s . It is worth to be said that the influence of nonlinear rigidity on the amplitude and phase characteristics can be omitted for $\omega_g/\omega_1 \gg 1$.

Substituting (58) into (50) and (49), we obtain

$$A_2 = \frac{F_0 q_2}{m_1 \omega_1^2 Q}, \quad (61)$$

$$A_3 = \frac{F_0 q_2 q_3}{m_1 \omega_1^2 Q} \quad (62)$$

Fig. 7 **a** $1/Q - \omega_g/\omega_1$ diagrams for various ζ_1 ; **b** $\varphi_1 - \omega_g/\omega_1$ diagrams for various ζ_1

The relations (61) represent the approximate amplitudes of vibration of masses m_2 and m_3 according to the amplitude of vibration of the mass m_1 . In the expressions (61), the influence of the nonlinear stiffness property is incorporated by correcting the amplitudes according to A_1 . However, the influence of the phase angle of the first mass has negligible influence on the corresponding phase angles of the masses m_2 and m_3 .

Numerical example

Let us consider a system with three equal masses $m_1 = m_2 = m_3 = 1$ g, with equal linear stiffness coefficients $s_1 = s_2 = s_3 = 1$ g/s² and without damping. The equations of oscillatory motion are

$$\ddot{x}_1 + x_1 + s_\alpha x_1 |x_1|^{1/2} + (x_1 - x_2) = \cos(2t), \quad (63)$$

$$\ddot{x}_2 + (x_2 - x_1) + (x_2 - x_3) = 0, \quad (64)$$

$$\ddot{x}_3 + (x_3 - x_2) = 0, \quad (65)$$

where the excitation amplitude and frequency are $F_0 = 1$ g cm/s² and $\omega_g = 2$ s⁻¹. The equations (63–65) are solved analytically and numerically. In Fig. 9 the $x_1 - t$,

Fig. 8 **a** $1/Q - \omega_g/\omega_1$ diagrams for various s ; **b** $\varphi_1 - \omega_g/\omega_1$ diagrams for various s

$x_2 - t$ and $x_3 - t$ diagrams for masses m_1 , m_2 and m_3 are numerically obtained. Using the analytical procedure suggested in this paper and the relations (58), (61) and (62) the steady-state amplitudes A_1 , A_2 and A_3 are computed. The calculated values are: $A_1 = 0.71439$ cm, $A_2 = 0.42857$ cm and $A_3 = 0.14286$ cm.

Comparing the analytical and numerical results, it is seen that the difference is negligible. The analytical solutions are on the top of the numerical ones.

5 Conclusion

The approximate method based on rotating vectors for solving multi-degree-of-freedom nonlinear oscillatory system is developed. The presented methodology enable representation in a complex plane of rotating vectors which correspond to each mass of the system. The dynamic balance between internal and external forcing for each mass of the oscillator yields the fundamental resonance of the linear and damped system. The nonlinearity in the system is introduced as the cor-

Fig. 9 **a** $x_1 - t$ diagram (full line) and $A_1 - t$ (dotted line) for m_1 ; **b** $x_2 - t$ diagram (full line) and $A_2 - t$ (dotted line) for m_2 ; **c** $x_3 - t$ diagram (full line) and $A_3 - t$ (dotted line) for m_3

rection to linear amplitudes of vibration. The suggested method is the generalization of the well-known procedure developed for a linear one degree-of-freedom oscillator. A difference in the multi degree of freedom system compared to one degree-of-freedom oscillator is that the force vector acts indirectly on all of masses in the system and the motion of each next mass in the system depends on the properties of the previous ones. The method is applied for a special case of a three mass

system with three degrees of freedom. The effect of nonlinearity on the vibration property of each of three mass is evident. It is concluded that the effect of nonlinearity on the amplitude of vibration decreases with higher values of the excitation frequency.

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