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On isospin and flavour of leptons and quarks

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Abstract Isospin emerges naturally from the Lorentz transformation of spinors, if they are based on the vector representation of the Lorentz group. The resulting extended Dirac equation for a massive spin-one-half fermion has two new additional degrees of freedom associated with the up and down components of isospin. This doublet is interpreted as describing the electron and neutrino. It is adjoined with the $SU(2)$ symmetry group. The extended Dirac equation appears in six versions which are connected by similarity transformations. It is argued that this trait may explain the occurrence of the three families of the leptons and suggested that flavour arises genuinely from the algebraic properties of the extended Dirac equation. Its solutions are discussed and the physical role of isospin is elucidated. Isospin symmetry can be gauged, which leads to a weak-interaction-type theory and is valid for finite initial mass. Breaking the isospin $SU(2)$ symmetry yields the correct electric charges of the particles by means of the electroweak unification procedures of the standard model.

1 Introduction

In the standard model (SM) of elementary particle physics [1, 2], all the involved spin-one-half fermions are assumed to have zero mass. This assumption ensures chiral symmetry at the outset, and thus it enables the subjection of only the left-handed doublet-fields of quarks and leptons to the weak interaction with the $SU(2)$ gauge bosons. Why nature has chosen just the left- but not the right-handed fermion fields to take part in the unified electroweak interactions remains unexplained theoretically. But accepting this fact in the SM is a successful theoretical adjustment (Weinberg [3]) to the empirical result of parity violation in the weak interactions.

However, the real neutrinos are known to have small but finite masses, which have become indirectly apparent by the oscillations between the three neutrino generations in astroparticle experiments (solar and atmospheric neutrinos) and ground-based experiments (short-baseline reactor and long-baseline accelerator neutrinos), see reviews by, e.g. Kajita [4] and Farzan and Tórtola [5]. Also, the recent discovery by high-precision measurements at CERN of the W boson mass (Aaltonen et al. [6]) possibly hints at the need for extensions of the SM. It seems therefore desirable to develop a theory, which assumes right from the start a finite mass of the free fermions and thus breaks the chiral symmetry.

Here, we shall ad hoc assume a very small mass initial m. It may in energy units be of the order of a few eV according to the presently accepted neutrino mass, or of the order of a few GeV for the free up and down quark. The real physical masses are introduced in the SM by the Yukawa-coupling to the Higgs field [2]. This procedure shifts the problem to a determination of the coupling constants for the different fermion species in the SM. Here, we consider the "bared" leptons and quarks as components of single-mass multiplets, their real masses will of course depend on their "dressing" by the gauge fields, a problem that remains unsolved in the SM.

Recently, Marsch and Narita [7] constructed an extended Dirac equation by exploiting the four-vector representation of the Lorentz group. Their equation reveals in addition to the particle/antiparticle and spin-up/spin-down degrees of freedom of a spin-one-half fermion an isospin-type degree of freedom. Here, we derive six equivalent versions of that equation. We begin with revisiting the standard four-vector Lorentz group generators and then derive the extended Dirac equations for the fermion doublet with eight spinor components. The associated spinorial rotation and rapidity operators are discussed.

As the key new feature of this equation, the concept of isospin emerges naturally, which is associated with the $SU(2)$ symmetry. Isospin is discussed in detail, as well as its application to an electroweak-style gauge theory which is one cornerstone of the SM. As

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shown in this paper, the doublet nature of leptons and quarks in terms of up and down flavours can be described by this appropriate extension of the standard Dirac equation.

In their pioneering work, Wigner [8] and then Bargman and Wigner [9] classified all possible relativistic states of elementary particles with any spin. Yet, only for fermions of spin one half a linear and causal relativistic wave equation could be derived by Dirac [10], who did not make use of the general insights of these researchers, as he published before them his famous equation employing what became to be known as Dirac gamma matrices. They accommodate spin one half together with the particle-antiparticle doublet for a charged fermion and established relativistic quantum mechanics and quantum field theory on the basis of the Clifford matrix algebra. The problem of deriving higher-spin equations has been addressed by many physicists with varying success (see, e.g. Marsch [11] and the review until 2012 by Esposito [12]). A rather formal mathematical treatment and comprehensive literature review as of 1994 of this difficult subject can be found in the book of Fushchich and Nikitin [13].

The difficult flavour problem [14] or puzzle indicates the current inability of the flavour-physics section of the SM to explain in a satisfying way the real physical masses of the free elementary particles, as well as the specific values of the angles involved in the important PMNS and CKM mass-mixing matrices [1, 2]. We do not intend to deliver in this work an explanation of these key parameters. However, we shall address the fundamental question as to why leptons and quarks come at all in charge doublets, and why there are three generations of quarks (up-down, charm-strange, and top-bottom) and leptons (electron, muon, tau and their related neutrinos).

The main intention of this paper is to provide mathematical arguments explaining the origin of flavour. In the extended Dirac equation, which is based on the vectorial Lorentz transformation, the fermion doublets naturally originate in isospin doublets. Since the extended Dirac equation comes in six versions connected by similarity transformations, we argue that this mathematical multiplicity corresponds to real physics and yields the empirical family structure of the fermions in the SM. These results are the new findings of our paper, which is motivated by the wish to better understand solely from Lorentz invariance the origin of flavour in quantum field theory. Similar arguments on the origin of flavour were previously put forward by Marsch [15] who used the standard Dirac equation coming in six equivalent versions, which were obtained by permutation of the Pauli matrices with respect to their different possible positions in that equation.

The outline of the paper is as follows. We discuss in Sect. 2 the well-known vectorial Lorentz group and its generators and provide a new concise formula for the Lorentz transformation in Minkowski space. Then, we present in Sect. 3 the extended Dirac equation that comes in six versions (fully quoted in the Appendix), which are mathematically equivalent and connected by similarity transformations. In the Appendix, we derive the eigenfunctions of the extended Dirac equation in the Weyl basis and briefly discuss the CPT symmetry. In Sect. 4, the spinorial expressions of spin and rapidity are discussed, and the concept of isospin is derived. Sect. 5 presents the gauge theory of isospin guided by common procedures of the SM, and we discuss further the details of the covariant derivative and the effects of the CPT symmetry. Section 6 contains the discussion of our results and conclusions.

2 The Lorentz group and its generators

In this section, we provide the basic material to make the present paper self-consistent. Concerning the Lorentz transformation, we will make use of the components of the matrix three-vector $\mathbf{J}$ and $\mathbf{K}$. The Hermitian rotation vector $\mathbf{J}$ is the generator of the $SO(3)$ rotation subgroup of the Lorentz group. The anti-Hermitian vector $\mathbf{K}$ is the boost. According to their usual definitions [1, 2], the rotation and boost vectors obey the linked three-vector equations of the Lorentz algebra, which can be written concisely as $\mathbf{J}\times\mathbf{J} = \mathrm{i}\mathbf{J}$, $\mathbf{K}\times\mathbf{K} = -\mathrm{i}\mathbf{J}$, $\mathbf{J}\times\mathbf{K} = \mathbf{K}\times\mathbf{J} = \mathrm{i}\mathbf{K}$, where the cross-product sign stands for the commutator [,]. We can then define the following linear combinations

$$\mathbf{J}_{\pm} = \frac{1}{2}(\mathbf{J} \pm \mathrm{i}\mathbf{K}), \tag{1}$$

which commute with each other and obey the corresponding commutator relations $\mathbf{J}_{\pm}\times\mathbf{J}_{\pm} = \mathrm{i}\mathbf{J}_{\pm}$ and $\mathbf{J}_{\pm}\times\mathbf{J}_{\mp} = 0$. These algebraic properties are constitutive of the Lie algebra $so(3,1) = su(2)\otimes su(2)$ of the Lorentz group (LG) and signify that it can be decomposed into two commuting $su(2)$ sub-algebras determining the generators of the related $SU(2)$ sub-groups.

Following Marsch [15] and Marsch and Narita [7], we will subsequently make extensive use of the above symmetric four-vector generators of the representation of the LG. We call $\mathbf{J}_+$ the right-chiral and $\mathbf{J}_-$ the left-chiral spin operator, respectively. The chiral operators are formulated as

$$\mathbf{J}_{\pm} = \frac{1}{2}\mathbf{\Sigma}^{\pm}, \tag{2}$$

involving the subsequent generalized 4×4 spin matrices,

$$\Sigma_{\mathrm{x}}^{\pm} = \begin{pmatrix} 0 & \pm 1 & 0 & 0 \\ \pm 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix}, \quad \Sigma_{\mathrm{y}}^{\pm} = \begin{pmatrix} 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ \pm 1 & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix}, \quad \Sigma_{\mathrm{z}}^{\pm} = \begin{pmatrix} 0 & 0 & 0 & \pm 1 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ \pm 1 & 0 & 0 & 0 \end{pmatrix}, \tag{3}$$

with the commutator $[\mathbf{\Sigma}^{\pm}, \mathbf{\Sigma}^{\mp}] = 0$. Also, $\mathbf{\Sigma}^{\pm} \times \mathbf{\Sigma}^{\pm} = 2\mathrm{i}\mathbf{\Sigma}^{\pm}$. By complex conjugation of the Sigma matrices in (3), we can see that they obey $(\mathbf{\Sigma}^{\pm})^* = -\mathbf{\Sigma}^{\mp}$. Moreover, the Sigma matrices fulfil, like the Pauli matrices, an important metric condition in coordinate space, namely

$$\Sigma_j^{\pm}\Sigma_k^{\pm} + \Sigma_k^{\pm}\Sigma_j^{\pm} = 2\delta_{j,k}1_4. \tag{4}$$

Thus, the Sigma component matrices squared give unity, and their sum yields, $(\mathbf{\Sigma}^{\pm})^2 = 3\,1_4$. Here, 1_4 means the 4×4 unit matrix (and similarly 1_2 the 2×2 unit matrix). With the help of these matrices, we can reformulate again the four-vector Lorentz transformation and cast it into a form that manifestly shows the $SU(2) \otimes SU(2)$ group structure. In addition, we define the new Delta matrix. It corresponds to the metric in Minkowski space-time and is defined as

$$\Delta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{5}$$

The square of the Delta matrix is the four-dimensional unit matrix, $\Delta^2 = 1_4$. Delta has the important property that $\mathbf{\Sigma}^{\pm} = \Delta\mathbf{\Sigma}^{\mp}\Delta$, which means by help of the Delta matrix one can flip the sign of the Sigma matrices.

When use is made of Eq. (1), and the boost-angle vector $\boldsymbol{\beta}$ and rotation-angle vector $\boldsymbol{\theta}$ are introduced [2], we can write the four-vector Lorentz transformation rather symmetrically as

$$\Lambda_{\mathrm{V}} = \exp\left(\mathrm{i}\boldsymbol{\theta}_+ \cdot \mathbf{J}_+ + \mathrm{i}\boldsymbol{\theta}_- \cdot \mathbf{J}_-\right) = \exp\left(\mathrm{i}\boldsymbol{\theta}_+ \cdot \mathbf{J}_+\right)\exp\left(\mathrm{i}\boldsymbol{\theta}_- \cdot \mathbf{J}_-\right), \tag{6}$$

where the complex angle vectors $\boldsymbol{\theta}_{\pm} = \boldsymbol{\theta} \mp \mathrm{i}\boldsymbol{\beta} = \boldsymbol{\theta}^*_{\mp}$ were defined. We also exploited Eq. (1), according to which $\mathbf{J}_+$ and $\mathbf{J}_-$ commute, so that their exponential functions can be separated. They are not exponential functions in a strict sense, but are Lie group operators defined through the Taylor expansion of the exponential functions. Thus, these operators do not return a value but rather change a state (or the spinor) into a different one by the Lorentz transformation. By help of the Sigma matrices, we can rewrite Λ_{V} as follows:

$$\Lambda_{\mathrm{V}} = \exp\left(\frac{\mathrm{i}}{2}\boldsymbol{\theta}_+ \cdot \mathbf{\Sigma}_+\right)\exp\left(\frac{\mathrm{i}}{2}\boldsymbol{\theta}_- \cdot \mathbf{\Sigma}_-\right) = \Lambda_{\mathrm{R}}\Lambda_{\mathrm{L}}. \tag{7}$$

The plus sign denotes the right-chiral (R) and the minus sign the left-chiral (L) Lorentz transformation, whereby this affiliation is conventional but arbitrary. We will continue using the indices R and L throughout the remainder of this paper. Note that when taking the complex conjugate of Eq. (6), we find that $\Lambda_{\mathrm{V}} = \Lambda_{\mathrm{V}}^*$ by means of the properties of the Sigma matrices in Eq. (3).

Therefore, the four-vector Lorentz transformation is a real 4×4 matrix operator, as it should since it operates on a real four-vector V^{μ} in Minkowski space. But in what follows, we will let the Lorentz transformation operators act on complex four-vectors, which we may call "Minkowski spinors" replacing the Pauli spinors of the standard Dirac equation.

3 Six realizations of the extended Dirac equation

Throughout we use the conventional units of QFT, with $c = \hbar = 1$. As usual, the particle mass is denoted as m and its spin quantum number as s. The quantum mechanical covariant four-momentum operator is $P_{\mu} = (E, -\mathbf{p})$, yielding $P_{\mu} = \mathrm{i}\partial_{\mu}$ with the covariant space-time derivative given as $\partial_{\mu} = (\partial/\partial t, \partial/\partial\mathbf{x})$. In their recent paper, Marsch and Narita [7] derived an extended Dirac equation by exploiting the above discussed algebra of the four-vector generators of the Lorentz group. They obtained this new equation for the bi-spinor $\Psi^{\dagger} = (\psi_{\mathrm{R}}^{\dagger}, \psi_{\mathrm{L}}^{\dagger})$ in a basis that is different from the Weyl basis, which while being more appropriate we are going to use here. To obtain this new result, we let the above Lorentz group generators act on complex four-component Minkowski spinors $\psi_{\mathrm{R,L}}^{\dagger}$. The equation for this extended Dirac spinor then reads

$$\Gamma^{\mu}P_{\mu}\Psi = m\Psi. \tag{8}$$

In what follows, we will derive six different versions of the extended Dirac equation, some of which we shall indicate by an appropriate index from now on. All the six versions are quoted in the Appendix. As compared to [7], we use here the Gamma matrices in the Weyl basis. They look explicitly as follows:

$$\mathbf{\Gamma}_{\mathrm{W}} = \begin{pmatrix} 0 & \Delta\mathbf{\Sigma}_{\mathrm{L}} \\ -\Delta\mathbf{\Sigma}_{\mathrm{R}} & 0 \end{pmatrix}, \quad \Gamma_{0\,\mathrm{W}} = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}, \quad \Gamma_{5\,\mathrm{W}} = \begin{pmatrix} -1_4 & 0 \\ 0 & 1_4 \end{pmatrix}. \tag{9}$$

These Gammas obey of course the Clifford algebra. Also the usual chiral projection operator based on $\Gamma_{5\,\mathrm{W}}$ can be defined. We have $\Gamma_{0\,\mathrm{W}}^2 = \Gamma_{5\,\mathrm{W}}^2 = 1_8$ and $\Gamma_{j\,\mathrm{W}}^2 = -1_8$, for $j = x, y, z$. Obviously, for a massless fermion the extended Dirac equation in the Weyl basis decomposes into the two independent Weyl equations for the right- and left-chiral spinor fields. It should be noted that Pauli matrices do not appear in Eq. (9).

In order to transform the above equation into its form given in the Dirac basis, we make now use of the following unitary similarity transformation

$$U = \frac{1}{\sqrt{2}}\left(1_8 - \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}\right), \tag{10}$$

and we recall that the real matrix $\Delta^2 = 1_4$. Obviously, U^{-1} is obtained by replacing the minus by a plus sign before the inner bracket in Eq. (10), and thus $U^{-1}U = UU^{-1} = 1_8$. When operating with U from the left on Eq. (8), we obtain the Dirac equation in the Dirac basis for the transformed spinor $\Psi_\mathrm{D} = U\Psi_\mathrm{W}$. Thereby, the three-vector matrix remains unchanged, i.e. $\boldsymbol{\Gamma}_\mathrm{D} = \boldsymbol{\Gamma}_\mathrm{W}$, and the other two matrices are interchanged: $\Gamma_{0\,\mathrm{D}} = -\Gamma_{5\,\mathrm{W}}$, and $\Gamma_{5\,\mathrm{D}} = \Gamma_{0\,\mathrm{W}}$. For the matrices in Dirac representation, see the Appendix.

In order to transform Eq. (9) into its third possible form, we make now use of the following unitary similarity transformation

$$V = \frac{1}{\sqrt{2}}\left(1_8 - \mathrm{i}\begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}\right), \tag{11}$$

and we again recall that the real matrix $\Delta^2 = 1_4$. Obviously, V^{-1} is obtained by replacing the minus by a plus sign in front of the inner bracket in Eq. (11), and thus $V^{-1}V = VV^{-1} = 1_8$. When operating with V from the left and with V^{-1} from the right on Eq. (12), we obtain the Dirac matrices in the Weyl basis for the transformed spinor $V\Psi_\mathrm{MN} = \Psi_\mathrm{W}$. Thereby, the time-like component and the chiral component are transformed in the same fashion as $V\Gamma_{0,\mathrm{MN}}V^{-1} = \Gamma_{0\,\mathrm{W}}$, and $V\Gamma_{5,\mathrm{MN}}V^{-1} = \Gamma_{5\,\mathrm{W}}$. Thus, we obtain the third version of the extended Dirac equation, whereby the new Gamma matrices look explicitly as follows:

$$\boldsymbol{\Gamma}_\mathrm{MN} = \mathrm{i}\begin{pmatrix} \boldsymbol{\Sigma}_\mathrm{R} & 0 \\ 0 & -\boldsymbol{\Sigma}_\mathrm{L} \end{pmatrix}, \quad \Gamma_{0\,\mathrm{MN}} = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}, \quad \Gamma_{5\,\mathrm{MN}} = \begin{pmatrix} 0 & -\mathrm{i}\Delta \\ \mathrm{i}\Delta & 0 \end{pmatrix}. \tag{12}$$

The Marsch–Narita representation of the Gammas differs from those in the Dirac or Weyl basis in the diagonal form of the space-like component. In contrast, the Dirac basis diagonalizes the time-like component, and the Weyl basis the chiral component. Thus, the Marsch–Narita basis logically completes the set of non-trivial spinor representations.

We note that the standard Dirac gamma matrices in the Weyl or Dirac basis are obtained from Eqs. (9) and (12) by replacing Δ by 1_2 and $\boldsymbol{\Sigma}_\mathrm{R,L}$ by the Pauli $\boldsymbol{\sigma}$ vector, and then Ψ by the Dirac spinor ψ. So the corresponding standard Dirac matrices in this somewhat unusual Marsch–Narita basis look as follows:

$$\boldsymbol{\gamma}_\mathrm{MN} = \mathrm{i}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}, \quad \gamma_{0\,\mathrm{MN}} = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \quad \gamma_{5\,\mathrm{MN}} = \begin{pmatrix} 0 & -\mathrm{i}1_2 \\ \mathrm{i}1_2 & 0 \end{pmatrix}. \tag{13}$$

This is the first key suggestion we want to make in this paper. The three-family structure of the spin-one-half fermions originates from a kind of permutation symmetry of the extended Dirac equation owing to similarity transformations. Similar ideas have been put forward for the standard Dirac equation by Marsch [15], in which he used six possible permutations of the Pauli matrices involved in the definition of the Dirac gamma matrices. In the present situation, in connection with our Gammas, we are lead to another non-trivial representation of the generators of the $SU(2)$ group, which are given by

$$\tilde{\boldsymbol{\sigma}}_\mathrm{x} = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}, \quad \tilde{\boldsymbol{\sigma}}_\mathrm{y} = \begin{pmatrix} 0 & -\mathrm{i}\Delta \\ \mathrm{i}\Delta & 0 \end{pmatrix}, \quad \tilde{\boldsymbol{\sigma}}_\mathrm{z} = \begin{pmatrix} 1_4 & 0 \\ 0 & -1_4 \end{pmatrix}. \tag{14}$$

In terms of these matrices, the set of three different similarity transformations is given as follows:

$$U = \frac{1}{\sqrt{2}}(1_8 - \mathrm{i}\tilde{\boldsymbol{\sigma}}_\mathrm{y}), \quad V = \frac{1}{\sqrt{2}}(1_8 - \mathrm{i}\tilde{\boldsymbol{\sigma}}_\mathrm{x}), \quad W = \frac{1}{\sqrt{2}}(1_8 - \mathrm{i}\tilde{\boldsymbol{\sigma}}_\mathrm{z}). \tag{15}$$

In essence, the different similarity transformations reflect different components of the Pauli matrices Eq. (14). With the help of Eq. (15), we can obtain even more representations of the Gamma matrices by various similarity transformations. The corresponding expressions are give in Appendix. Apparently, the six resulting versions of the extended Dirac equation obtained by this procedure are closely connected with the above three Pauli-type matrices as generators of the $SU(2)$ group. This notion reflects the three dimensions of the real physical space, and again suggests that the six Gamma versions have physical meaning as a kind of "optional" degrees of freedom. It may explain why in the SM there exist exactly three fermion generations or families, as they were found empirically but appear to be redundant, since the difference between fermion families is only in the mass. How the flavour doublets are realized in nature is another fundamental question. To answer it, we make a second key suggestion: the fermion generations or families are realization of the different bases of the extended Dirac equation. This suggestion is related to the isospin symmetry as a consequence of the $SU(2) \times SU(2)$ structure of the Lorentz group.

In conclusion of this section, we obtain six versions of the extended Dirac equation, which come in pairs (as given in the Appendix) that are connected by similarity transformations. In the subsequent section, it is shown that the related spin operator is the same for all six versions. Yet, the question arises, whether the existence of these versions has some physical meaning. Indeed the associated mathematical solutions can be transformed into each other, but there seems to appear a "threefold-similarity" degree of freedom related to the three basis transformations Eq. (15) based on the Pauli-type matrices of Eq. (14) quoted above. We therefore suggest that this threefoldness corresponds to the three different families of isospin doublets of leptons and quarks of the spin one-half fermions. Consequently, the three family structure simply reflects the three dimensions of coordinate space, since the Pauli matrices are just the generators of the spinor representation of the rotation group. The implication is that there are no further quark or lepton generations beyond the third one, which is in agreement with the estimated number of neutrino generations according to the particle data group collaboration [16] and cosmological constraints [17].

4 Spin, rapidity and isospin

For all previously defined sets of Gamma matrices, the related spin (spinorial rotation) operator and rapidity (spinorial boost) operator take the same form. The rapidity operator for spinors is defined generally as

$$\mathbf{R} = \frac{\mathrm{i}}{2}\Gamma_0\boldsymbol{\Gamma}, \tag{16}$$

and similarly the spin operator for spinors is generally defined as

$$S_{\mathrm{x}} = \frac{\mathrm{i}}{2}\Gamma_{\mathrm{y}}\Gamma_{\mathrm{z}}, \quad \text{(cyclic index permutation)}. \tag{17}$$

By insertion of the expressions for the Gamma and Sigma matrices one obtains for the spin and rapidity operators, for example in the Weyl basis, the result

$$\mathbf{R}_{\mathrm{W}} = \frac{\mathrm{i}}{2}\begin{pmatrix} -\boldsymbol{\Sigma}_{\mathrm{R}} & 0 \\ 0 & \boldsymbol{\Sigma}_{\mathrm{L}} \end{pmatrix}. \tag{18}$$

$$\mathbf{S}_{\mathrm{W}} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\Sigma}_{\mathrm{R}} & 0 \\ 0 & \boldsymbol{\Sigma}_{\mathrm{L}} \end{pmatrix}. \tag{19}$$

So both operators are fully determined by the Sigma matrix operators, and they employ the right-chiral spin as well as left-chiral spin operator. Applying the rules of commutation of the Sigma matrices, we find after straightforward algebra that $\mathbf{S}_{\mathrm{W}}$ and $\mathbf{R}_{\mathrm{W}}$ also obey the Lorentz algebra, i.e. we have $\mathbf{S}_{\mathrm{W}} \times \mathbf{S}_{\mathrm{W}} = \mathrm{i}\mathbf{S}_{\mathrm{W}}$, $\mathbf{R}_{\mathrm{W}} \times \mathbf{R}_{\mathrm{W}} = -\mathrm{i}\mathbf{S}_{\mathrm{W}}$, and $\mathbf{R}_{\mathrm{W}} \times \mathbf{S}_{\mathrm{W}} = \mathrm{i}\mathbf{R}_{\mathrm{W}}$. One can see that $\mathbf{S}_{\mathrm{W}}$ is Hermitian and corresponds to the rotation operator $\mathbf{J}$, and $\mathbf{R}_{\mathrm{W}}$ is anti-Hermitian and corresponds to the boost operator $\mathbf{K}$ of the four-vector Lorentz group generators.

It turns out that the spin operator in the Weyl, Dirac and Marsch–Narita basis has the same mathematical form as given in Eq. (19). So we can continue using the non-indexed symbol $\mathbf{S}$ for all six Gamma matrix versions. But the rapidity operators differ. We obtain in the Dirac and Marsch–Narita basis the results

$$\mathbf{R}_{\mathrm{D}} = \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & \Delta\boldsymbol{\Sigma}_{\mathrm{L}} \\ \Delta\boldsymbol{\Sigma}_{\mathrm{R}} & 0 \end{pmatrix}, \quad \mathbf{R}_{\mathrm{MN}} = \frac{1}{2}\begin{pmatrix} 0 & \Delta\boldsymbol{\Sigma}_{\mathrm{L}} \\ -\Delta\boldsymbol{\Sigma}_{\mathrm{R}} & 0 \end{pmatrix}. \tag{20}$$

There is another important operator that commutes with the rapidity operators as well as with the single spin $\mathbf{S}$ operator. It is the isospin operator that is defined as

$$\mathbf{I} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\Sigma}_{\mathrm{L}} & 0 \\ 0 & \boldsymbol{\Sigma}_{\mathrm{R}} \end{pmatrix}. \tag{21}$$

It obeys $\mathbf{I}^2 = \frac{3}{4}1_8$ and $\mathbf{I} \times \mathbf{I} = \mathrm{i}\mathbf{I}$. Importantly, we obtain $[\mathbf{I}, \Gamma^\mu] = 0$ for all Gamma matrices we discussed so far, and thus $[\mathbf{I}, \Gamma_5] = 0$ holds as well. Almost trivially, $[\mathbf{I}, \mathbf{S}] = 0$ is true, because all right- and left-chiral Sigma matrix vector components commute with each other. Therefore, the isospin operator has a unique nature and essentially reflects the basic $SU(2) \otimes SU(2)$ symmetry of the Lorentz group, and therewith it reveals the chiral trait of the Lorentz transformation.

Finally, we discuss the spinorial Lorentz transformation. In analogy to the discussion of the rotation and boost operator for a four-vector V^μ in Minkowski space, we can define the right-and left-chiral rotation-boost operators (omitting here the name indices) for spinors

$$\mathbf{S}_{\pm} = \frac{1}{2}(\mathbf{S} \pm \mathrm{i}\mathbf{R}) = \mathbf{S}P_{\pm}. \tag{22}$$

The last step is obtained by exploiting Eqs. (16) and (17). The projection operator is defined as

$$P_{\pm} = \frac{1}{2}(1_8 \mp \Gamma_5). \tag{23}$$

The projectors are idempotent, and their sum gives $P_+ + P_- = 1_8$. In analogy to Eq. (6) we can then generally write the spinorial Lorentz transformation as

$$\Lambda_{\mathrm{S}} = \exp\left(\mathrm{i}\boldsymbol{\theta}_+ \cdot \mathbf{S}_+ + \mathrm{i}\boldsymbol{\theta}_- \cdot \mathbf{S}_-\right) = \exp\left(\mathrm{i}\boldsymbol{\theta}_+ \cdot \mathbf{S}_+\right)\exp\left(\mathrm{i}\boldsymbol{\theta}_- \cdot \mathbf{S}_-\right), \tag{24}$$

involving again the complex angle vectors $\boldsymbol{\theta}_{\pm} = \boldsymbol{\theta}^{*}_{\mp}$. Here, we exploited the fact that $\mathbf{S}_{+}$ and $\mathbf{S}_{-}$ commute, so that the group operators can be separated in a multiplicative way. By use of Eq. (22) and the properties of the projection operators, we can finally write, for any of the basis realizations of the extended Dirac equation, the Lorentz transformation in form of the useful decomposition

$$\Lambda_{\mathrm{S}} = \exp(\mathrm{i}\boldsymbol{\theta}_{+} \cdot \mathbf{S})P_{+} + \exp(\mathrm{i}\boldsymbol{\theta}_{-} \cdot \mathbf{S})P_{-}. \tag{25}$$

For zero boost angle $\boldsymbol{\beta} = \mathbf{0}$, we simply get for Λ the spinorial rotation as determined by the operator $\exp(\mathrm{i}\boldsymbol{\theta} \cdot \mathbf{S})$. Although $\mathbf{S}$ is Hermitian, the Lorentz transformation is not, because of the complex angle vectors $\boldsymbol{\theta}_{\pm}$. These considerations are all rather general. Upon insertion of the specific results from the Weyl representation, we obtain the block-diagonal Lorentz transformation in the form

$$\Lambda_{\mathrm{S}} = \begin{pmatrix} \Lambda_{\mathrm{R}} & 0 \\ 0 & \Lambda_{\mathrm{L}} \end{pmatrix} = \begin{pmatrix} \exp(\frac{\mathrm{i}}{2}\boldsymbol{\theta}_{+} \cdot \boldsymbol{\Sigma}_{\mathrm{R}}) & 0 \\ 0 & \exp(\frac{\mathrm{i}}{2}\boldsymbol{\theta}_{-} \cdot \boldsymbol{\Sigma}_{\mathrm{L}}) \end{pmatrix}. \tag{26}$$

This acts on the bi-spinor $\Psi^{\dagger} = (\psi_{\mathrm{R}}^{\dagger}, \psi_{\mathrm{L}}^{\dagger})$. The two elements of the 2×2 matrix Λ_{S} are identical to the two factors appearing in the vectorial Lorentz transformation Λ_{V} in Eq. (7). Therefore, the left- and right-chiral components transform independently like

$$\Psi^{\Lambda} = \Lambda_{\mathrm{S}}\Psi = \begin{pmatrix} \Lambda_{\mathrm{R}}\psi_{\mathrm{R}} \\ \Lambda_{\mathrm{L}}\psi_{\mathrm{L}} \end{pmatrix}. \tag{27}$$

This simple form is only possible in the Weyl but not the Dirac or Marsch–Narita basis. The Weyl version of the extended Dirac equation with the Gamma matrices as in Eq. (9) can also be derived by starting from the Lorentz transformation of Eq. (27). One can then construct the related Lagrangian according to the procedure described by Schwartz [2] in his textbook to derive the standard Dirac equation in the Weyl basis. Moreover, we want to mention that the isospin operator commutes with the spinorial Lorentz transformation operator, which is obvious from their specific forms in Eqs. (21) and (27), but also true for all six versions of the extended Dirac equation (see again the Appendix), since the isospin three-vector operator $\mathbf{I}$ commutes with the projection operators and the spin operator as well, which are the ingredients in the general Lorentz transformation Eq. (25).

5 Isospin gauge theory and SU(2) symmetry

As we have shown in the previous section, the extended Dirac equation based on the vector representation of the Lorentz group has in addition to the usual spin the new isospin. Both act on a complex four-component Minkowski spinor. It is named in analogy to the two-component Pauli spinor which describes the physical spin in terms of the fundamental representation of $SU(2)$ as expressed by the three Pauli matrices. Thus, the fermion field in our model has not only spin as angular momentum but also isospin forming a doublet, which we interpret as being related to the up and down components of the leptons and quarks, under the condition that they still have equal masses and hypercharges and not yet acquired their individual charges by electro-weak symmetry breaking. We stress again that the isospin vector commutes with all five Gamma matrices in the respective different representations.

We can now define a non-trivial isospin phase operator which corresponds to $SU(2)$ symmetry in its extended representation as given by the three isopin matrices of Eq. (21). We may write $\mathbf{I} = \frac{1}{2}\boldsymbol{\Sigma}$, where the definition of the $\boldsymbol{\Sigma}$ is obvious from Eq. (21). It is block diagonal and involves both chiral matrices $\mathbf{6}_{\mathrm{R,L}}$. But it does not mix the two chiral components of Ψ. We also include the $U(1)$ hypercharge symmetry with the coupling constant g' and the adequately normalized hypercharge operator $Y = \frac{y}{2}1_8$, which commutes with all Gammas and the isospin $\boldsymbol{\Sigma}$. The hypercharge is denoted y. In conventional notation, we use g for the $SU(2)$ related coupling constant. As mentioned, the three isospin matrices defined here provide a non-fundamental representation of $SU(2)$. We can then write the general local phase operator acting on the spinor as

$$P(\alpha, \boldsymbol{\lambda}) = \exp\frac{\mathrm{i}}{2}(g'\alpha(x)y1_8 + g\boldsymbol{\lambda}(x) \cdot \boldsymbol{\Sigma}). \tag{28}$$

Here, α is a scalar number, and $\boldsymbol{\lambda}$ is a three-vector of real numbers. Because we want to transit to an isospin gauge theory, we let $\alpha(x)$ and $\boldsymbol{\lambda}(x)$ be functions of the space-time coordinate x, used as an abbreviation of $x^{\mu} = (t, \mathbf{x})$. The derivatives of these scalar and three-vector quantities give the gauge fields

$$B^{\mu}(x) = \partial^{\mu}\alpha(x), \quad \mathbf{W}^{\mu} = \partial^{\mu}\boldsymbol{\lambda}(x) = (W_{\mathrm{x}}^{\mu}, W_{\mathrm{y}}^{\mu}, W_{\mathrm{z}}^{\mu}). \tag{29}$$

The phase operator Eq. (28) acts on the chiral doublet $\Psi^{\dagger} = (\psi_{\mathrm{R}}^{\dagger}, \psi_{\mathrm{L}}^{\dagger})$. However, it does not mix the doublet components, and thus it conserves chirality. However, the individual matrices $\mathbf{6}_{\mathrm{R,L}}$, respectively, mix the four components of each $\psi_{\mathrm{R,L}}$, and thus mix the up and down components of spin and isospin. So the "spin quartet" states are completely mixed by the chiral spin operators. By including a general non-local phase, we obtain the spinor field $\Psi_{\mathrm{P}}(x) = P(\alpha(x), \boldsymbol{\lambda}(x))\Psi(x)$. Differentiation with ∂^{μ} requires to introduce the covariant derivative. Namely, to make the kinetic terms of the Lagrangian of the extended Dirac equation invariant under the local hypercharge and isospin symmetries, we must elevate the ordinary derivative to a covariant one [1–3] that is usually defined by

$$D^{\mu} = \partial^{\mu} - \mathrm{i}\frac{1}{2}(g'B^{\mu}(x)y1_8 + g\mathbf{W}^{\mu}(x) \cdot \boldsymbol{\Sigma}). \tag{30}$$

We recall that the isospin operator and thus $\mathbf{\Sigma}$ commutes with Γ^μ, and therefore the introduction of D^μ does not cause any algebraic problems. We may now rearrange the gauge fields by means of Weinberg mixing and introduce new ones by help of the linear combination

$$\begin{pmatrix} B^\mu \\ W_z^\mu \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} A^\mu \\ Z_z^\mu \end{pmatrix}. \tag{31}$$

Here, $c = \cos(\theta)$ and $s = \sin(\theta)$. The variable theta was already used for the Lorentz group transformation (rotation and boost). To avoid confusion, we stress that it in this section it refers to the mixing of the gauge fields. We introduce also the following isospin matrices $\Sigma_\pm = (\Sigma_x \pm i\Sigma_y)/2$, and correspondingly the complex conjugate gauge fields $W_\pm^\mu = W_x^\mu \pm iW_y^\mu$. In terms of these quantities, one can rewrite the covariant derivate as

$$D^\mu = \partial^\mu - i(Q_e A^\mu(x) + Q_w Z^\mu(x) + g\Sigma_- W_+^\mu(x) + g\Sigma_+ W_-^\mu(x)). \tag{32}$$

Here, we followed closely the usual procedures of the SM [1, 2] and introduced via Eq. (31) the electromagnetic vector field A^μ and the neutral-current boson gauge field Z^μ, as well as the charged boson field $W_\pm^\mu$. The electric charge 8×8 matrix Q_e is defined as

$$Q_e = \frac{1}{2}(yg' \mathbb{1}_8 c + g\Sigma_z s). \tag{33}$$

The weak "neutral charge" 8×8 matrix Q_w is defined as

$$Q_w = \frac{1}{2}(-yg' \mathbb{1}_8 s + g\Sigma_z c). \tag{34}$$

We remind the reader again of the fact that Q_e, Q_w, as well as $\Sigma_\pm$ are linear combinations of $\mathbb{1}_8$ and $\mathbf{\Sigma}$, and as such they still commute with Γ^μ! Conventionally, the electron has the hypercharge $y = -1$. For the quarks we assume the hypercharge $y = \frac{1}{3}$ [2], but we will here just discuss the leptons. When we now impose the condition, $g'c = gs = e$, we obtain the electric-charge matrix as

$$Q_e = -\frac{e}{2} \begin{pmatrix} \mathbb{1}_4 - \Sigma_{L\,z} & 0 \\ 0 & \mathbb{1}_4 - \Sigma_{R\,z} \end{pmatrix}. \tag{35}$$

Here, the charge of $-e$ corresponds to the electron and 0 to the neutrino. Furthermore, $\tan\theta_W = g'/g$ and $\cot\theta_W = g/g'$ fixes the angle by the so-called Weinberg angle θ_W [3] defined by the ratio of the coupling constants. Insertion of the above condition into the weak-charge matrix yields

$$Q_w = \frac{e}{2} \begin{pmatrix} \tan\theta_W\, \mathbb{1}_4 + \cot\theta_W\, \Sigma_{L\,z} & 0 \\ 0 & \tan\theta_W\, \mathbb{1}_4 + \cot\theta_W\, \Sigma_{R\,z} \end{pmatrix}. \tag{36}$$

We recall that according to Eq. (21), we have

$$\Sigma_\pm = \begin{pmatrix} \Sigma_L \pm & 0 \\ 0 & \Sigma_R \pm \end{pmatrix}. \tag{37}$$

For the calculation of the eigenvalues of the charge operators Q_e and Q_w, we require the eigenfunctions of the extended Dirac equation, which are provided in the Appendix. For that purpose, we have to reconsider the eigenvalue equations with respect to the isospin, which are given in Eq. (57) in Appendix. The index j numbers the four eigenvectors u_j of the particle and v_j of the antiparticle, which are derived in detail in Appendix. There we also introduce the isospin eigenvalues given by the row vector $e_{Lj} = (-1, -1, 1, 1)$, which appears in the eigenvalue equation

$$I_z u_j = \frac{1}{2} e_{Lj} u_j, \quad I_z v_j = \frac{1}{2} e_{Lj} v_j \tag{38}$$

Since the isospin commutes with all Gammas, we obtain the same eigenvalues for the eigenfunctions of the moving fermion, which are given in Eq. (59) in Appendix. A similar equation is obtained after Eq. (56) in Appendix for the spin in the rest frame

$$S_z u_{0j} = \frac{1}{2} e_{Rj} u_{0j}, \quad S_z v_{0j} = \frac{1}{2} e_{Rj} v_{0j}, \tag{39}$$

where the spin eigenvalue is given by the row vector $e_{Rj} = (1, -1, 1, -1)$. However, since the spin does not commute with the Gammas, this equation cannot be transferred to the moving frame. Yet, the fermion states may still be classified according to their spin values in the rest frame. We are now in the position to calculate the eigenvalues of the charge operator defined in Eq. (35) and Eq. (36). Thus, we obtain the electric charges from

$$Q_e u_j = -\frac{e}{2}(\mathbb{1}_8 - \Sigma_z) u_j = (-e)(1 - e_{Lj}) u_j = q_j u_j. \tag{40}$$

Similarly, we obtain for the weak-charge operator the eigenvalue equation

$$Q_w u_j = \frac{e}{2}(\tan\theta_{\mathrm{W}} \mathbf{1}_8 + \cot\theta_{\mathrm{W}} \Sigma_z) u_j = \frac{e}{2}(\tan\theta_{\mathrm{W}} + \cot\theta_{\mathrm{W}} e_{\mathrm{L}j}) u_j = w_j u_j. \tag{41}$$

For the spin quantum numbers we obtained above in Eq. (39), a simpler equation, $s_j = e_{\mathrm{R}j} = (1, -1, 1, -1)$. The electric charges are given by $q_j = -e(1 - e_{\mathrm{L}j}) = e(-1, -1, 0, 0)$, and the weak charges by the more lengthy expression $w_j = e(\tan\theta_{\mathrm{W}} - \cot\theta_{\mathrm{W}}, \tan\theta_{\mathrm{W}} - \cot\theta_{\mathrm{W}}, \tan\theta_{\mathrm{W}} + \cot\theta_{\mathrm{W}}, \tan\theta_{\mathrm{W}} + \cot\theta_{\mathrm{W}})$, which depends on the Weinberg angle.

Finally, in order to complete the transformation of the covariant derivative, we need explicit forms for the $\Sigma_{\mathrm{R,L}\pm}$ matrices. On the basis of Eq. (3), we obtain

$$\Sigma_{\mathrm{R}\pm} = \frac{1}{2}\begin{pmatrix} 0 & 1 & \pm\mathrm{i} & 0 \\ 1 & 0 & 0 & \mp 1 \\ \pm\mathrm{i} & 0 & 0 & -\mathrm{i} \\ 0 & \pm 1 & \mathrm{i} & 0 \end{pmatrix}, \quad \Sigma_{\mathrm{L}\pm} = \frac{1}{2}\begin{pmatrix} 0 & -1 & \mp\mathrm{i} & 0 \\ -1 & 0 & 0 & \mp 1 \\ \mp\mathrm{i} & 0 & 0 & -\mathrm{i} \\ 0 & \pm 1 & \mathrm{i} & 0 \end{pmatrix}, \tag{42}$$

Then, we can state explicitly the effects that these operators have on the Minkowski spinors χ_j (with $j = 1, 2, 3, 4$) of the basis as given in Eq. (55) in Appendix. After some lengthy calculation, we obtain

$$\begin{array}{cccc} \Sigma_{\mathrm{R}+}\chi_1 = 0 & \Sigma_{\mathrm{R}+}\chi_2 = \chi_1 & \Sigma_{\mathrm{R}+}\chi_3 = 0 & \Sigma_{\mathrm{R}+}\chi_4 = \chi_3 \\ \Sigma_{\mathrm{R}-}\chi_1 = \chi_2 & \Sigma_{\mathrm{R}-}\chi_2 = 0 & \Sigma_{\mathrm{R}-}\chi_3 = \chi_4 & \Sigma_{\mathrm{R}-}\chi_4 = 0 \\ \Sigma_{\mathrm{L}+}\chi_1 = -\chi_3 & \Sigma_{\mathrm{L}+}\chi_2 = -\chi_4 & \Sigma_{\mathrm{L}+}\chi_3 = 0 & \Sigma_{\mathrm{L}+}\chi_4 = 0 \\ \Sigma_{\mathrm{L}-}\chi_1 = 0 & \Sigma_{\mathrm{L}-}\chi_2 = 0 & \Sigma_{\mathrm{L}-}\chi_3 = -\chi_1 & \Sigma_{\mathrm{L}-}\chi_4 = -\chi_2 \end{array} \tag{43}$$

In conclusion, we have derived and discussed all key quantities appearing in the covariant derivative Eq. (32). We may summarize the results in a little formal table showing the connections between the various spin-one-half fermion states:

$$\begin{array}{cccc} \text{spin} & s_j & (1, -1, 1, -1) & (\uparrow, \downarrow, \uparrow, \downarrow) \\ \text{charge} & q_j & (-e, -e, 0, 0) & e, e, \nu, \nu \\ \text{function} & \chi_j & 1, 2, 3, 4 & \\ \text{up flip} & \Sigma_+\chi_j & 1, 2, 3, 4 \Rightarrow 3, 4, 0, 0 & \\ \text{down flip} & \Sigma_-\chi_j & 1, 2, 3, 4 \Rightarrow 0, 0, 1, 2 & \end{array} \tag{44}$$

As is well-known from the SM [1, 2], the operators $\Sigma_\pm$ do, according to the above table, change the nature of the fermion state from electron to neutrino and vice versa, but apparently leave their spins unchanged. Since the isospin operator commutes will all Gammas, including the chirality operator Γ_5 and the associated projection operators $P_\pm$, it cannot change chirality, and thus it cannot mix the states u and v. This means it does not mix particles with their antiparticles. They have different expectation or mean values of their charges, though, with $\bar{u} Q_e u = -e$ but $\bar{v} Q_e v = +e$.

As final topic of this section, we address some symmetry properties of the covariant derivative given in Eq. (32). The definitions of the symmetry operations charge conjugation, parity and time inversion for the extended Dirac equation can be found in the second subsection of Appendix. We abbreviate the expression in the bracket of Eq. (32) as δ^μ such that $\mathrm{i}D^\mu = \mathrm{i}\partial^\mu + \delta^\mu$. We quote here δ^μ again:

$$\delta^\mu = Q_e A^\mu(x) + Q_w Z^\mu(x) + g\Sigma_- W_+^\mu(x) + g\Sigma_+ W_-^\mu(x). \tag{45}$$

It turns out that the parity matrix P commutes with all terms of δ^μ, i.e. it commutes with the charge as well as the charge-flip operators. Therefore, the matrices in the covariant derivative do not violate parity. Breaking of the isospin $SU(2)$ symmetry does not affect parity which remains intact in the isospin gauge theory. This differs from the SM, where only the left-chiral spinor field couples to the weak gauge bosons.

It appears that the charge conjugation $\mathcal{C}$ does not commute with δ^μ. We find that $\mathcal{C}^{-1}(\Sigma_- W_+^\mu(x))\mathcal{C} = -\Sigma_+ W_-^\mu(x)$. Thus, the net effect of charge conjugation is to transform the weak coupling constant g into $-g$. Moreover, since $\mathcal{C}^{-1}\Sigma_z\mathcal{C} = -\Sigma_z$, we obtain that the electric charge $q_j = e(-1, -1, 0, 0)$ transforms into $q_j = e(0, 0, -1, -1)$. Similarly, the weak charge transforms into $w_j = e(\tan\theta_{\mathrm{W}} + \cot\theta_{\mathrm{W}}, \tan\theta_{\mathrm{W}} + \cot\theta_{\mathrm{W}}, \tan\theta_{\mathrm{W}} - \cot\theta_{\mathrm{W}}, \tan\theta_{\mathrm{W}} - \cot\theta_{\mathrm{W}})$. This means that charge conjugation interchanges the charges of the isospin up- and down-components, and not simply the sign of the electric charge.

Concerning the time-inversion, the related matrix T has the effect that it transforms the z-component of isospin into that of negative spin, and therefore $T^{-1} I_z T = -S_z$. Moreover, it does the same with the $I_\pm$ components. So we have $T^{-1} I_{z,\pm} T = -S_{z,\pm}$, and thus we can conclude that time inversion flips the isospin to the negative spin. When we analyse the effects of T on Γ_0 and Γ_5, we find that T commutes with both of them. However, for the Gamma matrix vector $\mathbf{\Gamma}$, one obtains $T^{-1}\mathbf{\Gamma}T = \tilde{\mathbf{\Gamma}}$, where in this latter quantity just the indices R and L are interchanged as compared with Eq. (9). So we may say that time inversion acts on the Gamma and isospin (or spin) matrices such that it exchanges the chiral indices, and thus it may be called "chirality exchange", whereby the spin as axial vector also changes its sign, whereas the polar Gamma vector does not.

Inspection of the general expression for the $\mathcal{CPT}$ operator in Eq. (64) in Appendix shows that the product of the three matrices C, P and T yields $CPT = \Gamma_5$, which commutes with $\mathbf{I}$. Moreover, we have $[\delta^\mu, \Gamma_5] = 0$, and thus CPT altogether has neither an effect on the covariant derivative nor on the phase operator Eq. (28), yet the single matrices have and of course the operators C, P and T, which also act on the gauge fields.

6 Summary and conclusion

We have established an extended Dirac equation which comes in six versions connected by similarity transformations. This equation for an elementary spin-one-half fermion has two new additional degrees of freedom associated with the up- and down-components of isospin. It emerges naturally from the Lorentz transformation of spinors which are based on the vector representation of the Lorentz group. It is argued that these traits may explain the occurrence of the three families of the leptons and quarks. We suggest that the origin of flavour arises genuinely from the properties of the extended Dirac equation. Isospin symmetry can be gauged and leads to a weak-interaction-type theory, which is also valid for finite initial mass and does not violate parity. Breaking the isospin $SU(2)$ symmetry yields the correct charges of the particles by the electroweak-style unification procedures of the SM.

These results are in contrast with the conventional SM approach, which introduces internal symmetries or gauge groups not related to the Lorentz transformation. Concerning $SU(2)$ symmetry and flavour, it seems to be sufficient to exploit that there are six variants of the extended Dirac equation. The mathematical multiplicity of that equation ultimately originates from the Lorentz group, which means it is obtained by maximally exploiting space-time symmetry. Furthermore, the related concept of isospin explains the values of electric charge for the electrons and neutrinos (as well as of the up and down quarks). The three generations of the quarks and leptons are suggested by the mathematical option of six versions of the extended Dirac equation, which reveals a kind of permutation symmetry established by similarity transformations.

The extended Dirac equation for a massive fermion isospin-doublet does not break chiral symmetry, which the SM does by construction, using at the outset left- and right-chiral Weyl fermions as elementary fields. Thus, the present theory requires a new physical mechanism to achieve this goal and to be in compliance with the measured parity violation in radioactive and mesonic decays [2]. In fact, this mechanism was perhaps proposed already in 1958 by Feynman and Gell-Mann [18] in their famous early theory of the Fermi interaction. To work this out further within the present framework is beyond the scope of our paper.

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Appendix

Gamma matrices

In this appendix, we fully quote the six sets of extended Gamma matrices. In the Weyl and then Dirac representation, we obtain

$$\boldsymbol{\Gamma}_{\mathrm{W}} = \begin{pmatrix} 0 & \Delta\boldsymbol{\Sigma}_{\mathrm{L}} \\ -\Delta\boldsymbol{\Sigma}_{\mathrm{R}} & 0 \end{pmatrix}, \quad \Gamma_{0\,\mathrm{W}} = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}, \quad \Gamma_{5\,\mathrm{W}} = \begin{pmatrix} -1_4 & 0 \\ 0 & 1_4 \end{pmatrix}. \tag{46}$$

$$\boldsymbol{\Gamma}_{\mathrm{D}} = \begin{pmatrix} 0 & \Delta\boldsymbol{\Sigma}_{\mathrm{L}} \\ -\Delta\boldsymbol{\Sigma}_{\mathrm{R}} & 0 \end{pmatrix}, \quad \Gamma_{0\,\mathrm{D}} = \begin{pmatrix} 1_4 & 0 \\ 0 & -1_4 \end{pmatrix}, \quad \Gamma_{5\,\mathrm{D}} = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}. \tag{47}$$

The two sets of Gamma matrices as given above in the representation of Marsch and Narita read

$$\boldsymbol{\Gamma}_{\mathrm{MN1}} = \mathrm{i}\begin{pmatrix} \boldsymbol{\Sigma}_{\mathrm{R}} & 0 \\ 0 & -\boldsymbol{\Sigma}_{\mathrm{L}} \end{pmatrix}, \quad \Gamma_{0\,\mathrm{MN1}} = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}, \quad \Gamma_{5\,\mathrm{MN1}} = \begin{pmatrix} 0 & -\mathrm{i}\Delta \\ \mathrm{i}\Delta & 0 \end{pmatrix}. \tag{48}$$

$$\boldsymbol{\Gamma}_{\mathrm{MN2}} = \mathrm{i}\begin{pmatrix} \boldsymbol{\Sigma}_{\mathrm{R}} & 0 \\ 0 & -\boldsymbol{\Sigma}_{\mathrm{L}} \end{pmatrix}, \quad \Gamma_{0\,\mathrm{MN2}} = \begin{pmatrix} 0 & -\mathrm{i}\Delta \\ \mathrm{i}\Delta & 0 \end{pmatrix}, \quad \Gamma_{5\,\mathrm{MN2}} = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}. \tag{49}$$

Finally, the two sets of Gamma matrices as previously given by Marsch and Narita [7] are read as follows:

$$\boldsymbol{\Gamma}_{\mathrm{MN3}} = \mathrm{i}\begin{pmatrix} 0 & \Delta\boldsymbol{\Sigma}_{\mathrm{L}} \\ \Delta\boldsymbol{\Sigma}_{\mathrm{R}} & 0 \end{pmatrix}, \quad \Gamma_{0\,\mathrm{MN3}} = \begin{pmatrix} 0 & \mathrm{i}\Delta \\ -\mathrm{i}\Delta & 0 \end{pmatrix}, \quad \Gamma_{5\,\mathrm{MN3}} = \begin{pmatrix} -1_4 & 0 \\ 0 & 1_4 \end{pmatrix}. \tag{50}$$

$$\boldsymbol{\Gamma}_{\mathrm{MN4}} = \mathrm{i}\begin{pmatrix} 0 & \Delta\boldsymbol{\Sigma}_{\mathrm{L}} \\ \Delta\boldsymbol{\Sigma}_{\mathrm{R}} & 0 \end{pmatrix}, \quad \Gamma_{0\,\mathrm{MN4}} = \begin{pmatrix} 1_4 & 0 \\ 0 & -1_4 \end{pmatrix} \quad \Gamma_{5\,\mathrm{MN4}} = \begin{pmatrix} 0 & \mathrm{i}\Delta \\ -\mathrm{i}\Delta & 0 \end{pmatrix}. \tag{51}$$

All these matrices are connected to each other mathematically by means of the three similarity transformations U, V and W as given in Eq. (15) and by their inverse matrices U^{-1}, V^{-1} and W^{-1}, or more explicitly in the main text for U and V in Eqs. (10) and (11). For example, U transforms Γ_{W} into Γ_{D}, and V transforms Γ_{MN1} into Γ_{W} or $\Gamma^{\mu}_{\mathrm{MN2}}$ into Γ_{D}, and W transforms Γ_{MN3} into Γ_{W}

or Γ_{MN4} into Γ_{D}. Furthermore, W transforms Γ_{MN1} into Γ_{MN2}, and V transforms Γ_{MN3} into Γ_{MN4} Of course, the six versions are mathematically equivalent and concerning the physics content, but it turns out that the version in the Weyl basis is advantageous, as it gives readily two independent equations for ψ_{R} and ψ_{L} for vanishing mass m. We recall that the spin and isospin operators are the same for all six versions. However, the rapidity operators differ. We find that $R_{\mathrm{MN1}} = R_{\mathrm{MN}}$, $R_{\mathrm{MN2}} = -R_{\mathrm{D}}$, $R_{\mathrm{MN3}} = R_{\mathrm{W}}$ and finally $R_{\mathrm{MN4}} = -R_{\mathrm{MN}}$, which were given in Eqs. (18) and (20).

Eigenfunctions and space-time symmetries in the Weyl basis

Here, we shall derive and discuss the eigenfunctions of the extended Dirac equation in the Weyl basis. In the rest frame of the fermion, we look for purely time dependent solutions for the particle, $\psi_{\mathrm{P}} = u_0 \exp(-\mathrm{i}mt)$, and $\psi_{\mathrm{A}} = v_0 \exp(\mathrm{i}mt)$ for the antiparticle. Then, the equations for the polarization spinors read

$$\begin{pmatrix} -1_4 & \Delta \\ \Delta & -1_4 \end{pmatrix} u_0 = 0, \quad \begin{pmatrix} 1_4 & \Delta \\ \Delta & 1_4 \end{pmatrix} v_0 = 0. \tag{52}$$

The solutions are

$$u_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1_4 \\ \Delta \end{pmatrix} \chi, \quad v_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1_4 \\ -\Delta \end{pmatrix} \chi. \tag{53}$$

Here, χ is an arbitrary Minkowski spinor to be determined later. We define the conjugate spinors required subsequently for Lorentz invariance as

$$\bar{u}_0 = (\Gamma_0 u)^\dagger = \frac{1}{\sqrt{2}}(\chi^\dagger, \chi^\dagger \Delta), \quad \bar{v}_0 = (\Gamma_0 v)^\dagger = \frac{1}{\sqrt{2}}(-\chi^\dagger, \chi^\dagger \Delta). \tag{54}$$

This gives $\bar{u}_0 v_0 = \bar{v}_0 u_0 = 0$, and the conventional normalization $\bar{u}_0 u_0 = 1$ and $\bar{v}_0 v_0 = -1$. These definitions have the consequence that $\bar{u}_0 \Gamma_0 u_0 = 1$ and $\bar{v}_0 \Gamma_0 v_0 = 1$, and furthermore $\bar{u}_0 \boldsymbol{\Gamma} u_0 = 0$ and $\bar{v}_0 \boldsymbol{\Gamma} v_0 = 0$, whereby $\Delta \boldsymbol{\Sigma}_{\mathrm{R}} = \boldsymbol{\Sigma}_{\mathrm{L}} \Delta$ has been exploited. We also assumed that the spinor χ is normalized to unity. Yet to be specific, we can make use of the four basis functions given in the subsequent Eq. (55), which were calculated in the paper by Marsch and Narita [7] as eigenfunctions of $\Sigma_{\mathrm{R,L}\,z}$. These basis spinors read

$$\chi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \chi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -\mathrm{i} \\ 0 \end{pmatrix} \quad \chi_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \mathrm{i} \\ 0 \end{pmatrix}, \quad \chi_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}. \tag{55}$$

They are orthogonal and normalized to unity. By insertion of χ_j into Eq. (53), we thus obtain the four polarization spinors u_{0j}, respectively v_{0j}, which form a complete orthogonal set of basis spinors for the extended Dirac equation in the rest frame. What are their eigenvalues with respect to the spin and isospin? Operation with S_z yields for example

$$S_z u_{0j} = \frac{1}{2} \begin{pmatrix} \Sigma_{\mathrm{Rz}} & 0 \\ 0 & \Sigma_{\mathrm{Lz}} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \chi_j \\ \Delta \chi_j \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} \Sigma_{\mathrm{Rz}} \chi_j & 0 \\ 0 & \Delta \Sigma_{\mathrm{Rz}} \chi_j \end{pmatrix} = \frac{1}{2} e_{\mathrm{R}j} u_{0j}. \tag{56}$$

Here, we introduced the spin eigenvalue $e_{\mathrm{R}j} = (1, -1, 1, -1)$, where as usual 1 means spin up and -1 spin down. Similarly, we can calculate the eigenvalues of the isospin as follows:

$$I_z v_{0j} = \frac{1}{2} \begin{pmatrix} \Sigma_{\mathrm{Lz}} & 0 \\ 0 & \Sigma_{\mathrm{Rz}} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \chi_j \\ -\Delta \chi_j \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} \Sigma_{\mathrm{Lz}} \chi_j & 0 \\ 0 & -\Delta \Sigma_{\mathrm{Lz}} \chi_j \end{pmatrix} = \frac{1}{2} e_{\mathrm{L}j} v_{0j}. \tag{57}$$

Here, we introduced the isospin eigenvalue $e_{\mathrm{L}j} = (-1, -1, 1, 1)$, whereby 1 means isospin up and -1 isospin down. It will turn out below that these values also determine the electric charge associated with the eigenfunctions u_{0j} and v_{0j}, whereby -1 corresponds to the electron and 1 to the neutrino in the isospin doublet.

In order to get the solutions of the extended Dirac equation in the moving frame, we can simply boost the above spinors by means of the following procedure. The standard ansatz is

$$\begin{aligned} \Psi_{\mathrm{P}} &= u \exp(-\mathrm{i}(Et - \mathbf{p} \cdot \mathbf{x})), \\ \Psi_{\mathrm{A}} &= v \exp(\mathrm{i}(Et - \mathbf{p} \cdot \mathbf{x})), \end{aligned} \tag{58}$$

where $E = \sqrt{m^2 + p^2}$. If we take

$$\begin{aligned} u_j &= (\Gamma^\mu p_\mu + m 1_8) u_{0j}, \\ v_j &= (\Gamma^\mu p_\mu - m 1_8) v_{0j}, \end{aligned} \tag{59}$$

then these spinors solve the Fourier-transformed extended Dirac equation, since $(\Gamma^\mu p_\mu)^2 = (\Gamma_0 E + \boldsymbol{\Gamma} \cdot \mathbf{p})^2 = m^2$. The spinors can be normalized to unity by division through the factor $\sqrt{2m(E+m)}$ for u_j, respectively, $\sqrt{2m(E-m)}$ for v_j. To obtain these factors we have, for example, to calculate $\bar{u}_j u_j$ and make use of the properties of the rest frame spinors, i.e. $\bar{u}_0 \Gamma_0 u_0 = 1$ and $\bar{u}_0 \boldsymbol{\Gamma} u_0 = 0$.

Finally, we emphasize that the boosted polarization spinors remain eigenfunctions of the isospin operator I_z, as it commutes with all Gammas; however, that is not true for S_z, and therefore in the boosted solutions the spin of a moving fermion is not defined any more.

Finally, we want to briefly discuss the CPT symmetries of the extended Dirac equation in the Weyl basis. Marsch and Narita [19] have exhaustively and generally discussed this issue, so we can refer to their previous discussion and present here the essentials. The three basic symmetries consist of the simple operations of complex conjugation, named $\mathtt{C}$, parity reflection or space inversion, named $\mathtt{P}$, and time inversion, named $\mathtt{T}$. They have the following effects on the spinor field: $\mathtt{C}\Psi = \Psi^*$, $\mathtt{P}\Psi(\mathbf{x}) = \Psi(-\mathbf{x})$, and $\mathtt{T}\Psi(t) = \Psi(-t)$. These operations are their own inversions. Furthermore, there are three 8×8 matrices involved, and one can generally define these symmetry operations as

$$\begin{aligned} \mathcal{C} &= C\,\mathtt{C}, \\ \mathcal{P} &= P\,\mathtt{P}, . \\ \mathcal{T} &= T\,\mathtt{T} \end{aligned} \tag{60}$$

The matrices (C, P, T) still need to be determined by explicit operation of $\mathcal{C}, \mathcal{P}, \mathcal{T}$ on the Dirac equation. The transformed fields are identified by a calligraphic subscript. We define $\Psi_{\mathcal{C}} = C\mathtt{C}\Psi$, $\Psi_{\mathcal{P}} = P\mathtt{P}\Psi$, and $\Psi_{\mathcal{T}} = T\mathtt{T}\Psi$. Following the general calculations and arguments in Marsch and Narita [19], we obtain the symmetry operations the subsequent expressions. For parity we require the real matrix Γ_0 and obtain

$$\mathcal{P} = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix} \mathtt{P}. \tag{61}$$

For charge conjugation, we obtain

$$\mathcal{C} = \begin{pmatrix} 0 & 1_4 \\ 1_4 & 0 \end{pmatrix} \mathtt{C}. \tag{62}$$

For time inversion, we obtain

$$\mathcal{T} = \begin{pmatrix} -\Delta & 0 \\ 0 & \Delta \end{pmatrix} \mathtt{T}. \tag{63}$$

As a result, we obtained the mathematical expressions for all three symmetry operations in the Weyl basis. Now, we can also evaluate their combined effects in the form of the CPT operator leading to the famous CPT theorem. The result is

$$\mathcal{CPT} = \begin{pmatrix} 0 & 1_4 \\ 1_4 & 0 \end{pmatrix} \mathtt{C} \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix} \mathtt{P} \begin{pmatrix} -\Delta & 0 \\ 0 & \Delta \end{pmatrix} \mathtt{T} = \Gamma_5 \mathtt{CPT}. \tag{64}$$

This operator then transforms the spinor field as follows:

$$\mathcal{CPT}\Psi(\mathbf{x}, t) = \Gamma_5 \Psi^*(-\mathbf{x}, -t). \tag{65}$$

We make use of these symmetry operations in the section on gauge theory.

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