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Gamma convergence and renormalization group: Two sides of a coin?

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Abstract We discuss, both from the point of view of Gamma convergence and from the point of view of the renormalization Group, the zero range strong contact interaction of three non-relativistic massive particles. *Formally*, the potential term is $g(\delta(x_3 - x_1) + \delta(x_3 - x_2))$, $g < 0$ and is the limit $\epsilon \to 0$ of approximating potentials $V_\epsilon(|x_i - x_3|) = \epsilon^{-3} V(\frac{|x_i - x_3|}{\epsilon})$, $V(x) \in L^1(R^3) \cap L^2(R^3)$. The presence of a delta function in the limit does not allow the use of standard tools of functional analysis. In the first approach (European Phys. J. Plus 136-363, 2021), (European Phys. J. Plus 1136-1161, 2021), we introduced a map $\mathcal{K}$, called Krein Map , from $L^2(R^9)$ to a space (Minlos space) $\mathcal{M}$) of *more singular functions*. In $\mathcal{M}$, the system is represented by a one parameter family of self-adjoint operators. In the topology of $L^2(R^9)$, the system is an ordered family of weakly closed quadratic forms. By Gamma convergence, the infimum is a self-adjoint operator, the Hamiltonian H of the system. Gamma convergence implies resolvent convergence (An Introduction to Gamma Convergence Springer 1993) *but not operator convergence!*. This approach is variational and non-perturbative. In the second approach, perturbation theory is used. At each order of perturbation theory, divergences occur when $\epsilon \to 0$. A finite *renormalized* Hamiltonian H_R is obtained by redefining mass and coupling constant at each order of perturbation theory. In this approach, no distinction is made between self-adjoint operators and quadratic forms. One expects that $H = H_R$, i.e., that "renormalization" amounts to the difference between the Hamiltonian obtained by quadratic form convergence and the one obtained by Gamma convergence. We give some hints, but a formal proof is missing. For completeness, we discuss briefly other types of zero-range interactions.

1 Introduction

We consider the strong contact (zero range) interaction of two non-relativistic particles of equal mass with a third massive particle.

In quantum mechanics, the Hamiltonian of separate strong contact of a particle with two identical ones [1, 2] is described by the limit, when $\epsilon \to 0$, of the Hamiltonians $H_\epsilon = H_0 + \sum_{i=1,2} V_\epsilon(|x_i - x_0|)$ where H_0 is the three-body non-relativistic free Hamiltonian and $V_\epsilon(|x_i - x_0|) = \epsilon^{-3} V(\frac{|x_i - x_0|}{\epsilon})$, $V(x) \in L^1(R^3) \cap L^2(R^3)$.

In the limit $\epsilon \to 0$, the interaction is represented *formally* by two delta functions $\delta(x_i - x_0)$ $i = 1, 2$. Formal perturbation theory leads to divergences.

In order to describe the system by a self-adjoint Hamiltonian, one may follow two approaches: Gamma convergence or renormalization.

The first approach, through Gamma convergence [1, 2, 6] , is variational and non-perturbative. The limit is obtained in the sense *strong resolvent convergence*.

We recall briefly elements of Gamma convergence [6], a tool of common use in the theory of composite materials but seldom used in quantum mechanics.

We introduce first a map $\mathcal{K}$ (*Krein map*) of the formal Hamiltonian to a Hilbert space $\mathcal{M}$ [3] of more singular functions; the map *is fractioning and mixing* and acts differently on H_0 (an operator) and on $\delta(x - x_j)$ (a quadratic form).

For historical reasons, we call *Minlos space* the space $\mathcal{M}$.

In $\mathcal{M}$, the kinetic energy and the interaction potential have opposite sign and the same degree of singularity. The system is described by a well order family of self-adjoint operators [4, 5].

Returning to the topology of $L^2(R^9)$ produces a sequence of well-ordered weakly closed quadratic forms. *Notice that we do not invert the Krein map*; this map is fractioning and mixing, and therefore, it is not invertible.

By Sobolev embeddings, compactness holds and the infimum of these quadratic forms can be closed strongly; its closure is a self-adjoint operator H, the Hamiltonian of our system.

This paper is dedicated to Elliott Lieb, an outstanding scientist and a friend, in celebration of his 90th birthday.

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Gamma convergence [6] implies *strong resolvent convergence* of the Hamiltonians H_ϵ to H.

Notice that the sequence of operators H_ϵ diverges in the strong operator topology

We give in the next Sections some details; for full proofs, we refer to [1, 2, 6].

Remark 1 Since the Krein map is mixing and fractioning, this approach is along the lines of *rearrangement inequalities* (notice that we take the infimum of a sequence of quadratic forms). Their role has been stressed in particular by E.Lieb.

Remark 2 This strategy of using as intermediate step a map to a space of more singular functions goes back to Friederichs [12] for the Laplacian in R^+; therefore, the map we use could be also called *Friederichs map*.

In the second approach (renormalization) [8][9][10], the potential term is represented as two delta "functions."

The Hamiltonian is also here a formal limit for $\epsilon \to 0$ of Hamiltonians $H_\epsilon = H_0 + V_\epsilon$ with potentials $V_\epsilon(|x_i - x_0|) = \epsilon^{-3} V(\frac{|x_i - x_0|}{\epsilon})$, $V(x) \in L^1(R^3) \cap L^2(R^3)$.

The limit is now considered as limit of quadratic forms.

Formal perturbation theory leads *at each order* to a quadratic form that divergences when $\epsilon \to 0$.

At each order in ϵ, these divergences are *renormalized* by redefining the parameters, mass and coupling constant.

This sequence of renormalizations defines the renormalization (semi-)group.

In quantum mechanics and in non-relativistic field theory, this sequence of operations converges to a limit (a fixed point) [8–10] .

2 Contact interaction in quantum mechanics; Krein map, minlos space and gamma convergence

We recall briefly the steps taken in [1, 2]. For further details, we refer to [1, 2] .

We consider in R^3 the strong contact interaction of two no relativistic identical massive particles of coordinates x_1, x_2 with a third massive particle of coordinate x_0.

Contact interactions are self-adjoint extension of the symmetric operator H_0^0, the free Hamiltonian of a *three body system* restricted to functions that vanish in a neighborhood of the *contact manyfold* $\Gamma \equiv \cup_{i=1,2}\{x_i - x_0 = 0\}$ $x_i \in R^3$.

The operators that describe strong contact are limits as $\epsilon \to 0$, *in strong resolvent sense*, of Hamiltonians H_ϵ with potentials that scale as $V_\epsilon(x_i - x_j) = \frac{1}{\epsilon^3} V(\frac{|x_i - x_j|}{\epsilon})$, $V \in L^1(R^3) \cap L^2(R^3)$.

We stress that convergence holds in the strong resolvent sense, i.e., the limit of the resolvents is the resolvent of a self-adjoint operator.

When $\epsilon \to 0$, the resolvent family $R_\epsilon = (z - H_\epsilon)^{-1}$ remains uniformly bounded and analytic outside any cone along the real axis and vertex in a convenient $C < 0$.

Resolvent identities are satisfied for $\epsilon \geq 0$. The limit is therefore the resolvent of a self-adjoint operator bounded below.

This operator is not the strong limit of H_ϵ as $\epsilon \to 0$. The sequence H_ϵ diverges to $+\infty$ as $\epsilon \to 0$.

The self-adjoint extension is constructed in [1, 2] through a non-perturbative procedure based on Gamma convergence [6] , a variational tool introduced by E. de Giorgi and of common use in the theory of composite materials.

As intermediate step we introduced in [1, 2] a map $\mathcal{K}$ from $L^2(R^9)$ to a space of *more singular functions*. We call this map *Krein map* $\mathcal{K}$, and we call the target space *Minlos space* $\mathcal{M}$ [3].

The map is *fractioning* (the functions in the new space are more singular) and *mixing* (the map does not preserve the channel stricture).

The target space is $\mathcal{K} \equiv H_0^{-\frac{1}{4}}(L^2(R^9))$; it is a space of *more singular functions*.

The Krein map acts differently on operators and on quadratic forms.

On the kinetic energy, it acts as $H_0 \to H_0^{-\frac{1}{4}} H_0 H_0^{-\frac{1}{4}}$ and on the potential term as $W = H_0^{-\frac{1}{2}}(\delta(x_1 - x_0) + \delta(x_2 - x_0))H_0^{-\frac{1}{2}}$.

Its action is *different* on the kinetic energy and on the "potential" ;notice that the former is a self-adjoint operator and the latter is a quadratic form.

The Krein map is mixing and fractioning and can be regarded as a microscope that permits to see fine details of the interaction.

Recall that W and $\frac{1}{H_0}$ commute as quadratic forms (as can be seen in Fourier space). Therefore, the system is abelian. If the potential is negative (attraction) in $\mathcal{M}$, the kinetic and potential part has the same singularity (a pole) *but with opposite signs* at the origin in the difference of the coordinates of the particles that are in strong contact.

Therefore, [4, 5] in $\mathcal{M}$, the system is represented by a one parameter ordered sequence of self-adjoint operators. The parameter is the angular momentum of the motion in the system in which the barycenter is a rest.

Each operator has an infinite sequence of bound states with eigenvalues that scale geometrically.

We have studied this system in [1, 2].

Remark Notice that we *do not invert* the Krein map; this map is fractioning and mixing, and therefore, it is not invertible. The Krein map is only an instrument (a microscope) to put in evidence the "optimal" macroscopic picture.

Finding the optimal structure was also the original purpose of renormalization.

To extract a self-adjoint operator (the Hamiltonian of our system) we make use of a variational procedure, Gamma convergence [6], introduced by E. de Giorgi and mostly used in the analysis of finely fragmented materials.

The Gamma limit F(y) of a set of quadratic form in a Banach space Y is the quadratic form defined by the relations

$$\forall y_n, y_n \to y : F(y) \leq liminf F(y_n) \tag{1}$$

$$\forall x \in Y \exists x_n \; x_n \to x : F(x) \geq limsup_n F(x_n) \tag{2}$$

In our case, due to the special form of the potential, the Gamma limit F(y) of a set of quadratic form is the quadratic form defined by the relations

$$\forall y \in Y, y_n \to y; F(y) = liminf F(y_n) \quad \forall x \in Y_n \;\; \forall \{x_n\} : F(x) \leq limsup_n F_n(x_n) \tag{3}$$

The first condition implies that $F(y)$ is a common lower bound for the forms F_n, the second implies that this lower bound is optimal.

The condition for the existence of the Gamma limit is that the sequence be contained in a compact set for the topology of Y (so that a Palais-Smale convergent subsequence exists).

In our case, the topology of Y is the Frechet topology defined by Sobolev semi-norms and compactness follows from the absence of zero energy resonances.

Therefore, in our case, the Gamma limit exists and it is a quadratic form which is bounded below. Notice that the Krein map is order preserving, and therefore, in our case, the sequence is monotone.

The limit form is strictly convex, and by a theorem of Kato [7], it can be closed strongly and provides the "physical" Hamiltonian" H

Gamma convergence implies *strong resolvent convergence* [6] and since the limit is a self-adjoint operator this implies also convergence of spectra and of wave operators.

Recall that the resolvent family of a self-adjoint operator H is defined as $R(z, H) = (z - H)^{-1}$ where $z \notin \sigma(H)$.

Consider now the approximating Hamiltonians H_ϵ. The Krein map is positivity preserving. Since the potential is negative and increasing in absolute value when ϵ decreases to zero, they form a decreasing sequence

It follows that the resolvents of the operators H_ϵ have a limit for $\epsilon \to 0$, and the resolvent of the operator H is the strong limit of the resolvents of the Hamiltonians H_ϵ. No rate of convergence can be given in the parameter ϵ.

Remark that resolvent converge means that the limit for $\epsilon \to 0$ of the resolvents is the resolvent of a self-adjoint operator H_{lim}.

We have noticed that H_ϵ diverges in norm for $\epsilon \to 0$.

It is likely that the difference $H_\epsilon - H_{lim}$ *is the diverging term that is present in a perturbative analysis.*

Since the potential is of finite range (in fact, zero range), the system described by H_{lim} is asymptotically free and the wave operator can be defined.

Remark In the two-dimensional case, the role of resolvent convergence has been stressed in [11].

3 Renormalization

In our analysis, of the interactions through Gamma convergence, we noticed that divergences occur because we consider the limit of the quadratic forms.

In renormalization theory, these divergencies are "cured" at each order in ϵ by introducing renormalization [8–10] , i.e., a modification of the diverging parameters of the theory (mass and charges). The guiding principle is to subtract divergencies. The redefinition is through the subtraction of countably many terms which are either quadratic (Kinetic energy) or cubic (interaction) and a redefinition of the Hilbert space.

At each order in perturbation theory, this procedure of renormalization provides a well-defined symmetric quadratic form bounded below (recall that in renormalization, no distinction is made between self-adjoint operators and symmetric quadratic forms).

This sequence of renormalizations forms an abelian semigroup with a limit (a fixed point, an *attractor*). The fixed point provides the "physical" value of the parameters (charges) for the "physical" Hamiltonian. The final result is the *renormalized Hamiltonian* a well-defined quadratic form. The guiding principle of renormalization is to subtract divergencies. The redefinition is through the subtraction of countably many terms which are either quadratic in the fields (Kinetic energy) or cubic (interaction) and a redefinition of the Hilbert space.

At each order in perturbation theory, this procedure of renormalization provides a well -defined symmetric quadratic form bounded below. No distinction is made between self-adjoint operators and symmetric quadratic forms).

This sequence of renormalizations forms an abelian semigroup with a limit (a fixed point, an *attractor*). This fixed point provides the "physical" value of the parameters (charges and masses) for the "physical" Hamiltonian. The final result is the *renormalized Hamiltonian*.

The proof requires many estimates. By construction, the limit is a symmetric quadratic form.

While in general, symmetry of the limit quadratic forms is evident, it is hard to verify that it is strongly closed and represents a self-adjoint operator.

In the formal process of renormalization, no distinction is made between self-adjoint operators and symmetric quadratic forms. And even if one were able to prove self-adjointness of the limit it would be hard to find its spectrum. (In our approach, through Gamma convergence, one finds the resolvent of the limit operator and therefore its spectrum.)

Notice that in perturbation theory, one renormalizes first and then takes the limit $\epsilon \to 0$, while in Gamma convergence, the Krein map is defined after taking the limit $\epsilon \to 0$ (at the cost of introducing temporarily a new space, *Minlos space*, of more singular functions).

In both approaches, there is no control on the rate of convergence in the parameter ϵ. The advantage of the approach through Gamma convergence, compared with the use of the renormalization group, is that it is within the theory of self-adjoint extensions and that the regularity of the wave functions plays a role *but not the statistics*. Therefore, the method applies to Bosons and to Fermions.

In both cases, one has *no rate of convergence in the parameter* ϵ. *It may be possible to prove convergence with parameter* $\frac{1}{log\epsilon}$

4 Weak contact

A weaker form of local contact is the *weak contact*. These interactions occur mostly in quantum mechanics, whereas strong contact is typical of quantum field theory.

A weak contact of the Hamiltonian is the limit *in strong resolvent sense* of Hamiltonians with potentials that scale as $V_\epsilon(|x_i - x_0|) = \frac{1}{\epsilon^2} V(\frac{|x_i - x_0|}{\epsilon})$.

In three dimensions, this implies the presence of a zero energy resonance.

Also, this interaction can be analyzed using perturbation theory and renormalization; the renormalization is weaker here.

In [1, 2], we studied weak contact making use of a Krein map and a Minlos space.

Weak contact is the limit $\epsilon \to 0$ of approximating Hamiltonians H_ϵ with potentials $V_\epsilon(|x_i - x_0|) = \epsilon^{-2} V(\frac{|x_i - x_0|}{\epsilon})$, $V(x) \in L^1(R^3) \cap L^2(R^3)$. We denote these limits with Θ_i; they are infinite step functions.

The Krein (rearrangement) map is also here mixing, but it is fractioning in a weaker form; it acts in the same way on the kinetic energy and on the potential.

On the kinetic energy, it acts as $H_0 \to H_0^{-\frac{1}{4}} H_0 H_0^{-\frac{1}{4}}$ and on the potential term as $W = H_0^{-\frac{1}{4}} (\Theta(x_1 - x_0) + \Theta(x_2 - x_0)) H_0^{-\frac{1}{4}}$.

Also here, the Krein map is mixing and fractioning and can be regarded as a microscope that permits to see fine details of the interaction.

Recall that W and $\frac{1}{H_0}$ commute as quadratic forms (as can be seen in Fourier space). Therefore, also here, the system is abelian.

Also here if the potential is negative (attraction), and in $\mathcal{M}$, the kinetic and potential parts have the same singularity (a pole) *but with opposite signs*.

Also here, the system is represented in $\mathcal{M}$ by an ordered family of self-adjoint operators.

This corresponds in the physical Hilbert space to a family of quadratic forms. Their infimum is a self-adjoint operator, the Hamiltonian of the system.

In [1, 2], we have studied the case of three particles in mutual weak contact; this system has as semiclassical limit the three-body problem in Newton mechanics.

We have also studied the case of two pairs of particles in which each particle of one pair is in weak contact with both particles of the other pair. We have proved that if there is a very weak repulsion between the pairs (a resonance), the structure is described by the Ginsburg-Landau functional.

If the repulsion is absent the energy functional is the version of the Gross-Pitaevski functional that has an essential singularity at contact. (In the literature, the resonances are often called "ghosts.")

This functional describes the Bose-Einstein condensate in the low density regime.

Weak contact of the barycenters of two pairs in strong contact gives the building block for a Bose-Einstein condensate in the high density regime.

Remark 1 Notice that semi-classically weak contact corresponds to Coulomb interaction.

Classically weak and strong contact is, respectively, holonomic and an-honolomic constrains.

We do not give here the details.

Remark 2 In two dimensions, there is only one type of contact interactions, the weak one. This interaction is studied in [11] using perturbation theory and renormalization.

In [11], it is remarked that it natural to study resolvent convergence.

5 Other non-relativistic quantum systems

Along the lines described above, one can compare renormalization and Gamma convergence for other systems.

Consider, e.g., a system composed of two pairs of identical particles. The particles can be either bosons or fermions.

This system was studied in [2] in quantum mechanics using resolvent convergence.

There is a strong contact attractive interaction between the particles in each pair, and there is a further weak contact interaction between their barycenters. Notice that identical fermions with opposite spin orientations can have a strong contact interaction.

The system can represent an element of a Bose-Einstein condensate (one can add a regular confining potential that does not interfere with zero-range interactions [2]).

Self-adjointness and the spectral properties of this system can be analyzed in quantum mechanics using Gamma convergence [2].

The eigenstates are critical points of a Gross-Pitaevskii energy functional. It is not essential that the "charges" of the particles be equal.

Notice that these systems and the system of two particles in strong contact with a third particle studied in [1, 2] are in three dimension the only irreducible local systems compatible with strong contact.

The analysis of this system using the renormalization group can be done . After suitable subtractions (renormalizations), a quadratic form is obtained, but no proof is available that represents a self-adjoint operator.

6 Renormalization group in non-relativistic field theory

We compare now our analysis using Gamma convergence with the analysis through the renormalization group [6] in non-relativistic field theory.

In quantum mechanics, the particles are elements of $L^2(R^3)$ and therefore can be localized. This allows for the definition of bound states and scattering states and of attractive point interactions.

On the contrary in field theory, the fields are extensive quantities.

Consider the local interaction of a massive particle with a non-relativistic field.

The interaction is linear in the field, and the potential is represented with a delta function.

Divergence occurs at every order of perturbation theory. These divergences are canceled by redefining the parameters of the theory (mass and charge) and the metric topology of the space (wave function renormalization).

We consider here renormalization in its original formulation [8–10] , i.e., as a mathematical version of the microscope. In this setting, Gamma convergence will be the counterpart of renormalization.

The interaction is linear in the field. We make use of Fock space.

Also here, we find diverging terms at each order of perturbation. At each order renormalization provides the Hamiltonian as a well defined symmetric quadratic form bounded below.

The guiding principe is to subtract divergencies. No distinction is made between self-adjoint operators and symmetric quadratic forms.

One renormalizes first and then the take the limit $\epsilon \to 0$. *Formally*, this sequence of renormalizations has a limit (a fixed point, an *attractor*).

This fixed point provides the "physical" value of the parameters (charges and masses).

The result of renormalization is a symmetric quadratic form.

It is *assumed* that this form represents a self-adjoint operator, the renormalized Hamiltonian.

While in general, symmetry of the limit quadratic form is evident, it is hard to verify that it represents a self-adjoint operator.

This method seeks *operator convergence*. Notice that Gamma convergence provides *resolvent convergence*.

It is likely that the "infinite mass and charge renormalization" is due to the fact that the limit resolvent is not the resolvent of the limit quadratic form (which is "infinite") .

7 Gamma convergence and fock space

In this section, we sketch a possible use of Gamma convergence instead the renormalization group in Fock space.

Notice that the construction of a Fock space over a Hilbert space is separable, and Gamma convergence can be defined for quadratic forms in Fock space.

Consider a system of two non-relativistic particles of mass M interacting via contact interactions with non-relativistic particles of mass m.

The interaction "creates" or "annihilates" the particle of mass m.

We introduce creation and annihilation operators $a(x)$, $a^*(x)$ of the particle with mass m.

To control the singular nature of the interaction, we introduce the Krein map $\mathcal{K}$ and Minlos space $\mathcal{M}$ making use of the free Hamiltonian which is quadratic in the field. This space is now a Fock-Minlos space.

Also here, the Krein map *acts differently* on the free Hamiltonian and on the interaction term. Also here, the map it is *mixing* and *fractioning*.

In this space $\mathcal{M}$, the creation and annihilation operators are *bounded operators*, and therefore, perturbation theory can be used.

In $\mathcal{M}$, the number of particles is not conserved by the Hamiltonian flow, but the free flow and the Krein map commute; therefore, the Krein map can be performed at any time and the resulting theory is stationary.

The Krein map changes the metric topology, returning to the original Fock space one has an ordered family of quadratic forms.

Again, since there are no zero energy resonances, Gamma convergence applies. The lowest form can be closed in the strong topology and defines a self-adjoint operator in the original Fock space.

From Gamma convergence, it follows that also, in Fock space, the renormalized Hamiltonian is the limit *in strong resolvent sense* of the approximate Hamiltonians H_ϵ. It is self-adjoint since the forms in $\mathcal{M}$ are symmetric.

Again the Hamiltonian is the limit *in strong resolvent sense* as $epsilon \to 0$ of a Hamiltonian in which the potential converges weakly to a delta function.

Reference to the *contact manyfold* is essential. One cannot obtain this extension if one does not construct first the contact manifolds and finds the boundary conditions.

Our approach is non-perturbative, in fact, it is "maximally" non-perturbative since in the space $\mathcal{M}$, the kinetic and the potential term has the same weight.

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Appendix: Tracks in a clouds chamber

An example of joint strong contact interactions is tracks in a cloud chamber or in photographic plates.

Consider a fast neutral particle (a "cosmic ray") which interacts with an atom: the interaction is a *separate* strong contact interaction with the nucleus and with an outer conduction electron.

Both are ejected. Since the interaction is of very short range, one may use a semiclassical description for the particles which are ejected, respectively, as a negatively charged particle (an electron) and positively charged particle (an ion). After the interaction, they follow "classical trajectories."

In a supersaturated environment, such as a cloud chamber or a properly treated photographic plate, the particles produce ionization tracks; in the presence of a magnetic field, the tracks have opposite curvatures. We describe in this appendix two possible instances of joint strong contact.

Added in proofs: Thanks are due to a referee for constructive criticisms.

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