Eur. Phys. J. Plus (2023) 138:354
https://doi.org/10.1140/epjp/s13360-023-03944-z

THE EUROPEAN PHYSICAL JOURNAL PLUS

Regular Article

On a neutral Dirac particle interacting with a magnetic field in a topological defect space-time and its hidden supersymmetry

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Received: 5 May 2022 / Accepted: 30 March 2023

Abstract In this paper, we study the relativistic quantum dynamics of a neutral Dirac particle with a permanent magnetic dipole moment that interacts with an external magnetic field in the background space-time of a linear topological defect called spiral dislocation. The generalized Dirac wave equation is derived from the full action of that model involving the Lagrangian density of the Dirac spinor field in the background and the interaction model. The energy eigenvalues and corresponding wave functions are found in closed form by reducing the problem to that of a non-relativistic particle moving freely on a plane with a hole at the origin whose radius is determined by the defect parameter. In the limit of vanishing external magnetic field we are also able to establish a hidden SUSY structure of the underlying Dirac Hamiltonian allowing us to discuss the non-relativistic limit in some detail.

1 Introduction

Over the past decades, the Dirac equation [1] has been studied in various subjects and fields of physics. For example, one can refer to study the Dirac oscillator [2, 3], a charged Dirac particle in a static electromagnetic field [4], the Pseudospin symmetry of a Dirac nucleon in the relativistic Manning-Rosen potential [5], the Dirac particle in the curved space-time [6–8], the Dirac particle in the framework of the Lorentz- and CPT-violating standard-model extension [9–14], the electronic properties of graphene obtained from a Dirac Hamiltonian for a massless fermion [15, 16], relativistic quantum dynamics of a Dirac particle confined in a quantum ring [17], the influence of dislocations, as a kind of topological defects in graphene, on the electronic properties of a Dirac particle [18], the Dirac particle in a chiral cosmic string space-time [19], the Dirac particle with arbitrary magnetic moment [20], bound-state dynamics of a neutron with an anomalous magnetic dipole moment interacting with certain external fields [21, 22], bound states and geometric quantum phases for the Dirac particle with a magnetic dipole moment in a cosmic string space-time [23, 24], on the second dipole moment of Dirac particle [25], and the developed form of particle-wave duality based on Einstein's geodesic equation in two-dimensional space-time versus the Dirac result [26].

Hence, it is not surprising that topological defects can affect the physical properties of various systems that have been analyzed in different areas of physics such as gravitation and condensed matter physics. Indeed, inspired by the linear topological defects appearing in the context of crystalline solids, as well as the relationship between defects in solids and defects in cosmology, a generalization of topological defects in gravitation has been presented [27–31]. This fascinating approach has been taken on in many subsequent papers. As examples, let us refer to studies on the structure of the curvature tensor of cosmic strings determined by a conical singularity in space-time [32], the behavior of a Klein-Gordon particle in a class of space-times generated by defects [33–36], in Kaluza-Klein theories [37, 38], on the behaviour of a spin-zero Duffin–Kemmer–Petiau's particle in a cosmic string space-time [39] and in a global monopole space-time [40], on the spin and pseudospin symmetries of Dirac particles in topological defect backgrounds [41, 42], on the interaction of a scalar field with a Coulomb-type potential in a space-time with a screw dislocation [43], and on non-relativistic quantum dynamics of a single particle interacting harmonically with conical singularities associated with either a cosmic string, a global monopole, a magnetic flux string, or a screw dislocation [44].

In this work, we present the exact analytical solution of a neutral Dirac particle with a permanent magnetic dipole moment (PMDM), which interacts with an external magnetic field in the background space-time generated by a spiral dislocation known as a topological defect in solid lattice corresponding to torsion. Thus, the study of such an interaction in the desired background can

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Published online: 24 April 2023

be started by considering a generalization of a linear topological defect in gravitation, which means that we should consider Dirac spinors in curved space-time background.

The organization of this paper is as follows. In Sect. 2, at first, an introduction of the line element of the spiral dislocation background and ingredients of the relevant spinor covariant derivative are presented. This is followed by a discussion about the action of a Dirac spinor field in Sect. 3, which results in a set of equations associated with a Dirac particle with a PMDM interacting with an external magnetic field in the spiral dislocation background. The eigenvalue problem for the aforementioned system is then solved exactly in Sect. 4. This is achieved by reducing it to a Schrödinger-like equation for a free particle on a punctured plane, which in essence is Bessel's differential equation. In Sect. 5 we are able to establish a hidden SUSY structure for a vanishing magnetic field and briefly discuss the non-relativistic limit. Finally, in Sect. 6, we conclude with a short summary of our results.

2 Quantum dynamics in the presence of a spiral dislocation

In view of the vast literature on non-relativistic and relativistic quantum dynamics in space-time,[1] with dislocation fields that are considered as deformation fields,[2] we intend to provide a quantum description of a neutral spin-half particle with a PMDM interacting with an external magnetic field configuration in the relativistic regime. In fact, we want to study such interaction in background space-time generated by a distortion of a circle into a spiral recognized as the spiral dislocation. In this case, the dislocation field is due to a parameter associated with the spiral dislocation denoted by χ and has the dimension of a length [45]. As pointed out, for example, by Bakke and Furtado [46], it is directly related to the associated Burger's vector $\mathbf{b}$ via $\chi = |\mathbf{b}|/2\pi$. As discussed in Ref. [45], the Burger vector is parallel to the plane $z =$ constant, which is in the radial direction. Inspired by this kind of linear topological defect described in the context of crystalline solids and also by considering the relationship between defects in solids and defects in cosmology, we establish a generalization of such linear defect in gravitation [46–48]. Thus, taking in mind the natural units, where $c = 1$ and $\hbar = 1$, the background space-time, produced by such a kind of topological defect, can be described by the following line element [41, 46]

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}r^2 - 2\chi\,\mathrm{d}r\,\mathrm{d}\varphi - \left(\chi^2 + r^2\right)\mathrm{d}\varphi^2 - \mathrm{d}z^2. \tag{2.1}$$

Here, the spatial part of the line element given by eq. (2.1) presents a distortion of a circle into a spiral known as the spiral dislocation. This distortion takes place in a plane orthogonal to the z axis. Note that we are using cylindrical coordinates, that is, r, φ, and z are the radial, azimuthal, and axial coordinates, respectively.

When analyzing the neutral Dirac particle subject to such topological defect space-time, it is appropriate to consider Dirac spinors in curved space-time background [49]. Therefore, in order to define the spinors in the local reference frame (LRF) of the observer, we need to construct a non-coordinate basis $\hat{\theta}^a$ given by

$$\hat{\theta}^a = e^a{}_\mu(x)dx^\mu. \tag{2.2}$$

Here the $e^a_\mu(x)$ are known as tetrads in the literature and obey the following condition:

$$g_{\mu\nu}(x) = e^a{}_\mu(x)e^b{}_\nu(x)\eta_{ab}, \tag{2.3}$$

where $\eta_{ab} = \mathrm{diag}(+, -, -, -)$ represents the Minkowski space-time metric tensor and the Latin indices $a, b = 0, 1, 2, 3$ are used for the LRF. Furthermore, we denote the space-time indices by μ with $\mu \in \{t, r, \varphi, z\}$. The inverse of the tetrads are determined by $dx^\mu = e^\mu{}_a(x)\hat{\theta}^a$. Hence, the relationships $e^\mu{}_a(x)e^a{}_\nu(x) = \delta^\mu_\nu$ and $e^a{}_\mu(x)e^\mu{}_b(x) = \delta^a{}_b$ are expected to be satisfied through the tetrads and the inverse tetrads. Here, it suffices to present the values of the inverse tetrads as

$$e^t{}_0 = e^r{}_1 = e^z{}_3 = 1, \qquad e^\varphi{}_2 = 1/r, \qquad e^r{}_2 = -\chi/r. \tag{2.4}$$

Since we study the behavior of a Dirac particle in the space-time background described by a particular topological defect, we must apply the spinor covariant derivative ∇_μ instead of the partial derivative ∂_μ such that the components of the spinor covariant derivative can be obtained by $\nabla_\mu = \partial_\mu + \Gamma_\mu(x)$, with the spinorial connection $\Gamma_\mu(x) = \omega_{\mu ab}[\gamma^a, \gamma^b]/8$ [7, 8, 49]. Here, the γ^a matrices, defined in the LRF, correspond to the standard Dirac equation (DE) applied to the relativistic quantum description of spin one-half particles in the Minkowski spacetime

$$\gamma^0 = \hat{\beta} = \begin{pmatrix} \mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{pmatrix}, \quad \gamma^i = \hat{\beta}\hat{\alpha}^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \Sigma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}, \tag{2.5}$$

where Σ^i represents the spin vector and σ^i being the Pauli matrices. The Pauli matrices obey the usual anti-commutation relations $\{\sigma^i, \sigma^j\} = 2\delta_{ij}\mathrm{I}$, with δ_{ij} being the Kronecker delta and I denotes the 2×2 identity matrix. We may also use the notation $\sigma^0 = \mathrm{I}$. Recall that $i, j = 1, 2, 3$ are indices denoting the spatial part of the LRF.

[1] In the previous section, we referred to some of the work done in this area.

[2] As stated in ref. [45], in a solid, the dislocation field is associated with the lack of connectivity in the deformed area of a body. The body becomes disconnected due to deformation if the curl of the local deformation field is non-null.

Now, according to the spinor theory in curved spacetime, we have to provide the generalized Dirac matrices through the relation $\gamma^{\mu} = e^{\mu}{}_{a}\gamma^{a}$. Thus, to obtain the non-null component of the spinorial connection, in addition to the standard Dirac matrices, we need to obtain the non-null components of the connection 1-form $\omega^{a}_{\mu\ b}(x)$ using the Maurer-Cartan structure equations, $\mathrm{d}\hat{\theta}^{a} + \omega^{a}{}_{b} \wedge \hat{\theta}^{b} = 0$, written in the absence of torsion, in which $\omega^{a}{}_{b} = \omega^{a}_{\mu\ b}(x)\mathrm{d}x^{\mu}$. Thereby, by solving the associated Maurer-Cartan structure equations, we arrive at

$$\omega^{2}_{\varphi\ 1} = -\omega^{1}_{\varphi\ 2} = 1, \tag{2.6}$$

and then we get $\Gamma_{\varphi} = -\mathrm{i}\Sigma^{3}/2$.

3 Action of a Dirac spinor field

In this section we will discuss the action of a neutral Dirac fermion with a PMDM interacting with an external magnetic field. In addition, we will consider the gravitational effects arising from a curved space-time caused by the aforementioned topological defect. The action for this interaction scenario can be expressed as

$$\mathcal{S}[\Psi] = \int \mathrm{d}^{4}x\sqrt{|g|}\mathcal{L}_{\mathrm{GDE-PMDM}}, \tag{3.1}$$

where $g = \det(g_{\mu\nu})$ is the determinant of the metric tensor and the Lagrangian densities is given by

$$\mathcal{L}_{\mathrm{GDE-PMDM}} = \frac{\mathrm{i}}{2}\bar{\Psi}\overleftrightarrow{\nabla\!\!\!/}\Psi - M\bar{\Psi}\Psi, \tag{3.2}$$

where

$$\nabla\!\!\!/ \equiv \gamma^{\mu}\nabla_{\mu}.$$

This Lagrangian provides a framework to describe a minimal interaction. In some cases, additional terms are added to the Lagrangian [50]. It is noteworthy that this Lagrangian consists of a nonminimal coupling,

$$\mathrm{i}\gamma^{\mu}\nabla_{\mu} \equiv \mathrm{i}\gamma^{\mu}\left(\partial_{\mu} + \Gamma_{\mu}\right) + \frac{\tilde{\mu}}{2}\bar{\Psi}\Sigma^{\mu\nu}\Psi F_{\mu\nu}, \tag{3.3}$$

which specifically describes the interaction of such a fermion with an external magnetic field. The PMDM of the neutral Dirac particle is represented by the coupling constant $\tilde{\mu} \geq 0$. Here, the electromagnetic field tensor is denoted by $F_{\mu\nu}$ having components $F_{0i} = -F_{i0} = -E_{i}$ and $F_{ij} = \epsilon_{ijk}B_{k}$ such that the electric and magnetic fields are given by $\mathbf{E} = (E_{1}, E_{2}, E_{3})^{T}$ and $\mathbf{B} = (B_{1}, B_{2}, B_{3})^{T}$, respectively. In addition the relevant spin tensor is given by $\Sigma^{\mu\nu} = \mathrm{i}[\gamma^{\mu}, \gamma^{\nu}]/2$ and obeys the relation $\Sigma^{\mu\nu} - \Sigma^{\nu\mu} = \mathrm{i}[\gamma^{\mu}, \gamma^{\nu}]$.

Equation (3.3), which includes a combination of the spinor covariant derivative due to a topological defect in background space-time and also an additional term that is responsible for describing the corresponding interaction, can be expanded in terms of the electromagnetic field components in the following form

$$\begin{aligned}\mathrm{i}\gamma^{\mu}\nabla_{\mu} \equiv\, & \mathrm{i}\gamma^{t}\partial_{t} + \mathrm{i}\gamma^{r}\partial_{r} + \mathrm{i}\gamma^{\varphi}\left(\partial_{\varphi} + \Gamma_{\varphi}\right) + \mathrm{i}\gamma^{z}\partial_{z}\\ & + \mathrm{i}\tilde{\mu}\hat{\alpha}^{1}E_{r} + \mathrm{i}\frac{\tilde{\mu}}{r}\hat{\alpha}^{2}\left(E_{\varphi} - \chi E_{r}\right) + \mathrm{i}\tilde{\mu}\hat{\alpha}^{3}E_{z}\\ & - \frac{\tilde{\mu}}{r}\Sigma^{1}\left(B_{r} + \chi B_{\varphi}\right) - \tilde{\mu}\Sigma^{2}B_{\varphi} - \frac{\tilde{\mu}}{r}\Sigma^{3}B_{z}.\end{aligned} \tag{3.4}$$

In the absence of an electric field, i.e. $\mathbf{E} = 0$, the full generalized Dirac equation (FGDE) can be obtained from the action $\mathcal{S}[\Psi]$ in the usual way and explicitly reads

$$\begin{aligned}\Bigg[&\mathrm{i}\gamma^{0}\partial_{t} + \mathrm{i}\gamma^{1}\left(\partial_{r} + \frac{1}{2r}\right) + \mathrm{i}\frac{\gamma^{2}}{r}\left(\partial_{\varphi} - \chi\partial_{r}\right) + \mathrm{i}\gamma^{3}\partial_{z}\\ & -\frac{\tilde{\mu}}{r}\Sigma^{1}\left(B_{r} + \chi B_{\varphi}\right) - \tilde{\mu}\Sigma^{2}B_{\varphi} - \frac{\tilde{\mu}}{r}\Sigma^{3}B_{z} - M\Bigg]\Psi(t, \mathbf{r}) = 0.\end{aligned} \tag{3.5}$$

From now on we limit ourselves to a magnetic field of the form $\mathbf{B} = B_{0}\, r\hat{z}$ with $B_{0} \in \mathbb{R}$, which represents a unidirectional magnetic field in the positive or negative z-direction with a linearly increasing field strength,[3] in the radial direction. With the ansatz

$$\Psi(t, \mathbf{r}) = \mathrm{e}^{-\mathrm{i}\mathsf{E}t + \mathrm{i}\left(\ell + \frac{1}{2}\right)\varphi + \mathrm{i}kz}\psi(r), \tag{3.6}$$

[3] Such axial magnetic fields with a radial gradient play, for example, an important role in modelling magnetized relativistic jets [51], but are also used in so-called fixed-field alternating-gradient particle accelerators [52].

for the solution of the FGDE, we obtain four coupled equations for the four components of the radial Dirac spinor $\psi(r) = (\psi_1(r), \psi_2(r), \psi_3(r), \psi_4(r))^{\mathrm{T}}$ of the following form

$$(\mathsf{E} - M - \tilde{\mu} B_0)\psi_1(r) - k\psi_3(r) + \left(\left(\mathrm{i} - \frac{\chi}{r}\right)\partial_r + \frac{\mathrm{i}}{r}(\ell+1)\right)\psi_4(r) = 0, \tag{3.7a}$$

$$(\mathsf{E} - M + \tilde{\mu} B_0)\psi_2(r) + \left(\left(\mathrm{i} + \frac{\chi}{r}\right)\partial_r - \frac{\mathrm{i}\ell}{r}\right)\psi_3(r) + k\psi_4(r) = 0, \tag{3.7b}$$

$$k\,\psi_1(r) - \left(\left(\mathrm{i} - \frac{\chi}{r}\right)\partial_r + \frac{\mathrm{i}}{r}(\ell+1)\right)\psi_2(r) - (\mathsf{E} + M + \tilde{\mu} B_0)\psi_3(r) = 0, \tag{3.7c}$$

$$-\left(\left(\mathrm{i} + \frac{\chi}{r}\right)\partial_r - \frac{\mathrm{i}\ell}{r}\right)\psi_1(r) - k\psi_2(r) - (\mathsf{E} + M - \tilde{\mu} B_0)\psi_4(r) = 0. \tag{3.7d}$$

In the above $k \in \mathbb{R}$ is the eigenvalue of the z-component of the linear momentum $\hat{p}_z = -\mathrm{i}\partial_z$ and $\ell \in \mathbb{Z}$ represents the eigenvalue of the angular momentum operator $\hat{L}_z = -\mathrm{i}\partial_\varphi$. Furthermore, E stands for the energy eigenvalue of the Dirac particle, that is, the eigenvalue of $\mathrm{i}\partial_t$

In the next section, we will decouple the set of eqs. (3.7) by deriving a second-order wave equation, who's solution represents those four radial spinor components.

4 Exact solutions for a Dirac particle with PMDM

In order to achieve the decoupling of eqs. (3.7) we start with the ansatz $\psi_1(r) = \alpha_1\,\psi_3(r)$ and $\psi_2(r) = \alpha_2\,\psi_4(r)$, where α_1 and α_2 are constant. This ansatz results in the two conditions

$$\alpha_1 = -\frac{(-M + \tilde{\mu} B_0 - \mathsf{E}) - \alpha_2 k}{(-M + \tilde{\mu} B_0 + \mathsf{E})\alpha_2 + k}, \qquad \alpha_2 = -\frac{\alpha_1 k + (-M - \tilde{\mu} B_0 - \mathsf{E})}{(-M - \tilde{\mu} B_0 + \mathsf{E})\alpha_1 - k}, \tag{4.1}$$

from which we obtain the following values

$$\alpha_1 = \frac{\mathsf{E}}{k} - \sqrt{\frac{\mathsf{E}^2}{k^2} - 1}, \qquad \alpha_2 = -\frac{\mathsf{E}}{k} - \sqrt{\frac{\mathsf{E}^2}{k^2} - 1}. \tag{4.2}$$

Inserting this result into eqs. (3.7) we obtain a third relation between $\psi_3(r)$ and $\psi_4(r)$

$$\psi_4(r) = \frac{\left(\mathrm{i} + \frac{\chi}{r}\right)\partial_r - \frac{\mathrm{i}\ell}{r}}{(M - \tilde{\mu} B_0 - \mathsf{E})\alpha_2 - k}\,\psi_3(r). \tag{4.3}$$

and an associated second-order wave equation for $\psi_3(r)$

$$\left(1 + \frac{\chi^2}{r^2}\right)\frac{\mathrm{d}^2\psi_3(r)}{\mathrm{d}r^2} + \left(\frac{1}{r} - \frac{\chi^2}{r^3} - \mathrm{i}\frac{2\chi\ell}{r^2}\right)\frac{\mathrm{d}\psi_3(r)}{\mathrm{d}r} - \frac{\ell^2}{r^2}\psi_3(r) + \mathrm{i}\frac{\chi\ell}{r^3}\psi_3(r) + \kappa^2\psi_3(r) = 0, \tag{4.4}$$

where κ is given by

$$\kappa^2 = \left(\sqrt{\mathsf{E}^2 - k^2} + \tilde{\mu} B_0\right)^2 - M^2. \tag{4.5}$$

Before we study the exact solution of this equation, let us briefly comment on the behavior of $\psi_3(r)$ for small r.

Assuming that for small r the solution of eq. (4.4) behaves like $\psi_3(r) \sim r^a$, we are led to the two solutions $a = 2$ and $a = 0$. Obviously, the first result $\psi_3(r) \sim r^2$ implies a vanishing wave function at origin, that is, Dirichlet boundary conditions are applied at $r = 0$. On the other side, for the second solution, we need to extend the ansatz to $\psi_3(r) \sim c + r^b$ with an arbitrary constant c and $b > 0$. This results in $b = 2$ and hence we are led to the boundary condition $\psi_3'(0) = 0$, which is a Neumann boundary condition at the origin. Both results are only valid for $\chi \neq 0$. It is obvious that $r = 0$ is a singular point due to the non-trivial topology characterized by the line element (2.1). Whereas the Dirichlet condition appears to be the natural choice, we will consider both options in our way forward. To conclude this part, let us mention that for the flat space limit $\chi = 0$ we have the usual result $\psi_3(r) \sim r^{|\ell|}$.

With this in mind we are now aiming at the exact solution of (4.4). In doing so we first make the ansatz $\psi_3(r) = \mathrm{e}^{\mathrm{i}\,\ell\,\mathrm{Arctan}(r/\chi)}\tilde{Q}(r)$ and eq. (4.4) leads us to a Schrödinger-type wave equation of the form

$$\left(1 + \frac{\chi^2}{r^2}\right)\frac{\mathrm{d}^2\tilde{Q}(r)}{\mathrm{d}r^2} + \left(\frac{1}{r} - \frac{\chi^2}{r^3}\right)\frac{\mathrm{d}\tilde{Q}(r)}{\mathrm{d}r} - \frac{\ell^2}{r^2 + \chi^2}\tilde{Q}(r) + \kappa^2\tilde{Q}(r) = 0. \tag{4.6}$$

In a second step we now change to a new radial variable $\rho = \sqrt{\chi^2 + r^2} \geq \chi$ and set $Q(\rho) = \tilde{Q}(r) = \tilde{Q}(\sqrt{\rho^2 - \chi^2})$. With this substitution we arrive at

$$\frac{\mathrm{d}^2 Q(\rho)}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}Q(\rho)}{\mathrm{d}\rho} + \frac{1}{\rho^2}\left(\kappa^2\rho^2 - \ell^2\right)Q(\rho) = 0. \tag{4.7}$$

Surprisingly, eq. (4.7) is the radial Schrödinger equation of a free particle of mass 1/2 with energy $\varepsilon = \kappa^2$ moving on a two-dimensional plane. Hence, $\kappa > 0$ can be interpreted as wave number for the radially outgoing and incoming wave. However, we do have the condition $\rho \geq \chi$, that is, the two-dimensional plane has a hole at the origin with radius χ. That is, we need to specify boundary conditions at the border of that hole. Our discussion on the small r above allows us to consider both, Dirichlet and Neumann conditions at $\rho = \chi$.

Noting that eq. (4.7) is the well-known Bessel differential equation, we can express its general solution in terms of two linearly independent Bessel functions. Here we opt for $H_\ell^{(1)}(\kappa\rho)$ and $H_\ell^{(2)}(\kappa\rho)$ being the Hankel functions of first and second kind, respectively. They show the asymptotic behaviour $H_\ell^{(1,2)}(\kappa\rho) \sim \exp\{\pm\mathrm{i}\kappa\rho\} \sim \exp\{\pm\mathrm{i}\kappa r\}$ of an outgoing and incoming radial wave for large $\rho \sim r$ with the expected wave number κ. Hence the exact solution of (4.7) is given by

$$Q^{D,N}(\rho) = \mathcal{A}^{D,N}\, H_\ell^{(1)}(\kappa\rho) + \mathcal{B}^{D,N}\, H_\ell^{(2)}(\kappa\rho), \tag{4.8}$$

where the superscripts D and N stand for the Dirichlet and Neumann conditions. That is $Q^D(0) = 0$ and $\frac{\mathrm{d}Q^N}{\mathrm{d}\rho}(0) = 0$. This is achieved via the proper selection of the constants $\mathcal{A}^{D,N}$ and $\mathcal{B}^{D,N}$ given by

$$\mathcal{B}^D = -\mathcal{A}^D\,\frac{H_\ell^{(2)}(\kappa\chi)}{H_\ell^{(1)}(\kappa\chi)} \qquad \text{and} \qquad \mathcal{B}^N = -\mathcal{A}^N\,\frac{H_\ell^{(2)'}(\kappa\chi)}{H_\ell^{(1)'}(\kappa\chi)} \tag{4.9}$$

together with the normalisation condition. For both boundary conditions the energy eigenvalues are given by

$$\mathsf{E}_{k,\kappa} = \pm\sqrt{k^2 + \left(\sqrt{M^2 + \kappa^2} - \tilde{\mu}B_0\right)^2}, \qquad k \in \mathbb{R}, \qquad \kappa \geq 0\cdot \tag{4.10}$$

The energy spectrum of a neutral Dirac particle in the topological defect background with a PMDM $\tilde{\mu}$ coupled to the magnetic field is continuous and bounded from below, respectively above, as follows

$$\mathsf{E}_{k,\kappa} \geq |M - \tilde{\mu}B_0|, \qquad \mathsf{E}_{k,\kappa} \leq -|M - \tilde{\mu}B_0|. \tag{4.11}$$

Obviously, the usual spectral gap of the free Dirac fermion is either decreased or increased by the presence of the PMDM in a magnetic field, when the latter is pointing towards the positive or negative z-axis, respectively. The deformation parameter χ does not have any effect on the spectrum but it has an obvious effect on the wave functions (4.8) via the parameters (4.9) and the r-dependent phase $\exp\{\mathrm{i}\ell\,\mathrm{Arctan}(r/\chi)\}$.

5 A hidden supersymmetry

In this section will show that the relativistic system discussed above exhibits, in the limit of a vanishing magnetic field or equivalently a vanish PMPD coupling constant, a hidden SUSY structure known from SUSY quantum mechanics [53]. In fact, when multiplying eq. (3.5) from the left with γ^0 we can put it into the form

$$\mathrm{i}\partial_t\Psi = \hat{\mathcal{H}}_{\mathrm{FGDE}}\Psi, \tag{5.1}$$

where the full generalized Dirac Hamiltonian is represented by 2×2 block matrices as follows

$$\hat{\mathcal{H}}_{\mathrm{FGDE}} = \mathcal{Q}_1 + \hat{\beta}\mathcal{M} = \begin{pmatrix} \mathcal{M} & A \\ A & -\mathcal{M} \end{pmatrix}. \tag{5.2}$$

In the above we have introduced the block matrices

$$\mathcal{M} = (M + \tilde{\mu}B_0\sigma_3) \qquad \text{and} \qquad \mathcal{Q}_1 = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}, \tag{5.3}$$

where

$$A := -\mathrm{i}\sigma^1\left(\partial_r + \frac{1}{2r}\right) - \mathrm{i}\frac{\sigma^2}{r}\left(\partial_\varphi - \chi\partial_r\right) - \mathrm{i}\sigma^3\partial_z = A^\dagger \tag{5.4}$$

is a self-adjoint operator acting on $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$. At this stage we note that in order to fulfill the requirements of a supersymmetric Dirac Hamiltonian [53] we need to require $[\mathcal{Q}_1, \mathcal{M}] = 0$. This can obviously be achieved by switching off the magnetic field or the PMPD coupling parameter. Hence, from now on we consider the case $\tilde{\mu} B_0 = 0$ only and observe that

$$\hat{\mathcal{H}}^2_{\mathrm{FGDE}} = \mathcal{Q}_1^2 + \mathcal{M}^2 = \left(A^2 + M^2\right) \otimes \mathrm{I}. \tag{5.5}$$

We will now establish an $N = 2$ SUSY structure of the system under discussion. In doing so we closely follow [53] and introduce the following operators

$$\mathcal{H}_{\mathrm{SUSY}} = \frac{1}{2M}\begin{pmatrix} A^2 & 0 \\ 0 & A^2 \end{pmatrix}, \qquad \mathcal{Q} = \frac{1}{\sqrt{2M}}\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{Q}^\dagger = \frac{1}{\sqrt{2M}}\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \tag{5.6}$$

which obviously obey the SUSY algebra

$$\mathcal{H}_{\mathrm{SUSY}} = \{\mathcal{Q}, \mathcal{Q}^\dagger\}, \qquad \{\mathcal{Q}, \hat{\beta}\} = 0, \qquad \mathcal{Q}^2 = 0 = (\mathcal{Q}^\dagger)^2, \tag{5.7}$$

where $\hat{\beta}$ acts as the Witten parity operator. The SUSY Hamiltonian introduced above is related to the free Dirac Hamiltonian in the background space-time (2.1) as follows

$$\hat{\mathcal{H}}^2_{\mathrm{FGDE}} = M^2 + 2M\mathcal{H}_{\mathrm{SUSY}}. \tag{5.8}$$

With this relation the eigenvalues of $\mathcal{H}_{\mathrm{SUSY}}$, denoted by $\varepsilon_{k,\kappa}$, immediately follow from those of $\hat{\mathcal{H}}_{\mathrm{FGDE}}$ given in (4.10) with $\tilde{\mu} B_0 = 0$,

$$\varepsilon_{k,\kappa} = \frac{1}{2M}\left(k^2 + \kappa^2\right). \tag{5.9}$$

Obviously, the ground-state energy $\varepsilon_{0,0}$ vanishes and SUSY is unbroken. However, when switching on the magnetic interaction SUSY is explicitly broken.

For supersymmetric Dirac Hamiltonians it is also known [53] that there exists a unitary operator $\hat{U}$, representing a Foldy-Wouthuysen transformation, which block-diagonalizes it, that is,

$$\hat{\mathcal{H}}_{\mathrm{FW}} = \hat{U}\hat{\mathcal{H}}_{\mathrm{FGDE}}\hat{U}^{-1} = \begin{pmatrix} \sqrt{A^2 + M^2} & 0 \\ 0 & -\sqrt{A^2 + M^2} \end{pmatrix}. \tag{5.10}$$

Above relation may be rewritten as

$$\hat{\mathcal{H}}_{\mathrm{FW}} = \hat{\beta} M \sqrt{1 + \frac{2\mathcal{H}_{\mathrm{SUSY}}}{M}} \tag{5.11}$$

and indicates that in the non-relativistic limit, where M becomes large, the SUSY Hamiltonian is indeed the one representing the non-interacting system under investigation in this limit as

$$\hat{\mathcal{H}}_{\mathrm{FW}} \sim \hat{\beta}(M + \mathcal{H}_{\mathrm{SUSY}}). \tag{5.12}$$

In concluding the discussion on SUSY we finally note that the above relation (5.11) has been found for several relativistic SUSY Hamiltonians not limited to spin-$\frac{1}{2}$ particles [54, 55].

6 Conclusion

In this paper, we started off with a (1 + 3)-dimensional space-time generated by the spiral dislocation background followed by a concise review of the mathematical formulation of the Dirac spinor in the context of quantum field theory in curved space-time. Then, the relativistic behaviour of a neutral spin-half particle with a PMDM that interacts with an external magnetic field in that background has been investigated. This interaction scenario has been designed by presenting the action of a Dirac spinor field involving the Lagrangian density of the Dirac spinor field in that background and the Lagrangian density corresponding to the interaction model. In search of analytical solutions, the wave equation deriving from the action of a Dirac spinor field corresponding to that possible scenario has been established.

We were able to reduce the relativistic wave equations to that of a radial Schrödinger problem of a free particle moving on a plane with a hole of radius given by the deformation parameter χ. The energy eigenvalue are found in closed form and we showed that this system has a continuous spectrum like that of free Dirac particle but with the gap between the positive and negative branch being increased or decreased due to the presence of the PMPM, i.e. the gap is given by $\left[-|M - \tilde{\mu} B_0|, |M - \tilde{\mu} B_0|\right]$. We were also able to present the Dirac energy eigenspinors in closed form for Dirichlet and Neumann boundary conditions along the z-axis, both of which are admissible.

Finally, we could show that the system under discussion, for vanishing magnetic interaction, exhibits a SUSY structure known from SUSY quantum mechanics. That is, the free Dirac Hamiltonian in a background space-time characterized by the line element

(2.1) exhibits a SUSY structure with unbroken SUSY. This is similar to the situation of a free Dirac particle moving in de Sitter and anti-de Sitter space [56, 57].

Acknowledgements The authors thank the referee for a thorough reading of the manuscript and invaluable suggestions, which led to significant improvements.

Funding Open Access funding enabled and organized by Projekt DEAL.

Data Availibility Statement No Data associated in the manuscript.

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