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Gravity, axion Chern-Simons Einstein, and black holes

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Abstract The low-energy limit of string theory involves an anomaly-removing correction of the action of Einstein-Hilbert, which defines a practical idea: Chern-Simons (CS) modified gravity. The CS correction composes of the product of a scalar field with the Pontryagin density, in which the scalar field is considered as a background field (non-dynamical formulation) or as an evolving field (dynamical formulation). Different solutions of Einstein's general relativity (GR) continue in the amended theory; a keyable exception is the Kerr metric, which has affected a search for rotating black hole solutions. In this study, we derive a black solution characterizing a rotating black hole within the non-dynamical and dynamical frame of CS axion Einstein gravity (CSAEG). The case of non-dynamical has no new physics different from GR, which is consistent with what is presented in the literature. The main merit of this study is that in the dynamical framework of CSAEG, we show that the scalar field has a dynamical value up to the first-order approximation and under some constraints between the unknown functions that characterize this theory. Moreover, we show that the function U that characterizes CS can help to weaken the singularity for the invariant $^{*}RR$.

1 Introduction

Spinning, supermassive black holes are foreseen to exist at the core of most galaxies. The gravitational field in the outer of these models plays an essential role in the development of apprehending compact objects and in the emission of gravitational waves. In general relativity (GR), stationary and axisymmetric solutions are characterized by a black hole mass and the Kerr metric spin angular momentum [1]. In amended gravitational theories, the Kerr metric is not necessary to be a solution to the field equations. For example, in the Einstein-Dilaton- Gauss-Bonnet gravity, slowly rotating black hole solution was shown recently to differ from Kerr [2]. Either the main tool deviation from the Kerr metric comes from purely gravitational wave ones [3, 4] or electromagnetic observations [5, 6] can thus supply insight into extensions of GR or lack from there [7]. As is known that the inflationary era can be described by single scalar field theories, and also in the frame of modified gravity [8–11].

In recent years CS modified gravity [12] has received remarkable attention. This theory does not have the Kerr metric as a solution [13]. The Lagrangian of CS gravitational theory is constructed by the Einstein-Hilbert action and a parity-violating, four-dimensional correction. The model of CS found that gravitational string theory relentlessly asks for this correction to preserve mathematical consistency [14, 15]. This correction is essential for perturbative string sector four-dimensional compactification [14, 16, 17]. Generally, this correction arises in the existence of Ramond- Ramond scalars due to duality symmetries [18].

The action's CS amendment is determined by the product of a CS scalar field, ζ, and the Pontryagin density, $^{*}RR$. The Pontryagin density consists of the contraction of the Riemann tensor and its dual. The dual of the Riemann tensor involves contractions of the Levi-Civita tensor, which is odd under a parity transformation, possibly enhancing gravitational parity-breaking. Indeed, this reality does not yield that parity-preserving solutions are not allowed in CS amended theory because as $\zeta \to const$. The amended approach coincides with GR. As an alternative, the CS correction inserts a means to increase parity-violation by a pure curvature term, contrary to the matter sector, as is well known to happen in GR.

In CS modified gravity, there are two formulations of independent theories in their own right, yielding different observables. The dynamical formulation, in which the CS scalar is considered a dynamical field, is provided by its stress-energy tensor and evolution equation; however, in the non-dynamical construction, the CS scalar field is described by function. Its evolution equation yields a differential constraint on the space of permitted solutions (called the Pontryagin constraint, defined as the vanishing of the Pontryagin density).

Recently, most studies have focused on the non-dynamical construction of the CS modified gravity. In particular, the CS amended theory has been employed to investigate the leptogenesis issue [14] and the flat rotation curves of galaxies [19]. Moreover, many applications of amended CS gravity have been carried out like torsion [20, 21], fermion interactions, gravitational waves [22, 23], the slow-rotation limit [24], the far-field behavior [25–28], exact solutions [29–31] and Schwarzschild BH perturbation theory [32].

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Moreover, cosmological and spherically symmetric solutions of Lovelock and AdS_5 CS gravity have been derived [33]. Moreover, in the frame of cosmology, AdS-invariant Chern-Simons gravity is obtained by using an appropriate selection of the coupling constants [33]. However, the dynamical construction of CS amended theory remains rich in research soil. The present research aims to study the dynamical and non-dynamical, (which gives no new physics), construction of the CS Axion Einstein theory (CSAET).

The construction of this study is as follows: In Sect. 2, we give a summary of CSAET. In Sect. 3, we apply the field equation of CSAET to a spherically symmetric spacetime and show that the scalar field of this theory always has a trivial value (constant value). In Sect. 4, we apply the non-dynamical formulation of the field equations of CSAET. In Sect. 5, we apply the field equations of CSAET in the dynamical formulation and show that the value of the scalar field always yield a constant value. In Sect. 6, we give our conclusion of this study.

2 A summary of CS Axion Einstein's theory

Before we discuss the CS-corrected misalignment axion in the frame of Einstein's gravity, we are going to give a brief survey of CS gravity, where its action is given by:

$$S_{\rm EH} = \kappa \int_V d^4x\sqrt{-g}R\,, \tag{1}$$

$$S_{\rm CS} = \frac{\alpha}{4}\int_V d^4x\sqrt{-g}\zeta R\tilde{R}\,, \tag{2}$$

$$S_\zeta = -\frac{\beta}{2}\int_V d^4x\sqrt{-g}\big[g^{\mu\nu}\big(\nabla_\mu\zeta\big)(\nabla_\nu\zeta)+2V(\zeta)\big]\,, \tag{3}$$

$$S_{\rm matter} = \int_V d^4x\sqrt{-g}L_{\rm matter}\,. \tag{4}$$

Here $S_{\rm EH}$ denotes the Einstein-Hilbert action defined, $S_{\rm CS}$ represents the Chern-Simons rectification, S_ζ is the kinetic term, and the potential of the (pseudo) scalar-field ζ and $S_{\rm matter}$ is the action of matters, where $L_{\rm matter}$ is the matter Lagrangian density. Throughout this study, the following conventions will be used: $\kappa = \frac{1}{16\pi G}$, g is the determinant of the metric, α and β are dimensional constants, ∇_μ is the covariant derivative, R is the Ricci scalar curvature, and $a, b, c, \cdots = 0, 1, 2, 3$. The expression $R\tilde{R}$ is the Pontryagin density, defined as:

$$R\tilde{R} = R^b{}_{acd}\tilde{R}^{a\ cd}_{\ b}\,, \tag{5}$$

with $\tilde{R}^{a\ cd}_{\ b}$ being the dual Riemann-tensor defined by

$$\tilde{R}^{a\ cd}_{\ b} = \frac{1}{2}\epsilon^{cdef}R^a{}_{bef}\,, \tag{6}$$

where ϵ^{abcd} is the 4-dimensional skew-symmetric Levi-Civita tensor, which is a completely with $\epsilon^{0123} = -1$. The Chern-Simons scalar field ζ is a function of the spacetime coordinates, and ζ parametrizes deviation from general relativity. When $\zeta = $ const., the Chern-Simons gravity theory coincides with Einstein's general relativity theory since the Pontryagin density is the total divergence of the Chern-Simons topological current.

Now, we are ready to study a CS-corrected misalignment axion in the frame of Einstein's gravity by modifying the action given by Eqs. (1–3).[1] For this purpose, we take the action to have the following form [34]:

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\bigg[R - \frac{\beta\omega(\zeta)}{2}\partial_\mu\zeta\partial^\mu\zeta - \beta V(\zeta) + \alpha U(\zeta)\tilde{\epsilon}^{\mu\nu\rho\sigma}R^\tau{}_{\lambda\mu\nu}R^\lambda{}_{\tau\rho\sigma}\bigg]. \tag{7}$$

Here ζ is the scalar field, ω and U are unknown functions of the scalar field, and V is the potential. It is important to stress that the above action of CS Axion Einstein reduces to CS actions when the two functions $\omega(\zeta)$ and $U(\zeta)$ are equal one [35–38]. Generally, the misalignment axion is considered a canonical scalar field, but here we assume a generalized kinetic term. Because of the identity:

$$\delta\Big(\sqrt{-g}U(\zeta)\tilde{\epsilon}^{\mu\nu\rho\sigma}R^\tau{}_{\lambda\mu\nu}R^\lambda{}_{\tau\rho\sigma}\Big) = 2\sqrt{-g}U(\zeta)\Big[\tilde{\epsilon}^{\zeta\eta\rho\mu}R^{\tau\nu}{}_{\zeta\eta} + \tilde{\epsilon}^{\zeta\eta\rho\nu}R^{\tau\mu}{}_{\zeta\eta}\Big]\nabla_\rho\nabla_\tau\delta g_{\mu\nu}\,, \tag{8}$$

the action (7) yields the following field equations:

$$E_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R - T_{\mu\nu} - R_{\mu\nu} + 2\alpha\big(g_{\mu\xi}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\xi}\big)\nabla_\tau\nabla_\rho\Big(U(\zeta)\tilde{\epsilon}^{\zeta\eta\rho\xi}R^{\tau\sigma}{}_{\zeta\eta}\Big) \equiv 0\,, \tag{9}$$

where T_munu is the energy-momentum tensor made up of the scalar field and defined as:

$$T_{\mu\nu} = \frac{1}{2}\beta\omega(\zeta)\partial_\mu\zeta\partial_\nu\zeta - \frac{1}{4}\beta[\omega(\zeta)\partial_\rho\zeta\partial^\rho\zeta + 2V]g_{\mu\nu}\,. \tag{10}$$

[1] We will disregard the action provided by Eq.(4) because we are attempting to derive a vacuum solution.

The variation of the action with respect to the scalar field, ζ, yields:

$$E = \beta\nabla^{\mu}\big(\omega(\zeta)\partial_{\mu}\zeta\big) - \frac{1}{2}\beta\omega'(\zeta)\partial_{\mu}\zeta\partial^{\mu}\zeta - \beta V'(\zeta) + \alpha U'(\zeta)\epsilon^{\mu\nu\rho\sigma}R^{\tau}{}_{\lambda\mu\nu}R^{\lambda}{}_{\tau\rho\sigma} \equiv 0\,. \tag{11}$$

Now we are ready to study the field equations (9) and (11) in the frame of the solar system. Therefore, we will first apply the field equations (9) and (11) to a spherically symmetric spacetime and a non-spherically symmetric spacetime.

3 Spherically symmetric black holes in CSAET of gravity

In this section, we are going to apply the field equations (9) and (11) to the following spherically symmetric line element [39]:

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{12}$$

where $A(r)$ is an unknown function. Using the above data we get the field equations of $E_{\mu\nu} = 0$, Eq. (9), as: the (t, t)-component of Eq. (9) is,

$$0 = \frac{A\big(4\,A'r - 4 + 4\,A + 2\,r^2 V + r^2\omega\zeta'^2 A\big)}{4r^2}\,, \tag{13}$$

the (r, r)-component is

$$0 = \frac{4 - 4\,A'r - 4\,A - 2\,r^2 V + r^2\omega\zeta'^2 A}{4Ar^2}\,, \tag{14}$$

the (θ, θ)=(ϕ, ϕ)-component is

$$0 = -\frac{r}{4}\big(2\,A''r + 4\,A' + 2\,Vr + A\omega\zeta'^2 r\big)\,, \tag{15}$$

and the field equation of the scalar field, Eq. (11), takes the form:

$$0 = \frac{2\,\omega A\zeta''r - \omega'\zeta'^2 Ar + \big(2\,\omega' Ar + 2\,\omega[A'r + 2\,A]\big)\zeta' - 2\,V'r}{2r}\,. \tag{16}$$

The above system of differential equations shows that the unknown function U has no effect, and this is consistent with the result presented in the literature because of the nature of the line element (12). The above system, Eqs. (13)$\sim$(16), consists of four nonlinear differential equations in four unknown A, ω, V and ζ. So this is a closed system that has the following exact solutions:

$$\begin{aligned}
&A(r) = 1 - \frac{c_0 r^2}{6}\,, \quad V = c_0\,, \quad \omega(r) = 0\,, \text{ and } \quad \zeta(r) = \zeta(r)\,, \quad \text{or}\\
&A(r) = 1 - \frac{c_0 r^2}{6}\,, \quad V = c_0\,, \quad \omega(r) = \omega(r)\,, \text{ and } \quad \zeta(r) = c_1\,, \quad \text{or}\\
&A(r) = 1 + \frac{c_2}{r}\,, \quad V = 0\,, \quad \omega(r) = \omega(r)\,, \text{ and } \quad \zeta(r) = c_3\,, \quad \text{or}\\
&A(r) = 1 + \frac{c_2}{r}\,, \quad V = 0\,, \quad \omega(r) = 0\,, \text{ and } \quad \zeta(r) = \zeta(r)\,.
\end{aligned} \tag{17}$$

We can analyze the above set of solutions as follows:

- The first two sets of Eq. (17) are not physical solutions because they describe a system without gravitational source, i.e., there is no term in the metric potential A that behaves as $\mathcal{O}\big(\frac{1}{r}\big)$. The term that behaves as $\mathcal{O}\big(\frac{1}{r}\big)$ is the responsible one for the gravitational system.
- The last two sets of Eq. (17) represent physical solutions and can be analyzed as follows:

The last two sets of Eq. (17) show that the scalar field ζ has no effect on the metric potential in the case of spherical symmetry. These results mean that the scalar field in the case of spherical symmetry will have no effect. Furthermore, if we set the function $\omega = 1$ in (9) and (11) and apply the output field equations to the metric (20), we get the same result, indicating that the scalar field is trivial in this case as well.

4 Two constructions of CSAET gravity

The CSAET gravity can be categorized into two types: dynamic and non-dynamical. The non-dynamical construction is figured out by putting in the dimensional constant $\beta = 0$. In this case, the field equations (9) and (11) yield the following form:

$$0 = \frac{1}{2}Rg_{\mu\nu} - R_{\mu\nu} + 2\alpha\big(g_{\mu\xi}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\xi}\big)\nabla_{\tau}\nabla_{\rho}\Big(U(\zeta)\bar{\epsilon}^{\zeta\eta\rho\xi}R^{\tau\sigma}{}_{\zeta\eta}\Big)\,,$$

$$0 = U'(\zeta)\epsilon^{\mu\nu\rho\sigma} R^{\tau}{}_{\lambda\mu\nu} R^{\lambda}{}_{\tau\rho\sigma} \,. \tag{18}$$

The second equation of (18), which can be used as an evolution equation for the scalar field ζ, is now an additional differential constraint on the space of allowed solutions. Moreover, Eq. (18) shows that either $U'(\zeta) = 0$ or $\epsilon^{\mu\nu\rho\sigma} R^{\tau}{}_{\lambda\mu\nu} R^{\lambda}{}_{\tau\rho\sigma} = 0$. In this study, we assume:

$$\epsilon^{\mu\nu\rho\sigma} R^{\tau}{}_{\lambda\mu\nu} R^{\lambda}{}_{\tau\rho\sigma} = 0 \,. \tag{19}$$

The canonical choice is limited and does not permit axisymmetric rotating BH solutions [24–26] and perturbations of a certain parity of Schwarzschild black hole [32].

There is another choice to leave the field equations of the non-dynamical case to fix the unknown functions. In this case, we assume the metric to have the form[2]:

$$\begin{aligned} ds^2 = & -A(r)\Big(1 + a(r,\theta)\Big)dt^2 + \frac{1}{A(r)}\Big(1 + b(r,\theta)\Big)dr^2 \\ & + r^2\Big(1 + d(r,\theta)\Big)\Big[d\theta^2 + \sin^2\theta\Big(d\phi - \Omega(r,\theta)dt\Big)^2\Big], \end{aligned} \tag{20}$$

where the functions $a(r,\theta)$, $b(r,\theta)$, $c(r,\theta)$ and $\Omega(r,\theta)$ are of order $\mathcal{O} \sim (\eta)$. Hereafter, we take account of equations up to $\mathcal{O} \sim (\eta)$. Using Eq. (20) in Eq. (19) we get:

$$\frac{\eta}{2r^2}\Big(\sin\theta\,\Omega_{\theta r} + 2\Omega_r\cos(\theta)\Big)\Big(A_{rr}r^2 - 2 - 2rA_r + 2A\Big) = 0 \,, \tag{21}$$

where $\Omega_r = \frac{\partial\Omega(r,\theta)}{\partial r}$, $\Omega_\theta = \frac{\partial\Omega(r,\theta)}{\partial\theta}$, $A_r = \frac{dA(r)}{dr}$ and $A(r) = 1 - \frac{2M}{r}$. The solution of the above differential equation has the form:

$$\Omega(r,\theta) = \frac{\Omega(r)}{\sin^2\theta} \,, \tag{22}$$

Using Eq. (22) in the first equation of the field equation (18) we get: the (t,ϕ) component:

$$\begin{aligned} & \frac{\eta\, r\big[8\,\Omega'(r)M - r^2\Omega''(r) + 2\,r\Omega''(r)M - 2\,\Omega(r) - 4\,\Omega'(r)r\big]}{2(2\,M - r)} = 0 \,, \\ \Rightarrow \quad & \Omega(r) = \frac{c_4}{r^2}\Big(1 - \frac{2M}{r}\Big), \end{aligned} \tag{23}$$

where c_4 is a constant of integration of order $\mathcal{O} \sim \eta$. The above solution satisfies the field equations (18) in exact way up to $\mathcal{O} \sim \eta$. Using the above data we get the line element in the form:

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big)dt^2 + \Big(1 - \frac{2M}{r}\Big)^{-1}dr^2 + r^2\Big\{d\theta^2 + \sin^2\theta\Big[d\phi - \frac{c_4}{r^2\sin^2\theta}\Big(1 - \frac{2M}{r}\Big)dt\Big]^2\Big\} \,. \tag{24}$$

Eq. (24) is consistent with the result presented in [32, 40], which ensures that any solution of the non-dynamical case will coincides with Einstein's GR. The invariants of the above solution did not look different from Schwarzschild black hole solution.

The dynamical formulation allows the dimensional constant β to be arbitrary, in which the modified field equations are given by Eqs. (9) and (11). Equation (11) lifts the Pontryagin constraint, and is now an evolution equation for the CSAET coupling field. Thus, no limitation is imposed on the allowed space of solutions. As an alternative to prescribing the entire history of the CSAET coupling, one specifies some initial conditions for the scalar field ζ, which then evolves consistently through Eq. (11).

The dynamical and non-dynamical constructions are unequal and independent theories, although they share similarities in action and a few of the exact solutions. Although we can assume $\beta \to 0$ in action (7) to derive the non-dynamical frame, we cannot expect the same pattern affecting the black hole solutions of the dynamical framework can return black holes to the non-dynamical structure. A practical way to understand this is to assume smaller β in the scalar field equation. Because Pontryagin density is always non-vanishing, a smaller β impose an immense scalar field, whose value then spaces to the metric through the C-tensor, yielding an equally significant back-reaction on the geometry.

We close the discussion of this section with a few sentences on the dimensions of the scalar field and the coupling constants. The selection of the units of one set (α, β, ζ) limits the units of the others. As an example, if we assume the units of the scalar field to be $[\zeta] = L^d$, then $[\beta] = L^{-2A}$ and $[\alpha] = L^{2-d}$ with L being a unit of the length and we have assume $[\kappa] = 1$. A more logic choice can be to demand the scalar field of CS to be dimensionless, as already done in scalar tensor theories, which then demand that β be dimensionless and $[\alpha] = L^2$.[3] Other general choice could be to set $\alpha = \beta$, therefore putting S_ζ and S_{CS} on the same footing; we could then have $[\zeta] = L^{-2}$. Neither construction asks that we can choose special units for ζ, therefore, we will leave these arbitrary.

[2] In this study, we will use the Schwarzschild background solution.

[3] In this study we use the geometric units where $G = c = 1$, and therefore, the action has units of L^2. If we used the natural units $h = c = 1$, then the action will be dimensionless and if $[\zeta] = L^A$ then $[\beta] = L^{-2A-2}$ and $[\alpha] = L^{-A}$.

5 Rotating black holes in CS modified gravity

In this section,[4] we will study a rotating black hole solution in CSAET gravity when the scalar field is non-vanishing in the dynamical frame. For such purpose we are going to apply the field equations (9) and (11) to the line-element (12). Also we assume:

$$\begin{aligned} A(r,\theta) &= \eta\, a_1(r,\theta) + \mathcal{O}\big(\eta^2\big), \\ b(r,\theta) &= \eta\, b_1(r,\theta) + \mathcal{O}\big(\eta^2\big), \\ d(r,\theta) &= \eta\, d_1(r,\theta) + \mathcal{O}\big(\eta^2\big), \\ \Omega(r,\theta) &= \eta\Omega_1(r,\theta) + \mathcal{O}\big(\eta^2\big), \\ U(r,\theta) &= \eta\, U_1(r,\theta) + \mathcal{O}\big(\eta^2\big), \\ \zeta(r,\theta) &= \eta\, \zeta_1(r,\theta) + \mathcal{O}\big(\eta^2\big), \end{aligned} \tag{25}$$

where

$$\Omega_1(r,\theta) = \frac{2Ma}{r}\,, \tag{26}$$

and a is the rotation parameter of the Kerr solution whose angular momentum is defined as $J = \frac{a}{M}$.

Applying the field equations (9) and (11) to the line element (20) and after using Eqs. (25) we get the form of $E_\mu{}^\nu = 0$ up to $\mathcal{O}(\eta)$ as: the (t,ϕ)-component of Eq. (9) is

$$0 = \frac{1}{r^7}\left\{576\,\alpha\, U'(\zeta)\eta\, \cos\theta a M^2 - \beta\, \omega(\zeta) r^3\left[(2M-r)r\zeta_{rr} - \zeta_{\theta\theta} - 2(r-M)\zeta_r\zeta_\theta \cot\theta\right]r^2\right\}, \tag{27}$$

where $U'(\zeta) = \frac{\partial U(\zeta)}{\partial\zeta}$. To be able to solve the above equation we assume that

$$U'(\zeta) = \omega(\zeta)\,.$$

Using the above equation in Eq. (28) we get:

$$0 = \frac{1}{r^7}\left\{576\,\alpha\,\eta\, \cos\theta a M^2 - \beta\, r^3\left[(2M-r)r\zeta_{rr} - \zeta_{\theta\theta} - 2(r-M)\zeta_r\zeta_\theta \cot\theta\right]r^2\right\}, \tag{28}$$

The above equation is a non-homogenous partial differential equation whose homogenous solution takes the form:

$$\zeta = \zeta_1(r)\zeta_2(\theta)\,, \tag{29}$$

where $\zeta_1(r)$ and $\zeta_2(\theta)$ have the form:

$$\begin{aligned} \zeta_1(r) &= c_5 H\left[\left[\frac{\gamma}{2}, \frac{\gamma}{2}\right], \gamma, \frac{2M}{r}\right] r^{-\gamma/2} + c_6 H_1\left[\left[\frac{\gamma_1}{2}, \frac{\gamma_1}{2}\right], \gamma_1, \frac{2M}{r}\right] r^{-\gamma_1/2}, \\ \zeta_2(\theta) &= c_7 L\left[\frac{-\gamma}{2}, \cos\theta\right] + c_8 L_1\left[\frac{-\gamma}{2}, \cos\theta\right]. \end{aligned} \tag{30}$$

Here $H\big[\big[\frac{\gamma}{2}, \frac{\gamma}{2}\big], \gamma, \frac{2M}{r}\big]$ and $H_1\big[\big[\frac{\gamma_1}{2}, \frac{\gamma_1}{2}\big], \gamma_1, \frac{2M}{r}\big]$ are hypergeometric functions of the first kind, $L\left[\frac{-\gamma}{2}, \cos\theta\right]$ and $L_1\left[\frac{-\gamma}{2}, \cos\theta\right]$ are Legendre polynomials of the second kind, c_i are constants of integration and γ_i are defined as:

$$\gamma = 1 - \sqrt{1-4c_{11}}\,, \qquad \gamma_1 = 1 + \sqrt{1-4c_{11}}\,. \tag{31}$$

Now we are going to study the solution (30) to extract information on the constants of integration. For this purpose, let us consider the far-field manner of the solution, i.e., $r \gg M$, in which $\zeta_1(r)$ becomes:

$$\zeta_1(r) \sim c_7\left[1 + \frac{M}{2r}\tilde{\alpha}\right] r^{-\gamma/2} + c_8\left[1 + \frac{M}{2r}\gamma_1\right] r^{-\gamma_1/2}. \tag{32}$$

Requiring $\zeta_1(r)$ to take real value, we can see that $\gamma \in \Re$ and $\gamma_1 \in \Re$, which yields $c_{11} < 1/4$. Additionally, if $\zeta_1(r)$ has limited total energy outside the horizon, then $\zeta_1(r)$ should decompose to a constant faster than $1/r$, which yields that $\gamma > 2$ and $\gamma_1 > 2$. The formal requirement, i.e., $\gamma > 2$ cannot be recognized for any real $c_{11} < 1/4$, which forcing the constant $c_{11} = 0$, while the requirement $\gamma_1 > 2$ implies $c_{11} < 0$. This means that asking for finite total energy also yields $\zeta_1(r)$ to be not proportional to $\ln(A)$. The above discussions inform us that the homogenous solution of Eq. (30) has the form:

$$\zeta = \text{const}\,. \tag{33}$$

[4] In this section, we will assume the vanishing of the potential V and put the function $\omega(\zeta) = 1$ to make the calculations more applicable.

The particular solution of Eq. (28) gives:

$$\zeta_{Particular} = \frac{5}{8}\frac{\alpha}{\beta}\frac{a}{M}\frac{\cos(\theta)}{r^2}\left(1 + \frac{2M}{r} + \frac{18M^2}{5r^2}\right) + \text{const}\,. \tag{34}$$

The properties of a scalar field in a Kerr background are studied when considering axion hair for Kerr [40–42] and dyon [43] BHs and cosmological scenarios [44], in the frame of string theory. The solution in this study is equivalent to the one found in [40] when $U'(\zeta) = \frac{\partial U(\zeta)}{\partial \zeta} = \omega(\zeta)$.

5.1 The physical properties of the solution (20)

In this subsection, we are going to calculate the invariants of the solution derived in the previous section. For this aim we get:

$$^{*}RR = 24\,\frac{U(\zeta)\cos\theta a M^2\left(-48\,r^3 M^2 + 48\,r^4 M - 12\,r^5\right)\epsilon}{\sin(\theta)(2\,M - r)^2 r^{12}}\,. \tag{35}$$

The other scalars are the same as of Schwarzschild spacetime. Now let us analyze Eq. (35) where it has a singularity at the origin of the axis of rotation in addition to the singularity at $r = 0$. To overcome this difficulty we will put $U(\zeta) = f(r)\tan\theta$. This form of $U(\zeta)$ will remove the singularity at the origin of the axis of rotation. Moreover, the form of $f(r)$ can make the invariant $^{*}RR$ has a week singularity at large r if it behaves as $f(r) \sim r^n$ where n has a positive value and when n has a negative value the singularity of $^{*}RR$ becomes stronger.

Before closing this section, we want to stress that all the above calculations indicate that the scalar field of Chern-Simons Axion theory will always yield a dynamical scalar field when $U(\zeta)$ and ω satisfy

$$U'(\zeta) - \omega(\zeta) = 0\,. \tag{36}$$

6 Conclusion and discussion

In the present work, we have tried a first study of the physical properties of slowly rotating black holes in non-dynamical and dynamical CSAET modified gravity. We divided this study into three classes:

i. In the first class, we applied the field equations of CSAET to a spherically symmetric spacetime and derived a set of solutions. All these solutions show that the scalar field of this theory has a trivial solution, i.e., $\zeta = const$. The physics of all these solutions is known in the literature [45].

ii. In the second class, we assume the non-dynamical case, i.e., $\zeta = 0$, and applied the field equations of this case and derived a slowly rotating approximate solution up to $\mathcal{O} \sim \eta$. This case coincides completely with the results presented in [40]. The invariants of this solution do not look different from the invariants of the Schwarzschild black hole solution.

iii. In this class, we study the dynamical case where $\zeta \neq 0$. We applied this class's field equation and derived an asymptote solution up to $\mathcal{O} \sim \eta$. This class's most curious result is the scalar field's has a non-constant value under some constraints between $U'(\zeta)$ and $\omega(\zeta)$. Moreover, we have shown that the function U that characterizes CS can play a role in making the singularity weaker for the invariant $^{*}RR$.

To summarize, we used the very restrictive constraint $U'(\zeta) = \omega(\zeta)$ to derive a slowly rotating black hole solution in the frame of CSAET. If we abounded this constrain we may get a new black hole in the frame of CSAET. This project will be carried out elsewhere.

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Declarations

Conflict of interest The author have no competing interests to declare that are relevant to the content of this article.

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