



Solution for cosmological observables in the Starobinsky model of inflation

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Abstract This paper focuses on the Starobinsky model of inflation in the Einstein frame. We derive solutions for various cosmological observables, such as the scalar spectral index n_s , the tensor-to-scalar ratio r and their runnings, as well as the number of e -folds of inflation, reheating, and radiation, with minimal assumptions. The impact of reheating on inflation is explored by constraining the equation of state parameter ω_{re} at the end of reheating. An equation linking inflation with reheating is established, which is solved for the spectral index n_s . Using consistency relations of the model, we determine the other observables while the number of e -folds during inflation N_k , and the number of e -folds during reheating N_{re} are determined by their respective formulas involving n_s . We find remarkable agreement between the Starobinsky model and current measurements of the power spectrum of primordial curvature perturbations and the present bounds on the spectrum of primordial gravitational waves.

1 Introduction

The Starobinsky model of inflation, proposed by Starobinsky [1], is a geometric model that incorporates linear and quadratic terms of the scalar curvature in its action, distinguishing it from other inflationary models (for reviews on inflation, see e.g., [2–5]). By expressing the action in the Einstein frame, a scalar field potential emerges, aligning the model with typical inflation models. However, despite being proposed more than 40 years ago, most of the literature has focused on determining ranges for cosmological quantities like the scalar spectral index n_s , tensor-to-scalar ratio r , and the number of e -folds during inflation, denoted by N_k . In this study, we adopt an approach that allows us to derive solutions for the observables, as well as the number of e -folds during inflation, reheating and radiation, with minimal assumptions. We impose reheating conditions on inflation and obtain equations that enable us to solve for the desired quantities (for reviews on reheating, see e.g., [6–8]). To understand the impact of reheating on inflation, we constrain the equation of state parameter (EoS) at the end of reheating, denoted as ω_{re} . By establishing a connection between inflation and reheating, we derive an equation that determines the scalar spectral index, n_s . Using consistency relations within the model, we determine the remaining observables. Furthermore, we calculate the number of e -folds during inflation, reheating, and radiation, denoted as N_k , N_{re} , and N_{rd} , respectively.

The Starobinsky model modifies Einstein's theory of general relativity by introducing additional terms in the action. The action consists of the Einstein-Hilbert term, which is proportional to the Ricci scalar R , and an extra term proportional to the square of the Ricci scalar, R^2 . The original Starobinsky model's action is given by

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{Pl}^2}{2} \left(R + \frac{1}{6M^2} R^2 \right) + L_m \right], \quad (1.1)$$

where g is the determinant of the metric tensor, and $M_{Pl} = 2.44 \times 10^{18}$ GeV is the reduced Planck mass. The parameter M is related to the energy scale of inflation, and L_m represents the Lagrangian density for matter fields. The inclusion of the R^2 term leads to modified equations of motion and a modified theory of gravity. During inflation, the R^2 term dominates over the Einstein-Hilbert term, resulting in exponential expansion. The R^2 term introduces a repulsive interaction that counteracts the attractive gravitational behavior, driving the accelerated expansion of the universe. This unique feature of the Starobinsky model provides a mechanism for inflation based solely on modifications to the gravitational sector, without the need for additional scalar fields. The dominance of the R^2 term during inflation has implications for the dynamics of gravity and the resulting accelerated expansion. It allows for a prolonged period of inflation, resolving significant cosmological puzzles such as the horizon problem and the flatness problem. This extended exponential expansion is responsible for the observed large-scale homogeneity and isotropy of the universe.

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The model, originally defined in the Jordan frame, is equivalent to a single-field model with an asymptotically flat potential when transformed through a conformal transformation to the Einstein frame. Additionally, by considering the Standard Model fields as minimally coupled to gravity in the Jordan frame, the transformation to the Einstein frame induces a coupling between these fields and the inflaton, which provides a natural mechanism for graceful exit and reheating [9–11].

The model in the Einstein frame is obtained through a conformal transformation applied to the metric. This transformation is given by

$$g_{\mu\nu} \rightarrow e^{\sqrt{\frac{2}{3}} \frac{\phi}{M_{Pl}}} g_{\mu\nu}. \quad (1.2)$$

Applying this transformation to the Starobinsky model, we obtain the action

$$S = \int d^4x \sqrt{-g} \left(\frac{M_{Pl}^2}{2} R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right). \quad (1.3)$$

Within this formulation, the potential of the scalar field takes the following form

$$V(\phi) = V_0 \left(1 - e^{-\sqrt{\frac{2}{3}} \frac{\phi}{M_{Pl}}} \right)^2, \quad (1.4)$$

where V_0 is defined as $\frac{3}{4} M_{Pl}^2 M^2$. By having the model expressed in the Einstein frame and identifying a scalar potential $V(\phi)$, we can employ the usual expressions for single-field inflation under the slow-roll approximation.

The Starobinsky model is remarkably consistent with current measurements of the power spectrum of primordial curvature fluctuations and the constraints on primordial gravitational waves. These measurements are derived from collaborations such as Planck and BICEP/Keck [12–14]. We consider the bounds provided by the Table 3 of [12] for the cosmological model Λ CDM+ r + $dn_s/d \ln k$ with the data set Planck TT,TE,EE+lowE+lensing +BK15+BAO. These bounds offer constraints on the parameters and observables within the specific cosmological model and data combination

$$n_s = 0.9658 \pm 0.0040 \quad (68\% \text{ C.L.}), \quad (1.5)$$

$$r < 0.068 \quad (95\% \text{ C.L.}). \quad (1.6)$$

The Starobinsky model predicts specific patterns in the anisotropies of the cosmic microwave background (CMB), including a characteristic damping scale in the power spectrum. Future observations from experiments such as the Simons Observatory [15] and the CMB-S4 [16] collaboration are anticipated to yield more accurate measurements of the CMB. The Simons Observatory comprises four telescopes positioned at an elevation of 5200 m in the Atacama Desert of Chile, while the CMB-S4 collaboration aims to enhance our understanding of key cosmological parameters by conducting precise measurements of the CMB's temperature and polarization. Specifically, the improved measurements of the CMB are expected to provide additional constraints on the Starobinsky model and other inflationary models, refining our understanding of the early universe and its evolution. These experiments aim to study the CMB with high sensitivity and precision using advanced detectors, larger arrays, multiple frequencies, improved angular resolution, and careful site selection. These experiments will provide valuable data to refine our understanding of the early universe, test inflationary models, and investigate fundamental physics.

The paper is organized as follows: Sect. 2 introduces the concept of reheating and establishes a general equation that connects inflation with reheating. This equation is then used to solve for n_s in the context of the Starobinsky model, as presented in Sect. 3, where we also provide definitions for relevant quantities. Additionally, we derive an expression for the number of e -folds during inflation in terms of the spectral index n_s . Furthermore, we employ the consistency relations of the model to determine other observables. The analysis also encompasses the calculation of the number of e -folds during inflation, reheating, and the radiation era. Finally, Sect. 4 provides the conclusion of our paper.

2 Reheating constraints

Models of inflation can be related to cosmological observables, which, to first order in the slow-roll (SR) approximation, are expressed as (see, for example, [3, 17])

$$n_t = -2\epsilon = -\frac{r}{8}, \quad (2.1)$$

$$n_s = 1 + 2\eta - 6\epsilon, \quad (2.2)$$

$$n_{sk} \equiv \frac{dn_s}{d \ln k} = 16\epsilon\eta - 24\epsilon^2 - 2\xi_2, \quad (2.3)$$

$$n_{tk} \equiv \frac{dn_t}{d \ln k} = 4\epsilon(\eta - 2\epsilon), \quad (2.4)$$

$$A_s = \frac{1}{24\pi^2 \epsilon} \frac{V}{M_{Pl}^4}. \tag{2.5}$$

Here, $M_{Pl} = 2.43568 \times 10^{18} \text{GeV}$ is the reduced Planck mass, r denotes the tensor-to-scalar ratio, n_s represents the scalar spectral index, n_{sk} its running (which is commonly denoted as α), n_t represents the tensor spectral index, and n_{tk} represents its running, in a self-explanatory notation. The amplitude of density perturbations at a particular wave number k is denoted by A_s . All quantities are evaluated at the moment of horizon crossing at wavenumber $k = 0.05/\text{Mpc}$. The SR parameters involved in the above expressions are

$$\epsilon \equiv \frac{M_{Pl}^2}{2} \left(\frac{V'}{V}\right)^2, \quad \eta \equiv M_{Pl}^2 \frac{V''}{V}, \quad \xi_2 \equiv M_{Pl}^4 \frac{V'V'''}{V^2}, \tag{2.6}$$

where primes on V denote derivatives with respect to the inflaton ϕ .

Expanding on earlier work [17–19], it is possible to derive an equation for the number of e -folds during reheating [20, 21] by relating the comoving Hubble scale wavenumber k at horizon crossing to the present scale wavenumber $k_0 = a_0 H_0$ as follows (also see [22, 23] for further details)

$$N_{re} = \frac{4}{1 - 3\omega_{re}} \left(-N_k - \frac{1}{3} \ln \left[\frac{11g_{s,re}}{43} \right] - \frac{1}{4} \ln \left[\frac{30}{\pi^2 g_{re}} \right] - \ln \left[\frac{k}{a_0 T_0} \right] - \frac{1}{4} \ln \left[\frac{\rho_e}{H_k^4} \right] \right). \tag{2.7}$$

In the above equation, the number of degrees of freedom of species at the end of reheating is denoted by g_{re} , while $g_{s,re}$ represents the entropy number of degrees of freedom after reheating. The energy density at the end of inflation is denoted by ρ_e , with a_0 and T_0 representing the scale factor and temperature today, respectively. The energy density above is model-dependent and can be expressed as $\rho_e = \frac{3}{2} V_e$. Here, V_e represents the potential of the model at the end of inflation, while H_k is the Hubble function at the comoving Hubble scale wavenumber k .

An expression for the number of e -folds during reheating, in terms of energy densities, can be obtained [20] by solving the fluid equation assuming a constant EoS ω_{re}

$$N_{re} = \frac{1}{3(1 + \omega_{re})} \ln \left[\frac{\rho_e}{\rho_{re}} \right] = \frac{1}{3(1 + \omega_{re})} \ln \left[\frac{\frac{3}{2} V_e}{\frac{\pi^2 g_{re}}{30} T_{re}^4} \right], \tag{2.8}$$

where T_{re} is the reheating temperature. From Eqs. (2.7) and (2.8) we get

$$N_k = -\frac{1 - 3\omega_{re}}{12(1 + \omega_{re})} \ln \left[\frac{45V_e}{\pi^2 g_{re} T_{re}^4} \right] - \frac{1}{3} \ln \left[\frac{11g_{s,re}}{43} \right] - \frac{1}{4} \ln \left[\frac{30}{\pi^2 g_{re}} \right] - \ln \left[\frac{k}{a_0 T_0} \right] - \frac{1}{4} \ln \left[\frac{27V_e M_{Pl}^4}{2V_k^2} \right], \tag{2.9}$$

where $V_k \equiv V(\phi_k)$ is the potential at the comoving Hubble scale wavenumber k . We can express the potential as $V(\phi) = V_0 f(\phi)$, where V_0 represents the overall scale and $f(\phi)$ contains all the terms of the potential that depend on ϕ . This choice does not introduce any loss of generality. Equation (2.9) simplifies as follows

$$N_k = \ln \left(\frac{\left(\frac{43}{11g_{s,re}}\right)^{1/3} g_{re}^{1/4} \sqrt{\pi} a_0 T_0 \left(\frac{\pi^2 g_{re} T_{re}^4}{45M_{Pl}^4}\right)^{\frac{1-3\omega_{re}}{12(1+\omega_{re})}} \left(\frac{V_0}{M_{Pl}^4}\right)^{\frac{1+3\omega_{re}}{6(1+\omega_{re})}} \frac{f^{1/2}(\phi_k)}{f^{\frac{1}{3(1+\omega_{re})}}(\phi_e)} \right). \tag{2.10}$$

We can further simplify Eq. (2.10) eliminating V_0 . By using Eq. (2.5) we can solve for V_0 in terms of ϕ_k and the amplitude of scalar perturbations A_s at horizon crossing

$$V_0 = \frac{12A_s \pi^2 (f'(\phi_k) M_{Pl})^2}{f(\phi_k)^3} M_{Pl}^4, \tag{2.11}$$

where $f'(\phi_k)$ is the derivative of $f(\phi)$ with respect to ϕ evaluated at $\phi = \phi_k$. Finally, Eq. (2.10) can be written as

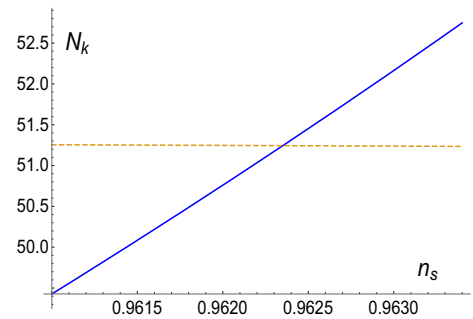
$$N_k = T_1 + \frac{1}{3(1 + \omega_{re})} \ln \left(\frac{(f'(\phi_k) M_{Pl})^{1+3\omega_{re}}}{f(\phi_k)^{3\omega_{re}} f(\phi_e)} \right), \tag{2.12}$$

where the term T_1 is given by

$$T_1 = \ln \left(\frac{\left(\frac{43}{11g_{s,re}}\right)^{1/3} g_{re}^{1/4} \sqrt{\pi} a_0 T_0 \left(\frac{\pi^2 g_{re}}{45}\right)^{\frac{1-3\omega_{re}}{12(1+\omega_{re})}} (12A_s \pi^2)^{\frac{1+3\omega_{re}}{6(1+\omega_{re})}} \right) + \frac{1 - 3\omega_{re}}{3(1 + \omega_{re})} \ln \frac{T_{re}}{M_{Pl}}. \tag{2.13}$$

We immediately notice that, for $\omega_{re} = 1/3$, T_1 (and N_k) is independent of T_{re} . Equation (2.12) is a general equation connecting reheating and inflation, valid for any single field potential and any ω_{re} . In the following section we show how to effectively use this approach by applying it to the Starobinsky potential.

Fig. 1 The solid (blue) curve represents the lhs of Eq. (3.4) while the dashed (orange) curve represents the slowly varying function of the rhs of the same equation. For the Starobinsky model in the Einstein frame defined by Eq. (3.1) we approximate the EoS by $\omega_{re} = 0$, and $T_{re} = 3.1 \times 10^9$ GeV [11]. The intersection point gives us the value $n_s = 0.96235$ for the scalar spectral index and, from the consistency relations for the Starobinsky model given by Eqs. (2.1), (3.6), (3.7), and (3.8), we obtain the other observables



3 The Starobinsky model, observables and the number of e-folds

The potential for the Starobinsky model in the Einstein frame is given by

$$V(\phi) = V_0 \left(1 - e^{-\sqrt{\frac{2}{3}} \frac{\phi}{M_{Pl}}} \right)^2. \tag{3.1}$$

By expressing the model in the Einstein frame and identifying a scalar potential $V(\phi)$, we can use standard expressions for slow-roll single-field inflation. This enables us to establish relationships between cosmological observables such as r , n_{sk} , n_{tk} , and the scalar spectral index n_s . Here, n_{sk} represents the running of the spectral index, while n_{tk} denotes the tensor running.

To find relations between n_s and the rest of the quantities of interest, it is convenient to derive a closed-form expression for the inflaton field at horizon crossing, $\phi = \phi_k$. This can be accomplished by solving Eq. (2.2), which results in

$$\phi_k = \sqrt{\frac{3}{2}} M_{Pl} \ln \left(\frac{4 + 3\delta_{n_s} + 4\sqrt{1 + 3\delta_{n_s}}}{3\delta_{n_s}} \right), \tag{3.2}$$

where $\delta_{n_s} \equiv 1 - n_s$. We can determine the number of e-folds during inflation N_k , after the pivot scale of wavenumber $k \equiv a_k H_k$ left the horizon, using the SR approximation

$$N_k = -\frac{1}{M_{Pl}^2} \int_{\phi_k}^{\phi_e} \frac{V}{V'} d\phi = \frac{3}{4} \left(e^{\sqrt{\frac{2}{3}} \frac{\phi_k}{M_{Pl}}} - e^{\sqrt{\frac{2}{3}} \frac{\phi_e}{M_{Pl}}} - \sqrt{\frac{2}{3}} \left(\frac{\phi_k}{M_{Pl}} - \frac{\phi_e}{M_{Pl}} \right) \right). \tag{3.3}$$

Where ϕ_e is the field evaluated at the end of inflation which, following [24], we approximate as $\phi_e \approx 0.615 M_{Pl}$. Given the horizon exit value for ϕ_k by Eq. (3.2), it is possible to express N_k in terms of the spectral index n_s .

At the origin the Starobinsky model is well approximated by a quadratic potential, in this case $\omega_{re} = 0$. Also, for the Starobinsky model T_{re} has been determined to be 3.1×10^9 GeV [11]. In this case $T_1 \approx 58.7261 + \frac{1}{3} \ln \frac{T_{re}}{M_{Pl}} \approx 51.8988$. From Eq. (2.12) we get

$$N_k = T_1 + \frac{1}{3} \ln \left(\frac{f'(\phi_k) M_{Pl}}{f(\phi_e)} \right), \tag{3.4}$$

where $f'(\phi_k) = \frac{2}{M_{Pl}} \sqrt{\frac{2}{3}} e^{-\sqrt{\frac{2}{3}} \frac{\phi_k}{M_{Pl}}} \left(1 - e^{-\sqrt{\frac{2}{3}} \frac{\phi_k}{M_{Pl}}} \right)$ and, at the end of inflation, $f(\phi_e) = \left(1 - e^{-\sqrt{\frac{2}{3}} \frac{\phi_e}{M_{Pl}}} \right)^2$. Having obtained N_k in terms of ϕ_k , and ϕ_k in terms on n_s , we can solve Eq.(3.4) directly for n_s (see Fig. 1). This yields the following result

$$n_s = 0.96235, \tag{3.5}$$

with the last digit rounded off.ss

The consistency relations for the Starobinsky model [25] provide the following expressions for the tensor-to-scalar ratio r , the running of the spectral index n_{sk} , and the running of the tensor n_{tk} , in terms of n_s

$$r = \frac{4}{3} \left(2 - 2\sqrt{1 + 3\delta_{n_s}} + 3\delta_{n_s} \right), \tag{3.6}$$

$$n_{sk} = \frac{1}{18} \left(4 + 9\delta_{n_s} - 9\delta_{n_s}^2 - \sqrt{1 + 3\delta_{n_s}} (4 + 3\delta_{n_s}) \right), \tag{3.7}$$

$$n_{tk} = \frac{1}{36} \left(8 + 3(4 - 3\delta_{n_s})\delta_{n_s} - 8\sqrt{1 + 3\delta_{n_s}} \right), \tag{3.8}$$

Table 1 Above, we have listed the parameter values used in the calculations, while below, we present the values obtained for the observables and number of e -folds. Note that for $g_{re} \approx g_{s,re}$, Eq. (3.4) is practically independent of both g_{re} and $g_{s,re}$

Parameter	Value	Parameter	Value
k_p	$0.05 Mpc^{-1}$	T_{re}	$3.1 \times 10^9 \text{ GeV}$
T_0	$2.7255 K$	ϕ_e	$0.615 M_{pl}$
A_s	2.1×10^{-9}	a_{eq}	2.9×10^{-4}
Observable	Value	e -folds	Value
n_s	0.96235	N_k	51.2
r	0.00403	N_{re}	18.0
n_t	-0.00050	N_{rd}	43.9
n_{sk}	-0.00072	$N_k + N_{re} + N_{rd}$	113.2
n_{tk}	-0.000019	N_{keq}	113.2

We solve Eq. (3.4) for the spectral index n_s , and by using the consistency relations given by (2.1), (3.6), (3.7), and (3.8), we obtain the other observables. The number of e -folds during inflation N_k , reheating N_{re} , and radiation N_{rd} are calculated using the Eqs. (3.3), (2.8), and (3.9), respectively. For an explanation about the presence of the term N_{keq} , see the last paragraph of Sect. 3. We observe that all the values obtained are within ranges calculated previously with a different procedure [25] (see also [28]) and are in remarkable agreement with current measurements of the power spectrum of primordial curvature fluctuations and the present bounds on the spectrum of primordial gravitational waves

where $\delta_{n_s} \equiv 1 - n_s$, as before. Using these equations, we can compute the values for the other observables. We can also determine the number of e -folds of inflation, reheating and, from entropy conservation after reheating [26], the number of e -folds of radiation

$$N_{rd} \equiv \ln\left(\frac{a_{eq}}{a_r}\right) = \ln\left(\frac{a_{eq} T_{re}}{\left(\frac{43}{11g_{s,re}}\right)^{1/3} a_0 T_0}\right), \tag{3.9}$$

where a_r denotes the scale factor at the end of reheating or, equivalently, at the beginning of the radiation epoch and a_{eq} is the scale factor at radiation-matter equality. Values of observables as well as number of e -folds are given in the Table 1.

Finally, we can use the expression

$$N_{keq} \equiv \ln\left(\frac{a_{eq}}{a_k}\right) = \ln\left(\frac{a_e}{a_k}\right) + \ln\left(\frac{a_r}{a_e}\right) + \ln\left(\frac{a_{eq}}{a_r}\right) = N_k + N_{re} + N_{rd}, \tag{3.10}$$

as a consistency check. This equation can be written more concisely as $N_{keq} = \ln\left(\frac{a_{eq} H_k}{k_p}\right) = \ln\left(\frac{a_{eq} \pi \sqrt{A_s r}}{\sqrt{2} k_p}\right)$ [23]. Thus, the number of e -folds from the time scales of wavenumber $k = a_k H_k$ leave the horizon at a_k to the time of radiation-matter equality at a_{eq} is essentially given by the parameter r , equivalently, by the value of the scale factor at horizon crossing a_k [27]. We find that $N_k + N_{re} + N_{rd} \approx 113.182$, and the same value using the formula $N_{keq} = \ln\left(\frac{a_{eq} \pi \sqrt{A_s r}}{\sqrt{2} k_p}\right)$.

4 Conclusions

Our approach allows us to find solutions for the cosmological observables, including the number of e -folds during inflation, reheating, and radiation, with minimal assumptions. By imposing the reheating conditions, we establish a connection between inflation and reheating and derive Eq. (2.12) which, for the Starobinsky model and $\omega_{re} = 0$, is solved for the spectral index n_s . We use the consistency relations of the model to determine the values for the other observables. The number of e -folds during inflation N_k and the number of e -folds during reheating N_{re} are also determined by their respective formulas involving n_s , while the number of e -folds during radiation N_{rd} is determined by the reheating temperature T_{re} . The results show remarkable agreement between the Starobinsky model and current measurements of the power spectrum of primordial curvature fluctuations and the present bounds on the spectrum of primordial gravitational waves.

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