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On the analogy between stochastic electrodynamics and nonrelativistic quantum electrodynamics

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Abstract I expose nonrelativistic quantum electrodynamics in the Weyl–Wigner representation. Hence, I prove that an approximation to first order in Planck constant has a formal analogy with stochastic electrodynamics (SED), that is classical electrodynamics of charged particles immersed in a random radiation filling space. The analogy elucidates why SED agrees with quantum theory for particle Hamiltonians quadratic in coordinates and momenta, but fails otherwise.

1 Stochastic electrodynamics

Stochastic electrodynamics (SED in the following) is a theory that studies, within classical electrodynamics, the motion of charged particles interacting with electromagnetic fields. The difference with the standard classical theory is the assumption that there is a random electromagnetic radiation filling space. This radiation is treated as a stochastic field whose statistical properties are homogeneous, isotropic and Lorentz invariant. SED may be labelled a classical (i.e., not quantum) theory, although the assumption of a random background radiation is alien to, but compatible with, classical electrodynamics.

A reasoning leading to SED is as follows. When Rutherford proposed the nuclear atom, a common argument against the model was that such atom could not be stable. In fact the accelerated electron should radiate whence the atom would collapse. But this argument is misleading [1], the atom could not be stable *if isolated*. However, the hypothesis of isolation is not appropriate if there are many atoms in the universe. It is more plausible to assume that there is some amount of radiation filling space, whence every atom would radiate but it could also absorb energy from the radiation, eventually arriving at a dynamical equilibrium. This might explain the stability of the atom. Of course, Bohr had solved the problem by introducing postulates that were actually incompatible with classical laws. Early workers believed that SED, after some elaboration, might give rise to a reinterpretation of quantum mechanics. However, the attempt has not been successful.

In order to study the assumed radiation of SED we may expand it in plane waves, within a large volume, and represent the amplitudes of the waves by dimensionless complex numbers $\{a_l\}$. It is plausible that different amplitudes are statistically uncorrelated and the probability of every one is Gaussian. Hence, the probability distribution of the radiation amplitudes is as follows

$$W_{\text{vac}} = \prod_{l=1}^{n} \frac{2}{\pi} \exp\left(-2|a_l|^2\right), \tag{1}$$

normalized with respect to the integration in $d\,\text{Re}\,a_l\; d\,\text{Im}\,a_l$.

In free space it is plausible that the spectral energy density, $\rho_{\text{SED}}(\omega)$, (energy per unit volume and unit frequency interval, in short "spectrum") of the random radiation is homogeneous, isotropic and Lorentz invariant in the following sense. In a given inertial frame the frequency of one plane wave may change when we observe the radiation in a different inertial frame so that, fixing a small frequency interval (ω_1, ω_2), some plane wave frequencies will enter the interval and other frequencies will leave it with the change of frame. We define the spectrum as Lorentz invariant if the number of frequencies that enter the interval (ω_1, ω_2) balances the number that leave it, this for all frequency intervals. A necessary and sufficient condition for Lorentz invariance, in the said sense, is that the spectrum is proportional to the cube of the frequency [2, 3]. Thus, the spectrum of the random radiation may be written

$$\rho_{\text{SED}}(\omega) = C\omega^3, C = \frac{\hbar}{2\pi^2 c^3}, \tag{2}$$

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whence Planck constant $\hbar$ enters the theory via fixing the scale of the assumed universal radiation. The spectrum Eq. (2) corresponds to an average energy

$$\langle E_j \rangle = \frac{1}{2}\hbar\omega_j, \tag{3}$$

for the j plane wave in the expansion. Then, we may write the total energy, $E_{\text{SED}}\{a_j\}$, of the SED random field in terms of the amplitudes, demanding that the average of E_{SED}, taking Eq. (1) into account, gives Eq. (3) . This leads to the following

$$E_{\text{SED}} = \sum_l E_j = \sum_l \hbar\omega_l |a_l|^2. \tag{4}$$

The spectrum Eq. (2) implies a divergent total energy density and any cutoff would break Lorentz invariance, but Eq. (2) is assumed to be valid for low enough frequencies and the effect of high frequencies is neglected (it would produce on any charged particle a rapid shaking superposed to a more smooth path). Ultraviolet divergences are well known to exist also in QED and they are associated with the quantum vacuum fluctuations [2].

A review of the work on SED made until 1995 is the book by de la Peña and Cetto [3] and new results are included in more recent reviews [4–7].

In practice SED has been studied within the nonrelativistic approximation ignoring spin. In order to find the evolution equations of the particles we shall solve the dynamical (Newtonian) equation of motion that takes the radiation into account. I will write it in one dimension but might be extended to 3N dimensions for N-particle systems. That is

$$m\ddot{x} = -\frac{\mathrm{d}V(x)}{\mathrm{d}x} + m\tau\dddot{x} + eE(t), \tag{5}$$

where $m(e)$ is the particle mass (charge) and E is the x component of the electric field of the radiation that acts on the particle. The field $E(t)$ is treated as a stochastic process whence Eq. (5) is a stochastic differential equation of Langevin type with non-white noise. We see that the Newton equation is modified by two terms which correspond to the matter-radiation interaction. The second term on the right side is the radiation reaction force due to emission from the charged particle. This term does not contain $\hbar$ and it is well known in standard classical electrodynamics. The third term is the (Lorentz) force of the radiation on the particle in the electric dipole approximation, which is consistent with the nonrelativistic treatment. The approximation consists of neglecting the action of the magnetic field and the dependence on position of the electric field. This term is specific of SED and it is proportional to the square root of Planck constant because the average of $E(t)^2$ is proportional to the spectrum Eq. (2). The parameter τ is given by

$$\tau = \frac{2e^2}{3mc^3} \Rightarrow \tau\omega_0 = \frac{2}{3}\frac{e^2}{\hbar c}\frac{\hbar\omega_0}{mc^2} << 1. \tag{6}$$

so that the dimensionless quantity $\tau\omega_0$ is very small, it being the product of two small numbers namely the fine structure constant, $\alpha \equiv e^2/\hbar c \sim 1/137$, and the nonrelativistic ratio $\hbar\omega_0/mc^2 \simeq v^2/c^2 << 1$. It may be shown that the latter term of Eq. (5) is also small, which simplifies the solution, but I shall not study further this approach which may be seen in the references quoted above.

The results obtained from Eq. (5) agree, in the limit $\alpha \to 0$, with the quantum predictions for quadratic Hamiltonians. For instance, the squared mean position and momentum or the energy in the ground state are fairly well predicted. Corrections of order α may be obtained from Eq. (5) which correspond to radiative corrections of QED (e.g., Lamb shift). The solution of Eq. (5) is easy when it is linear, i.e., for quadratic Hamiltonians. It is not so easy for nonlinear systems, although many papers have been devoted to approximate solutions in that case. However, the problem with nonlinear systems is not just the difficulty of the solution, but the disagreement of the results with the quantum prediction as commented on below. Explaining this feature is one of the purposes of the present article.

The interest on SED is due to the fact that it provides an interpretation of several typically quantum phenomena in the spirit of classical physics. For instance, the mentioned stability of the classical (Rutherford) atom, Heisenberg uncertainty relations, and some examples of entanglement. SED provides a classical-like interpretation of phenomena where the spectrum of free space is modified by boundary conditions derived from macroscopic bodies, but the average energy $\frac{1}{2}\hbar\omega$ per normal mode still holds, e.g., the Casimir effect and the behavior of atoms in cavities. SED may be also studied at a finite temperature, where the thermal Planck spectrum is added to Eq. (2) , thus correctly predicting the specific heats of solids.

As the study of some simple systems provides an intuitive picture of several quantum phenomena SED has been proposed as a clue in the search for a realistic interpretation of quantum theory [7]. A different approach is to consider that SED is the closest classical approximation to quantum theory [5]. Furthermore, SED, with appropriate generalizations, has been believed a possible alternative to quantum mechanics [8]. However, there are many examples where SED predicts results in contradiction with quantum mechanics and experiments. Another shortcoming is that SED deals only with charged particles whilst QM laws are valid also for neutral particles.

In this article I will show that SED might be taken as a semiclassical theory. In fact it has similarity with nonrelativistic QED approximated to first order in Planck's constant. This would elucidate why SED gives predictions in agreement with quantum mechanics (and experiments) for linear systems (i.e., Hamiltonians quadratic in positions and momenta like the harmonic oscillator), but not for nonlinear systems.

In order to prove the assertion I will start revisiting the standard Hilbert space formalism of QED. (Throughout this article I will use QED for *nonrelativistic* quantum electrodynamics, the relativistic theory will not be studied here). After that I will briefly review the Weyl–Wigner formalism and derive an equation similar to Eq. (5) of SED as an approximation of order $O(\hbar)$, zeroth order corresponding to classical electrodynamics.

2 Nonrelativistic quantum electrodynamics

2.1 The standard Hilbert space formulation

For later convenience I will revisit the well-known Hilbert space formulation of nonrelativistic quantum mechanics coupled to the quantized electromagnetic field.

The fundamental equation should provide the evolution of the state of a system of particles, those charged interacting with each other, eventually with external fields, and with the quantized electromagnetic field. The state is represented by a density operator (or density matrix) $\hat{\rho}$ whose evolution is given by the equation

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = \left[\hat{H}, \hat{\rho}\right] \equiv \hat{H}\hat{\rho} - \hat{\rho}\hat{H}, \tag{7}$$

where the Hamiltonian consists of 3 terms, that is

$$\hat{H} = \hat{H}_{\text{part}} + \hat{H}_{\text{rad}} + \hat{H}_{\text{int}}, \tag{8}$$

for the particles, the radiation field and the interaction, respectively. For the sake of clarity I shall represent the operators with a "hat," e.g., $\hat{x}_j$, $\hat{p}_j$, and the classical quantities (c-numbers) without "hat," e.g., x_j, p_j.

A typical, but not general, particle Hamiltonian is the following

$$\hat{H}_{\text{part}} = \sum_{j=1}^{N}\left[\frac{\hat{\mathbf{p}}_j^2}{2m_j} + V\left(\left\{\hat{\mathbf{x}}_j\right\}\right)\right], \tag{9}$$

where m_j, $\hat{\mathbf{x}}_j$ and $\hat{\mathbf{p}}_j$ are the position and momentum of the particle j, and the potential V should include the interaction with given external fields and the instantaneous electrostatic interactions amongst the charged particles, which after the coupling with the vector potential of the radiation field would provide all electromagnetic interactions in the Coulomb gauge. Throughout the paper I will use that gauge, rather than Lorentz's, as appropriate for a non-relativistic theory.

For the study of radiation a popular approach is to start with a plane waves expansion of the vector potential operator in a large volume V. Hence, the Hamiltonian, representing the radiation energy in that volume, becomes

$$\hat{H}_{\text{rad}} = \frac{1}{2}\sum_{l} \hbar\omega_l (\hat{a}_l \hat{a}_l^\dagger + \hat{a}_l^\dagger \hat{a}_l), \tag{10}$$

where the operators fulfil the commutation rules

$$\left[\hat{a}_k, \hat{a}_l\right] = \left[\hat{a}_k^\dagger, \hat{a}_l^\dagger\right] = 0, \left[\hat{a}_k, \hat{a}_l^\dagger\right] = \delta_{kl},$$

and the ground state of the field is given by the vector state $\mid 0\rangle$ fulfilling

$$\hat{a}_j \mid 0\rangle = \langle 0 \mid \hat{a}_j^\dagger = 0, \tag{11}$$

where 0 means the null vector in the Hilbert space. This defines the minimal energy state of the field.

The interaction Hamiltonian may be written in terms of the vector potential, that is

$$\hat{H}_{\text{int}} = \sum_j \left[-\frac{e}{mc}\hat{\mathbf{p}}_i \cdot \hat{\mathbf{A}}\left(\hat{\mathbf{x}}_j, t\right) + \frac{e^2}{2mc^2}\hat{\mathbf{A}}\left(\hat{\mathbf{x}}_j, t\right) \cdot \hat{\mathbf{A}}\left(\hat{\mathbf{x}}_j, t\right)\right], \tag{12}$$

where the sum runs over the different particles, $\hat{\mathbf{x}}_j$ and $\hat{\mathbf{p}}_j$ being the position and momentum of particle j at time t. For simplicity I will assume that all particles have the same mass m and charge e from now on. The formalism sketched allowed nonrelativistic QED computations able to reproduce empirical results like the Lamb shift (e.g., the celebrated calculation by Bethe in 1947) [2]. Of course, the theory was superseded with the advent of relativistic QED.

2.2 Weyl–Wigner formulation

The Hilbert space formulation of quantum theory is not convenient for our purpose of studying the analogy with SED, a better procedure being to work in the Weyl–Wigner formalism (WW in the following). I stress that both are different forms of quantum theory that should predict the same results for experiments. In 1927 Hermann Weyl [9] proposed a general procedure for the quantization of (nonrelativistic) classical mechanics of particles via a transform that provides operators corresponding to positions and momenta of particles. Here, we are interested in the inverse Weyl transform that may be applied to a nonrelativistic system of particles as follows. For any trace-class operator $\hat{f}$ in the quantum Hilbert space of the particles the transform gives a function W in the classical phase space as follows [10, 11]

$$W\left(\left\{x_j, p_j\right\}\right) = (\pi)^{-6N} \int Tr\left\{\hat{f} \exp\left[i \sum_{j=1}^{3N}\left(\lambda_j \hat{x}_j + \mu_j \hat{p}_j\right)\right]\right\} \times \exp\left[\left(-i \sum_{j=1}^{3N}\left(\lambda_j x_j + \mu_j p_j\right)\right)\right] \prod_{j=1}^{3N} \mathrm{d}\lambda_j \mathrm{d}\mu_j. \tag{13}$$

In the particular case when $\hat{f}$ is the density operator of a quantum state the transform Eq. (13) gives the Wigner function of the state [12]. Another application of Eq. (13) is to derive observables from the corresponding operators in the Hilbert space, in particular the particle Hamiltonian, which for Eq. (9) gives simply

$$H_{\text{part}} = \sum_{j=1}^{N}\left[\frac{\mathbf{p}_j^2}{2m} + V\left(\left\{\mathbf{x}_j\right\}\right)\right]. \tag{14}$$

However, if $\hat{H}_{\text{part}}$ contained products of $\hat{\mathbf{x}}_i$ times $\hat{\mathbf{p}}_i$ the result of the Weyl transform would not be so simple due to the non-commutativity of these operators.

The WW formalism also allows getting the evolution of the Wigner function of a system of particles. For the Hamiltonian Eq. (14) it is the following (for a single particle, but the generalization to N particles is straightforward)

$$\frac{\partial W(\mathbf{x}, \mathbf{p})}{\partial t} = -\frac{1}{m}\mathbf{p} \cdot \nabla W - \frac{1}{\hbar}\int \frac{\mathrm{d}\mathbf{p}'}{(2\pi)^3}\widetilde{V}\left(\mathbf{x}, \mathbf{p}'\right) W\left(\mathbf{x}, \mathbf{p} + \mathbf{p}', t\right)$$
$$\widetilde{V}\left(\mathbf{x}, \mathbf{p}'\right) \equiv \int d\mathbf{u} \sin\left(\mathbf{p}' \cdot \mathbf{u}\right)\left[V(\mathbf{x} + \hbar\mathbf{u}/2) - V(\mathbf{x} - \hbar\mathbf{u}/2)\right]. \tag{15}$$

In summary the WW formalism allows to study nonrelativistic quantum mechanics providing identical predictions than the standard Hilbert space, that is both formalisms are physically equivalent. However, the Wigner function not always can be interpreted as a probability distribution in phase space because it is not positive in general.

For our purpose another form of the evolution is more convenient than Eq. (15) which is valid for any particle Hamiltonian, not just Eq. (14). It is given by the Moyal equation for any Wigner function $W\left(\left\{x_j, p_j\right\}\right)$, that is

$$\frac{\partial W}{\partial t} = \frac{2}{\hbar}\sin\left[\frac{\hbar}{2}\left(\frac{\partial}{\partial x_j}\frac{\partial}{\partial p_j'} - \frac{\partial}{\partial p_j}\frac{\partial}{\partial x_j'}\right)\right]\left[W\left(\left\{x_j, p_j\right\}\right) H_{\text{part}}\left(\left\{x_j', p_j'\right\}\right)\right]$$
$$\equiv \frac{2}{\hbar}\sum_{n=0}^{3N}\frac{(-1)^n}{(2n+1)!}\left[\frac{\hbar}{2}\left(\frac{\partial}{\partial x_j}\frac{\partial}{\partial p_j'} - \frac{\partial}{\partial p_j}\frac{\partial}{\partial x_j'}\right)\right]^{2n+1} \times \left[W\left(\left\{x_j, p_j\right\}\right) H_{\text{part}}\left(\left\{x_j', p_j'\right\}\right)\right] \equiv \left\{W, H_{\text{part}}\right\}_M, \tag{16}$$

where we should identify $\left\{x_j', p_j'\right\} = \left\{x_j, p_j\right\}$ after performing the derivatives. $\left\{W, H_{\text{part}}\right\}_M$ is a simplified notation to be used in the following, the subindex M standing for Moyal bracket. For the Hamiltonian Eq. (14) the Moyal equation may be derived from Eq. (15) via an expansion in powers of $\hbar$.

Equation (16) is valid for any (finite) set of particles with a general Hamiltonian written in terms of generalized coordinates and momenta, but it involves derivatives of infinite order, which makes it just a formal (not practical) equation. However, it may provide useful approximations if truncated at a finite order in Planck constant $\hbar$. In the limit $\hbar \to 0$ Eq. (16) reduces to the classical Liouville equation, Moyal bracket becoming the Poisson bracket.

Weyl transform Eq. (13) may be extended to the radiation field [13, 14], taking into account that some linear combinations of the "creation and annihilation operators of photons" in the normal modes (i.e., plane waves in free space) may play the role of "coordinates and momenta" operators. We may write the following (inverse) Weyl transform for the radiation field

$$f(\{a_l\}) = \left(\frac{2}{\pi}\right)^{2n}\prod_{l=1}^{n}\int_{-\infty}^{\infty}\mathrm{d}\lambda_l\int_{-\infty}^{\infty}\mathrm{d}\mu_l \exp\left[-2i\left(\lambda_l \mathrm{Re}a_l + \mu_l \mathrm{Im}a_l\right)\right] \times Tr\left\{\hat{f}\exp\left[i\lambda_l\left(\hat{a}_l + \hat{a}_l^{\dagger}\right) + \mu_l\left(\hat{a}_l - \hat{a}_l^{\dagger}\right)\right]\right\}, \tag{17}$$

where a_l and a_l^* are c-number field amplitudes that in the WW formalism are substituted for the Hilbert space operators $\hat{a}_l$ and $\hat{a}_l^\dagger$.

The transform Eq. (17) allows getting the quantum states and observables in the standard Hilbert space formalism as functions of the amplitudes $\{a_l\}$.. Then, the quantum field looks like a classical field in the WW formalism. In particular, we may describe the field by a vector potential, whence we might derive the electric and magnetic fields and hence the Hamiltonian, that is

$$H_{\text{rad}} = \sum_l \hbar\omega_l |a_l|^2 = \sum_l \hbar\omega_l \left[(\operatorname{Re} a_l)^2 + (\operatorname{Im} a_l)^2 \right], \tag{18}$$

to be compared with Eq. (10) . As is well known the evolution of $\operatorname{Re} a_l$ and $\operatorname{Im} a_l$ is formally similar to the position and momentum of a harmonic oscillator, which may be stressed with the following change of variables which I will use from now on

$$y_l = \sqrt{\frac{2\hbar}{\omega_l}} \operatorname{Re} a_l,\ q_l = \sqrt{2\hbar\omega_l} \operatorname{Im} a_l \Rightarrow H_{\text{rad}} = \sum_l \left[\frac{1}{2}\omega_l^2 y_l^2 + \frac{1}{2} q_l^2 \right]. \tag{19}$$

Hence, the Hamilton equations give, for free radiation,

$$\frac{\mathrm{d}}{\mathrm{d}t} y_l = \frac{\partial H_{\text{rad}}}{\partial q_l} = q_l,\ \frac{\mathrm{d}}{\mathrm{d}t} q_l = -\frac{\partial H_{\text{rad}}}{\partial y_l} = -\omega_l^2 y_l. \tag{20}$$

An advantage of the new variables is that the Hamiltonian H_{rad}, Eq. (19) , does not contain Planck constant $\hbar$, at a difference with Eq. (18) . The presence of $\hbar$ might mislead us to believe that Eq. (18) is a specifically quantum relation. Actually the introduction of $\hbar$ in Eq. (10) is currently made in order that the quantum operators $\left\{ \hat{a}_l, \hat{a}_l^\dagger \right\}$ are dimensionless but it has no deep meaning. However, the choice Eq. (18) is inconvenient in this paper where we want to distinguish quantum from classical features.

2.3 The quantum vacuum or zeropoint field

In the standard, Hilbert space, formalism the vacuum corresponds to the absence of "photons." The vacuum state $W_{\text{vac}}\left(\left\{ a_j, a_j^* \right\} \right)$ in the WW formalism should be derived from the vacuum density operator in Hilbert space, that is $|0\rangle\langle 0|$, via the Weyl transform Eq. (17) . To do that I shall start calculating the trace operation putting $\hat{f} = |0\rangle\langle 0|$ in Eq. (13) , taking the Campbell–Hausdorff formula into account that is

$$\exp\left(\hat{A} + \hat{B} \right) = \exp\left(\hat{A} \right) \exp\left(\hat{B} \right) \exp\left(-\frac{1}{2}\left[\hat{A}, \hat{B} \right] \right), \tag{21}$$

which is valid if the operator $\left[\hat{A}, \hat{B} \right]$ commutes with $\hat{A}$ and with $\hat{B}$. Hence, the trace involved in Eq. (13) becomes, for a single mode,

$$\begin{aligned} &Tr\left\{ |0><0| \exp\left[i\lambda\left(\hat{a} + \hat{a}^\dagger \right) + \mu\left(\hat{a} - \hat{a}^\dagger \right) \right] \right\} \\ &= \langle 0 \mid \exp\left[(i\lambda - \mu)\hat{a}^\dagger \right] \exp\left[(i\lambda + \mu)\hat{a} \right] \mid 0 \rangle \exp\left[-\lambda^2/2 - \mu^2/2 \right] \\ &= \exp\left[-\lambda^2/2 - \mu^2/2 \right]. \end{aligned}$$

If this is inserted in Eq. (17) we get

$$\begin{aligned} W_{\text{vac}} &= \prod_{l=1}^n \frac{4}{\pi^2} \int_{-\infty}^{\infty} \mathrm{d}\lambda_l \int_{-\infty}^{\infty} \mathrm{d}\mu_l \\ &\quad \times \exp\left[-2i(\lambda_l \operatorname{Re} a_l - \mu_l \operatorname{Im} a_l) - \frac{1}{2}\left(\lambda_l^2 + \mu_l^2 \right) \right] \\ &= \prod_{l=1}^n \frac{2}{\pi} \exp\left(-2|a_l|^2 \right). \end{aligned} \tag{22}$$

Equation (1) is the Wigner function of the "vacuum" state of the field. The corresponding normalized function in terms of the field variables $\{y_l, q_l\}$ is

$$W_{\text{vac}}(\{y_l, q_l\}) = \prod_{l=1}^n \frac{\omega_l}{\pi\hbar} \exp\left(-\frac{\omega_l^2 y_l^2 + q_l^2}{\hbar\omega_l} \right). \tag{23}$$

Vacuum expectations of the standard quantum (Hilbert space) treatment become integrals weighted by the function Eq. (23) in WW. In particular, the expectation value of the Hamiltonian H_{rad} Eq. (18) reproduces the spectrum Eq. (2) .

The quantum Eqs. (22) and (18) for the "vacuum" radiation are identical to the SED Eqs. (1) and (4) for the assumed radiation filling space. However, there is a conceptual difference: In SED Eq. (1) is interpreted as the probability distribution of a stochastic field, but in quantum physics it is the Wigner function of the vacuum state.

2.4 The evolution in phase space

The (quantum) evolution of any Wigner function $Z(\{y_l, q_l\}, t)$ of free radiation should be governed by Moyal Eq. (16) , but Moyal bracket becomes Poisson bracket in this case because all derivatives of $H_{\rm rad}$, Eq. (19), of order higher than 2 are nil. Then, the evolution is provided by the following Liouville equation

$$\frac{\partial Z}{\partial t} = \frac{\partial Z}{\partial y_l}\frac{\partial H_{\rm rad}}{\partial q_l} - \frac{\partial Z}{\partial q_l}\frac{\partial H_{\rm rad}}{\partial y_l} \equiv \{Z, H_{\rm rad}\}_{P,rad}, \tag{24}$$

where $\{Z, H_{\rm rad}\}_{P,rad}$ is the radiation Poisson bracket .

Now we are in a position to get the Hamiltonian of a system of charged particles coupled to electromagnetic radiation in the WW formalism. The Hamiltonian may be derived from the standard quantum one (involving operators in a Hilbert space) via the transform Eq. (17) [14]. It consists of 3 terms, that is

$$H = H_{\rm part} + H_{\rm rad} + H_{\rm int}. \tag{25}$$

A particular instance of mechanical (particles) Hamiltonian $H_{\rm part}\left(\left\{\mathbf{x}_j, \mathbf{p}_j\right\}\right)$ was given in Eq. (14). The radiation Hamiltonian appears in Eq. (18) and the interaction Hamiltonian may be written in terms of the vector potential, which may be got from Eq. (12) via the Weyl transform. That is

$$H_{\rm int} = \sum_j \left[-\frac{e}{mc}\mathbf{p}_j \cdot \mathbf{A}\left(\mathbf{x}_j, t\right) + \frac{e^2}{2mc^2}\mathbf{A}\left(\mathbf{x}_j, t\right) \cdot \mathbf{A}\left(\mathbf{x}_j, t\right) \right], \tag{26}$$

where the sum runs over the different particles, $\mathbf{x}_j$ and $\mathbf{p}_j$ being the position and momentum of particle j at time t. It is straightforward to write the vector potential in terms of the field variables, but it is not necessary for our purpose and I skip it.

After that we may treat the evolution of the system of particles plus radiation in a unified way analogous to the treatment of the particles alone. That is we define $F\left(\left\{\mathbf{x}_j, \mathbf{p}_j, y_l, q_l\right\}, t\right)$ to be the joint Wigner function in the space consisting of the phase space of the particles times the space spanned by the numerable set of field coordinates $\{y_l, q_l\}$. The evolution is given by the following generalized Moyal equation

$$\frac{\partial F}{\partial t} = \left\{F, H_{\rm part} + H_{\rm int}\right\}_M + \{F, H_{\rm rad}\}_P\ , \tag{27}$$

where the Moyal bracket (subindex M) reduces to Poisson's (subindex P) in the second term because $H_{\rm rad}$ is quadratic in the field amplitudes. Equation (27) might be the starting point for calculations in nonrelativistic QED treated in the WW formalism.

3 The approximation leading to analogy with SED

3.1 The Liouville equation

Solutions of the quantum evolution equation of nonrelativistic QED in the WW formalism, Eq. (27) , will not be studied in the present article. We shall deal only with the approximation to first order in Planck constant. It involves neglecting all terms of the expansion defining the Moyal bracket in Eq. (27), except the first one. Actually taking into account that the Moyal expansion consists of terms with even powers of $\hbar$, the approximation means neglecting terms of order $\mathrm{O}\left(\hbar^2\right)$. The radiation and the interaction Hamiltonians, Eqs. (19) and (26), respectively, do not contain $\hbar$, but Planck constant appears in the vacuum state Eq. (23) whence the approximation of Eq. (27) just described, leading to

$$\frac{\partial F}{\partial t} = \left\{F, H_{\rm part} + H_{\rm int}\right\}_P + \{F, H_{\rm rad}\}_P = \{F, H\}_P\ , \tag{28}$$

may be labelled an approximation to first order in $\hbar$. It is formally a classical Liouville equation for particles and field together.

The relevant point is that Eq. (28) is equally valid for both classical electrodynamics (CED) and "nonrelativistic QED in the WW formalism approximated to order $O(\hbar)''$. For simplicity, and also clarity, the expression in inverted commas will be labelled QEDWC in the following, where C stands for the approximation leading from Eq. (27) to the *classical* Liouville Eq. (28) . Thus, the evolution equations are classical in both CED and QEDWC, that is both rest on Maxwell–Lorentz theory. However, there is a difference in the initial conditions appropriate for the integration of the (Liouville) partial differential Eq. (28). In CED the initial state corresponds to absence of radiation, but in QEDWC it should be the quantum vacuum state for the radiation, that is ZPF given by Eq. (23). For the particles the initial state may be any probability distribution in phase space in CED, but it should be a Wigner function in QEDWC. I stress that the Wigner function for particles is *not* interpreted as a probability distribution too.

QEDWC is a mixture of classical evolution with quantum initial conditions. The interest for us is that SED has formal similarity with QEDWC. In fact Eq. (28) is valid in both cases and the initial state for the radiation should be also the same, that is the ZPF Eq. (23) , but maybe not for the particles. Also there is a conceptual difference between SED and QEDWC. In SED Eq. (23) *is* *interpreted* as a real random radiation filling space whence Eq. (28) is the evolution equation of particles interacting with the random field ZPF. Hence, we may state

Proposition 1 *From a formal (mathematical) point of view SED is similar to QEDWC except that the initial state of the particles may be any probability distribution in phase space in the former but it should be a Wigner function in the latter.*

This justifies the success of SED when dealing with linear systems, that is particle Hamiltonians quadratic at most in the coordinates and momenta of the particles. In fact, in this case no approximation takes place when we go from Eqs. (27) to (28) because only the first term contributes in the Moyal Eq. (16). In sharp contrast we should expect that SED fails badly for Hamiltonians not quadratic. Indeed the neglect of terms $O\left(\hbar^2\right)$, in going from Moyal to Poisson brackets, may produce errors of order $\hbar^2$, more properly order $(\hbar\omega/E)^2$, where the ratio of energy E by frequency ω may be taken as the typical action. In the microscopic (quantum) domain $\hbar\omega/E$ is of order unity, whence we conclude that SED applied to nonlinear systems is a rather bad approximation to quantum theory, which has been shown in actual calculations [3, 6, 7].

3.2 The equations of motion in QED and in SED

In SED it is common to start from Eq. (5) rather than Eq. (28) . In the following I study the relation between these two approaches. This will prove more clearly the formal analogy between QEDWC and SED.

In classical dynamics there is a close connection between the Liouville equation and the Hamilton equations of motion. The latter are coupled ordinary differential equations that, when integrated, provide the final positions and momenta of the particles in terms of the initial ones. The solutions may be written

$$\mathbf{x}_j(t) = \mathbf{X}_j(\{\mathbf{x}_k(0), \mathbf{p}_k(0)\}, t), \mathbf{p}_j(t) = \mathbf{P}_j(\{\mathbf{x}_k(0), \mathbf{p}_k(0)\}, t), \tag{29}$$

where $\left\{\mathbf{X}_j, \mathbf{P}_j\right\}$ are functions obtained from the integration of the Hamilton equations. In principle from Eq. (29) we might get the evolution of a distribution in phase space, which would amount to solving the Liouville equation. In fact, if we are given a function F_t in phase space at time t we may get the function at time 0 as follows:

$$\begin{aligned} F_t\left(\left\{\mathbf{x}_j(t), \mathbf{p}_j(t)\right\}\right) &= F_t\left(\mathbf{X}_j(\{\mathbf{x}_k(0), \mathbf{p}_k(0)\}), \mathbf{P}_j(\{\mathbf{x}_k(0), \mathbf{p}_k(0)\})\right) \\ &= F_o\left(\left\{\mathbf{x}_j(0), \mathbf{p}_j(0)\right\}\right). \end{aligned} \tag{30}$$

Similarly, we might obtain F_t from F_o using the equations obtained inverting Eq. (29) .

An analogous relation exists between the Liouville and the Hamilton equations in the two theories that we are considering in this article, namely CED and QEDWC. In fact they are formally classical dynamical problems for the variables of both the particles and the electromagnetic field. In the following I will compare both theories via the Hamilton equations of motion.

Let us start studying the solution of dynamical equations for charged particles in classical electrodynamics (CED). From the Hamiltonian Eq. (25) we have derived Eq. (28) and it is also easy to derive the canonical equations of motion. For simplicity I will study a single particle with Hamiltonian

$$H_{\text{part}} = \left[\frac{\mathbf{p}^2}{2m} + V(\mathbf{x})\right].$$

For the particle coupled to radiation we obtain from Eq. (25) the following Hamilton equations

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \frac{\mathbf{p}}{m} + \frac{\mathrm{d}H_{\text{int}}}{\mathrm{d}\mathbf{p}}, \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} = -\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}}V(\mathbf{x}), \\ &\frac{\mathrm{d}}{\mathrm{d}t}y_l = q_l + \frac{\mathrm{d}H_{\text{int}}}{\mathrm{d}q_l}, \frac{\mathrm{d}}{\mathrm{d}t}q_l = -\omega_l^2 y_l - \frac{\mathrm{d}H_{\text{int}}}{\mathrm{d}y_l}. \end{aligned} \tag{31}$$

where H_{int} was defined in Eq. (26) . From the former two equations we get

$$m\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = -\frac{\mathrm{d}V(\mathbf{x})}{\mathrm{d}\mathbf{x}} + e\mathbf{E}, \tag{32}$$

where $\mathbf{E}$ is the electric field of the radiation acting on the particle. The action of the magnetic field has been neglected, a usual approximation in nonrelativistic theory

Classical electrodynamics corresponds to solving Eq. (31) with nil radiation at the initial time, that is

$$y_l^0 = q_l^0 = 0 \text{ for all } l.$$

Solving the coupled differential Eq. (31) is lengthy, but the subject has been studied from long ago and the result is well known. In fact with a fair approximation one obtains [2]

$$m\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = -\frac{\mathrm{d}V(\mathbf{x})}{\mathrm{d}\mathbf{x}} + e\mathbf{E}_{RR}, \mathbf{E}_{RR} = \frac{2e}{3c^3}\dddot{\mathbf{x}}, \tag{33}$$

where $\mathbf{E}_{RR}$ is named radiation reaction field and $e\mathbf{E}_{RR}$ it is interpreted as the effective friction force acting on the particle due to the radiation emitted by it in its accelerated motion. (I point out that the approximation involving $\dddot{\mathbf{x}}$ presents well-known problems like the existence of runaway unphysical solutions, that I will not discuss here). Incidentally the radiation reaction causes that matter is unstable according to classical electrodynamics. The instability does not appear in QED as is well known and it is instructive to see how this happens in QEDWC, which is shown in the following.

As explained above *the same Hamiltonian* may be used in both QEDWC and CED. The difference does not pertain to the equations of motion, that are formally identical, but to the required initial conditions. QEDWC is a quantum theory whence the initial condition should be a Wigner function representing the state of both particles and field. In analogy with classical dynamics the evolution of the Liouville function might be obtained in principle via the solution of the Hamilton equations of motion, similar to the passage from Eqs. (29) to (30) . We should start solving Eq. (31) in order to get the variables at time t, that is

$$\mathbf{x}(\lambda, t), \mathbf{p}(\lambda, t), y_l(\lambda, t), q_l(\lambda, t); \lambda \equiv \left(\mathbf{x}_0, \mathbf{p}_0, \left\{y_l^0, q_l^0\right\}\right) \tag{34}$$

λ being the set of initial conditions. Hence, we might obtain in principle the evolution of the Wigner function of both particles and field, $\mathrm{F}\left(\mathbf{x}, \mathbf{p}, \{y_l, q_l\}, t\right)$ Eq. (28) , from the initial Wigner function $\mathrm{F}(\lambda, 0)$.

The problem of QEDWC is to find the electric field $\mathbf{E}(\lambda, t)$ that results from the solution of Eq. (31) with the initial conditions given in Eq. (34) . The solution is lengthy but a plausible approximation is that the initial conditions $\left\{y_l^0, q_l^0\right\}$ for the field just gives rise to an electric field $\mathbf{E}_{ZPF}$ additional to $\mathbf{E}_{RR}$ (see Eq. 33). That is the electric field of Eq. (32) should lead to

$$m\ddot{\mathbf{x}} = -\frac{\mathrm{d}V(\mathbf{x})}{d\mathbf{x}} + \frac{2e^2}{3c^3}\dddot{\mathbf{x}} + e\mathbf{E}_{ZPF}(\lambda, t). \tag{35}$$

Furthermore, we may approximate the evolution of $\mathbf{E}_{ZPF}(\lambda, t)$ by a *free evolution* of the field with initial conditions $\left\{y_l^0, q_l^0\right\}$.

If we solved Eq. (35) for a family of initial conditions $\lambda \in \Lambda$ we would get a family of solutions

$$\mathbf{x} = \mathbf{x}(\lambda, t), \mathbf{p} = \mathbf{p}(\lambda, t), \tag{36}$$

whence in principle we might obtain the Wigner function of both particle and radiation at time t from the initial Wigner function. That initial Wigner function should consist of the product of a Wigner function for the particle, that may be chosen at will, times the Wigner function of the radiation which should be the vacuum Wigner function Eq. (23) . This would provide the solution of the problem in the QEDWC formalism in principle.

The functions that appear in Eq. (36) have formal analogy with stochastic processes, which are currently defined as functions of time t and "chance" λ. This suggests that we could solve more efficiently the problem assuming that λ is actually a multidimensional random variable with probability distribution Eq. (23) for the field. In QEDWC the initial condition for the particle should be a Wigner function, but we may consider a similar theory where we use definite initial conditions for the particle, say $\mathbf{x}(0) = \mathbf{x}^0$, $\mathbf{p}(0) = \mathbf{p}^0$. This is precisely the choice made in SED, where Eq. (36) is taken as stochastic processes and Eq. (35) leads to the stochastic differential Eq. (5) .

4 The problems of interpretation

Quantum mechanics works yet despite its unprecedented success there is notorious disagreement for the interpretation. Everybody who has learned quantum mechanics agrees how to use it but we do not understand the meaning of the strange conceptual apparatus that each of us uses so effectively to deal with our world. For decades some physicists have been searching for modifications of quantum mechanics that either maintain its testable predictions or lead to changes too small to have yet been observed. Such modifications are motivated not by failures of the existing theory, but by philosophical discomfort with one or another of the prevailing interpretations of that theory. One of these attempts has been SED.

In my view a correct understanding of quantum theory should acknowledge that "the physical concepts with which the theory operates are intended to correspond with the objective reality, and by means of these concepts we picture this reality to ourselves" [15], an epistemology which I would label *realistic interpretation*. SED was, from the very beginning in the 1960', an attempt to get a realistic interpretation of quantum mechanics or even to replace it by a deeper theory. However, the difficulties for a generalization of SED beyond nonrelativistic electrodynamics and the failure to interpret even simple systems, like the hydrogen atom [16], led many authors to less ambitious expectations. In my present opinion SED is a kind of approximation to nonrelativistic QED with validity in a narrow domain.

This does not mean that the effort to develop SED has been useless. At present I think that it has been a clue or guide in the search for a better understanding of quantum theory, which now I see as a stochastic theory resting on the interpretation of the quantum vacuum fields like real stochastic fields [14]. Furthermore, the interpretation of the electromagnetic field within the Weyl–Wigner formalism provides a realistic picture for many quantum effects [17].

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