Beam blow up due to beamstrahlung in circular e^+e^- colliders

M. A. Valdivia Garcia^{1,a} , F. Zimmermann²

¹ U. Guanajuato, 36000 Guanajuato, Mexico

² CERN, 1211 Geneva 23, Switzerland

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Abstract After the discovery of the Higgs boson in 2012, several future circular colliders — Higgs factories — are proposed, such as FCC-ee and CEPC. At these e^+e^- colliders, beamstrahlung can greatly affect the equilibrium bunch length and energy spread. If the dispersion function at the collision point is not zero, beamstrahlung will also increase the transverse emittances. In this paper, we derive analytical formulae for the equilibrium beam parameters, considering the varying electromagnetic field during the collision, and taking into account hourglass effect, crossing angle, and crab waist. We illustrate the importance of beamstrahlung, including the increase of bunch length and the implied tolerance on the spurious dispersion function at the collision point, by considering a few examples.

1 Introduction

In most electron storage rings operated so far, the equilibrium transverse emittances, energy spread and bunch length were, or are, determined by a balance of quantum excitation and radiation damping, both arising from the synchrotron radiation emitted when the charged ultra-relativistic beam particles pass through the accelerator magnets, in particular through the bending magnets [1]. Future high-energy circular colliders, like FCC-ee [2] or CEPC [3], are proposed as high-precision Higgs factories, to study the Higgs boson discovered at the CERN Large Hadron Collider [4], or, more generally, as “electroweak factories”. In these future circular colliders, for the first time, the synchrotron radiation emitted during the collision in the electromagnetic field of the opposing beam becomes important [5]. This particular type of synchrotron radiation is called “beamstrahlung” [6–12]. A beam particle is lost whenever, during the collision, it radiates a photon of an energy high enough that the emitting particle falls outside the momentum acceptance. Through this process, the high-energy tail of the beamstrahlung can severely limit the beam lifetime [13, 14]. Design parameters for FCC-ee and CEPC are taking into account this lifetime limitation along with additional constraints imposed by a coherent beam-beam instability [15].

There is yet another novel effect of beamstrahlung in circular Higgs factories, which we will analyse in this article. Namely, at the aforementioned colliders, the beamstrahlung significantly increases the equilibrium bunch length and energy spread of the colliding beams. An increase of the energy spread had already been pointed out in Refs. [5, 13], where this

^a e-mail: valdivia@fisica.ugto.mx (corresponding author)

was estimated under some rough approximations, e.g., by multiplying the mean number of beamstrahlung photons and the average square of their energy, or by evoking a not fully accurate dependence on the crossing angle. However, the establishment of a new self-consistent equilibrium bunch length was not yet mentioned in Ref. [5]. The new equilibrium was, however, addressed in Refs. [13, 16–19], where the beamstrahlung was again treated in various approximate manners. Furthermore, with a nonzero dispersion at the interaction point (IP), beamstrahlung can also affect the transverse beam emittance [17, 20]. Such nonzero dispersion can either be due to uncorrected optics errors (“spurious dispersion”) or be intentionally introduced for the purpose of reducing the centre-of-mass energy spread (“monochromatization”) [21]. In this article, we derive exact formulae describing the impact of classical beamstrahlung on the beam distribution, including crossing angle, hourglass effect, nonzero dispersion at the IP, and the crab-waist collision scheme [22], which is an integral part of all future circular e^+e^- colliders. We also present approximate solutions, that are valid in practically important limiting cases.

2 Beamstrahlung photons

We consider collisions of 3-dimensional Gaussian bunches with transverse IP rms beam sizes σ_x^* and σ_y^* , under a horizontal crossing angle θ_c and with non-negligible hourglass effect, $\sigma_z > \beta_y^*$, where β_y^* denotes the vertical IP beta function and σ_z the rms longitudinal bunch length. The strength of the beamstrahlung is characterized by the parameter Υ , defined as [12, Chapter 3] $\Upsilon \equiv \gamma(|B| + |E|/c)/B_c = (2/3)\hbar\omega_c/E_e$, with $B_c = m_e^2 c^2/(e\hbar) \approx 4.4$ GT the Schwinger critical field, $\hbar\omega_c = (3/2)\hbar c\gamma^3/\rho$ the critical photon energy as introduced by Sands [1], E_e the electron (or positron) energy before radiation, B (E) the local magnetic (electric) field of the opposing bunch, $\rho = e/(pB)$ the local bending radius, γ the relativistic Lorentz factor corresponding to E_e , $p \approx E_e/c$ the particle momentum, e the electron charge, m_e the electron mass, $\hbar$ the Planck constant (divided by 2π), and c the speed of light. From [1], the average Υ during the collision is $\langle \Upsilon \rangle = (r_e \gamma^2/\alpha) \langle 1/\rho \rangle_{x,y,z,s}$, where α denotes the fine structure constant ($\alpha \approx 1/137$), $r_e \approx 2.8 \times 10^{-15}$ m the classical electron radius, and the average $\langle \rangle$ is taken over the transverse coordinates (x, y) , longitudinal coordinate s (along the beam line), and $z = (s - ct)$, i.e., a co-moving coordinate along the bunch, with $z = 0$ referring to the centre of the bunch, and t denoting the time. We note that the local bending radius ρ varies with the transverse and longitudinal positions of the colliding particle, and with the time during the collision [23, 24], or, equivalently, with the coordinate z .

For all proposed high-energy circular e^+e^- colliders, $\langle \Upsilon \rangle \ll 1$. In this case we can approximate the average number of photons per collision as [1]

$$n_\gamma \approx \frac{5}{2\sqrt{3}} \alpha \gamma \int ds \left\langle \frac{1}{\rho} \right\rangle_{x,y,z}, \quad (1)$$

where the integral over s extends through the collision region. The mean emission rate is given by n_γ divided by the average time interval between collisions, e.g., half the revolution period in case of two interaction points.

The average relative energy loss due to beamstrahlung (subscript ‘BS’) is [1]

$$\delta_{BS} \approx \frac{2}{3} r_e \gamma^3 \int ds \left\langle \frac{1}{\rho^2} \right\rangle_{x,y,z}. \quad (2)$$

Fig. 1 Bunch orientation and coordinates in the laboratory frame (a) and in the boosted frame (b). Locations of the vertical focusing plane without and with crab waist are also indicated

The average photon energy normalized to the beam energy, $\langle u \rangle$, is obtained as the ratio $\langle u \rangle = \delta_B / n_\gamma$.

The quantum excitation [1] from beamstrahlung (BS) emitted in a single collision can be written

$$\{n_\gamma \langle u^2 \rangle\}_{\text{BS}} \approx \frac{55}{24\sqrt{3}} \frac{r_e^2 \gamma^5}{\alpha} \int ds \left\langle \frac{1}{\rho^3} \right\rangle_{x,y,z}. \quad (3)$$

3 Collision scheme and inverse bending radius

For all future circular e^+e^- colliders the crab-waist collision scheme [22] has been adopted. In this scheme bunches are collided under a horizontal crossing angle θ_c with an effective overlap region $l_{\text{eff}} \equiv \sigma_z / \sqrt{1 + \Phi^2}$ [5, 14], with the “Piwinski angle” $\Phi \equiv \theta_c \sigma_z / (2\sigma_x^*)$. The crab-waist scheme implies $\Phi \gg 1$ and $\beta_y^* \geq l_{\text{eff}}$.

The classical formulae for synchrotron radiation would also be modified for l_{eff} shorter than the classical formation length ρ/γ [25, 26], with ρ the local bending radius. This “short-magnet” regime is characterized by an “undulator parameter” $K_{\text{max}} \equiv 2r_e N_b / (\sigma_z \theta_c) < 1$. As we show below, for all examples of interest, we find $K_{\text{max}} \gtrsim 3$, so that the classical radiation spectrum applies.

To treat the case of a nonzero horizontal crossing angle, we consider the collision in a co-moving (boosted) frame [27], where the collision is “head-on”, but both bunches are tilted by an opposite angle of magnitude $\theta_c/2$. This transformation is accomplished by a horizontal Lorentz boost with $\beta_{\text{boost}} = p_x c / E_e = \sin(\theta_c/2) \ll 1$, with p_x the horizontal momentum due to the crossing angle, considering that in the laboratory system $E_e \approx p_z c$ (ultrarelativistic particles, with p_z denoting the dominant longitudinal momentum). The beam energy in the boosted system is $E'_e \approx \gamma_{\text{boost}} (E_e - c p_x \beta_{\text{boost}}) = \gamma_{\text{boost}} E_e (1 - \beta_{\text{boost}}^2) = E_e / \gamma_{\text{boost}} \approx E_e$, since $\gamma_{\text{boost}} \approx 1/\sqrt{1 - \theta_c^2/4} \approx (1 + \theta_c^2/8)$. Hence, for typical crossing angles of 30 mrad, the beam energy in the boosted frame is within about a per mil equal to the beam energy in the laboratory system. The same is true for the energy of the beamstrahlung photons, which are emitted in the beam direction. In the following we neglect the tiny differences in beam and photon energies between boosted frame and laboratory frame. Figure 1 illustrates the two frames. We have dropped the apostrophe for the coordinates in the boosted frame. The pictures also indicate the x -dependent location of the minimum vertical beta function with and without crab waist.

In addition, to first approximation, for the future circular colliders considered, we may ignore the disruption effects [12], and we also neglect the horizontal rms angular beam divergence compared with the crossing angle θ_c . We do take into account the vertical hourglass effect by letting the vertical rms beam size change with positions s and x as

$$\sigma_y(s, x) = \varepsilon_y^{1/2} \left(\beta_y^* + \frac{(s \pm k_{cw}x/\theta_c)^2}{\beta_y^*} \right)^{1/2}, \quad (4)$$

where ε_y denotes the vertical rms emittance and β_y^* the vertical beta function at the focal point ($s = 0$, IP), and $k_{cw} = 1$ for a crab waist optics, with opposite sign for the strong (+) and weak beam (−), respectively, 0 otherwise. Introducing the nominal vertical beam size at the IP, $\sigma_y^* = \sqrt{\varepsilon_y \beta_y^*}$, the normalized coordinates $\bar{y} = x/\sigma_y^*$, $\bar{x} = x/\sigma_x^*$, $\bar{z} = z/\sigma_z$, $\bar{s} = s/\sigma_z$, and the function

$$G_y(\bar{s}, \bar{x}; \Phi) \equiv \sqrt{1 + \frac{\sigma_z^2}{\beta_y^{*2}} \left(\bar{s} \pm \frac{k_{cw}}{2\Phi} \bar{x} \right)^2}, \quad (5)$$

the vertical beam size at location (x, s) can be expressed in more compact form as

$$\sigma_y(\bar{s}, \bar{x}; \Phi) = \sigma_y^* G_y(\bar{s}, \bar{x}; \Phi), \quad (6)$$

where Φ is considered a parameter.

Under the above assumptions, and assuming $\sigma_x^* \gg \sigma_y(\bar{s}, \bar{x}; \Phi)$ in the collision region of interest, the inverse local bending radius ρ at normalized space-time coordinates $(\bar{x}, \bar{y}, \bar{s}, \bar{z})$ can be approximated as [28]

$$\frac{1}{\rho(\bar{x}, \bar{y}, \bar{s}, \bar{z}; \Phi)} = |\mathcal{F}(\bar{x} - 2\Phi\bar{s} + \Phi\bar{z}, \bar{y}, \bar{s}; \Phi)| \frac{2N_b r_e}{\gamma \sigma_z} \exp\left(-\frac{(2\bar{s} - \bar{z})^2}{2}\right) \quad (7)$$

where Φ enters both as a parameter for the vertical beam size (in case of a crab-waist collision), and as part of the first argument (where it reflects the horizontal separation of the opposite beam's centroid position due to the crossing angle), and the function $\mathcal{F}(\bar{x}, \bar{y}, \bar{s}; \Phi)$ may be expressed in terms of the Faddeeva function w (a scaled complementary complex error function defined as $w(z) = \exp(-z^2)\text{erfc}(-iz) = \exp(-z^2) \left(1 + (2i/\sqrt{\pi}) \int_0^z \exp(t^2) dt\right)$) as [29]

$$\mathcal{F}(\bar{x}, \bar{y}, \bar{s}; \Phi) = \sqrt{\frac{\pi}{2}} \frac{1}{\sigma_\Delta(\bar{s}, \bar{x}; \Phi)} \left(w \left[\frac{\bar{x}\sigma_x^* + i \frac{\bar{y}\sigma_y^*}{G_y(\bar{s}, \bar{x})}}{\sqrt{2}\sigma_\Delta(\bar{s}, \bar{x}; \Phi)} \right] - e^{-\frac{\bar{x}^2}{2} - \frac{\bar{y}^2}{2G_y^2(\bar{s}, \bar{x})}} w \left[\frac{\bar{x}\sigma_y^* G_y(\bar{s}, \bar{x}) + i \frac{\bar{y}\sigma_x^*}{G_y(\bar{s}, \bar{x})}}{\sqrt{2}\sigma_\Delta(\bar{s}, \bar{x}; \Phi)} \right] \right), \quad (8)$$

with

$$\sigma_\Delta(\bar{s}, \bar{x}; \Phi) \equiv \sqrt{\sigma_x^{*2} - \sigma_y^2(\bar{s}, \bar{x}; \Phi)}. \quad (9)$$

Equation (7) is valid for head-on collisions, of ultrarelativistic Gaussian beams, where the electric and magnetic fields are perpendicular to the direction of motion, which is the case in our boosted reference frame. The bunch horizontal tilt with longitudinal position in this reference frame has been taken into account in Eq. (7) through the change of the centroid location of the opposing beam, namely, by the term $(\bar{x} - 2\Phi\bar{s} + \Phi\bar{z})$, equivalent to $(x - (s - z/2)\theta_c)$, in the first argument of $\mathcal{F}$.

Fig. 2 Left picture: Variation of $\int ds \langle 1/\rho^n \rangle$ ($n = 1, 2$ and 3) with Φ , all normalized to the value at $\Phi = 0$ (subindex “0”) for two different values of β_y^* . Two analytical curves are superimposed. Right picture: Ratios of $\int ds \langle 1/\rho^n \rangle / (\int ds \langle 1/\rho \rangle)^n$, for n equal 2 and 3, and considering two different values of β_y^* .

Including the crossing angle and the hourglass effect, the average n th power of the inverse bending radius at location s is obtained as a triple integral of (7) over the three bunch dimensions:

$$\left\langle \frac{1}{\rho^n}(\bar{s}; \Phi) \right\rangle_{\bar{x}, \bar{y}, \bar{z}} = \frac{1}{(2\pi)^{3/2}} \int_{\bar{x}, \bar{y}, \bar{z}} \frac{d\bar{x} d\bar{y} d\bar{z}}{G_y(\bar{s}, -\bar{x}; \Phi)} \frac{1}{\rho(\bar{x}, \bar{y}, \bar{s}, \bar{z}; \Phi)^n} \exp \left(-\frac{(\bar{x} + \bar{z}\Phi)^2}{2} - \frac{\bar{y}^2}{2G_y^2(\bar{s}, -\bar{x}; \Phi)} - \frac{\bar{z}^2}{2} \right), \quad (10)$$

which can be evaluated numerically, for any integer n .

As is evident from Eqs. (1), (2) and (3), the integral of $\langle 1/\rho^n(\bar{s}; \Phi) \rangle_{\bar{x}, \bar{y}, \bar{z}}$ over s is our primary quantity of interest. Figure 2 (left picture) illustrates the variation of the integral $\int ds \langle 1/\rho^n \rangle_{\bar{x}, \bar{y}, \bar{z}}$ with the Piwinski angle Φ , for $n = 1, 2$ and 3 . Two possible analytical dependencies are superimposed. The simple $1/\sqrt{1 + \Phi^2}$ curve is not a good representation for any n . The more complex second-to-last curve (whose full expression is indicated in the legend) was given in [30, Eq. (2.59)] to describe the Φ dependence of δ_B , Eq. (2). Indeed, it appears to be a good description for the Φ variation of $\int ds \langle 1/\rho^2 \rangle$ at either value of β_y^* . The right picture of Fig. 2 presents numerically computed ratios $\int ds \langle 1/\rho^n \rangle / \int ds \langle 1/\rho \rangle^n$, for $n = 2$ and 3 . Curves for different values of β_y^* reveal the impact of the hourglass effect. The other beam parameters in both pictures correspond to the FCC-ee ZH production, i.e., to the far right column in Table 1.

Instead of applying the Bassetti-Erskine expression for the field of a two-dimensional Gaussian line charge distribution [29], as we did in Eq. (8), an (approximate) analytical treatment could also be based on the (modified) potential of a three-dimensional Gaussian charge distribution [31]. Such an approach was adopted in Refs. [18, 19].

4 Flat-Beam approximation

In the limit of flat beams (FB), $\sigma_x^* \gg \sigma_y(x, s)$, we drop terms of order σ_y/σ_x^* , and can then simplify the function $\mathcal{F}$ of Eq. (8) as follows:

$$\begin{aligned}
\mathcal{F}_{\text{FB}}(\bar{x}, \bar{y}, \bar{s}) &= \sqrt{\frac{\pi}{2}} \frac{1}{\sigma_x^*} \left(w \left[\frac{1}{\sqrt{2}} \bar{x} \right] - \exp \left[-\frac{1}{2} \left(\bar{x}^2 + \frac{\bar{y}^2}{G_y^2(\bar{x}, \bar{s}; \Phi)} \right) \right] w \left[\frac{i}{\sqrt{2}} \frac{\bar{y}}{G_y(\bar{x}, \bar{s}; \Phi)} \right] \right) \\
&= \sqrt{\frac{\pi}{2}} \frac{\exp \left[-\frac{\bar{x}^2}{2} \right]}{\sigma_x^*} \left[\operatorname{erf} \left(\frac{\bar{y}}{\sqrt{2} G_y(\bar{x}, \bar{s}; \Phi)} \right) - \operatorname{erf} \left(-\frac{i \bar{x}}{\sqrt{2}} \right) \right] \\
&= \frac{\exp \left[-\frac{\bar{x}^2}{2} \right]}{\sigma_x^*} \int_{-i \bar{x}}^{\frac{\bar{y}}{G_y(\bar{x}, \bar{s}; \Phi)}} e^{-\frac{1}{2} \tau^2} d\tau
\end{aligned} \tag{11}$$

where we have used the definition $w(\tau) = e^{-\tau^2} \operatorname{erfc}(-i\tau)$, the relation $\operatorname{erfc}(\tau) \equiv (1 - \operatorname{erf}(\tau))$, and the identity

$$\left(\frac{c}{\pi} \right)^{\frac{1}{2}} \int_p^q e^{-cx^2} dx = \frac{1}{2} [\operatorname{erf}(q\sqrt{c}) - \operatorname{erf}(p\sqrt{c})] . \tag{12}$$

We can make a further, more daring approximation. Namely, assuming $|\bar{x}| \lesssim 1$ and $|\bar{y}/G_y| \lesssim 1$, which we justify by the Gaussian weighting of $\bar{x}$ and $\bar{y}$ in Eq. (10), we expand the integrand as $\exp(-\tau^2/2) \approx (1 - \tau^2/2 + \dots) \approx 1$, to obtain

$$\frac{\exp \left[-\frac{\bar{x}^2}{2} \right]}{\sigma_x^*} \int_{-i \bar{x}}^{\frac{\bar{y}}{G_y(\bar{x}, \bar{s}; \Phi)}} e^{-\frac{1}{2} \tau^2} d\tau \approx \left| \left(\frac{\bar{y}}{G_y(\bar{x}, \bar{s}; \Phi)} + i \bar{x} \right) \right| \frac{\exp \left[-\frac{\bar{x}^2}{2} \right]}{\sigma_x^*} . \tag{13}$$

We will later compare all results obtained using this and the following approximation with those obtained by numerically computing the original complete expression (8), finding differences at the few percent level.

In addition, for crab-waist collisions with a large Piwinski angle Φ , we may also neglect the hourglass effect, by approximating $G_y(\bar{x}, \bar{s}; \Phi) \approx 1$. This leads to our final, most radical approximation for the function $\mathcal{F}$:

$$|\mathcal{F}_{\text{approx}}(\bar{x}, \bar{y}, \bar{s}; \Phi)| \approx |\bar{y} + i \bar{x}| \frac{\exp \left[-\frac{\bar{x}^2}{2} \right]}{\sigma_x^*} \quad \text{and} \quad G_y(\bar{x}, \bar{s}; \Phi) \approx 1 . \tag{14}$$

In the following, we compute various integrals involving the function $\mathcal{F}$ either using its fully general form, Eq. (8), or using the approximation $\mathcal{F}_{\text{approx}}$ of (14), that is much faster to evaluate, and compare the respective results.

5 Number of Beamstrahlung photons n_γ for head-on collision

Using the formulae we have derived so far, we can calculate the integral $\int ds \langle 1/\rho \rangle_{x,y,z}$, that enters in the average Υ parameter and in the average number of beamstrahlung photons emitted per collision, N_γ . To benchmark our formalism and computation techniques, we consider a special case often studied in the literature, namely the collision of flat beams with zero crossing angle, which implies $\Phi = 0$. The integral of interest becomes

$$\begin{aligned}
\int ds \left\langle \frac{1}{\rho} \right\rangle_{x,y,z} &= \frac{\sigma_z}{(\sqrt{2\pi})^3} \frac{Q\sigma_x^*}{\sigma_z} \int d\bar{x} d\bar{y} d\bar{s} d\bar{z} |\mathcal{F}(\bar{x}, \bar{y}; 0)| \\
&\quad \exp \left[-\frac{1}{2} \{ (2\bar{s} - \bar{z})^2 + \bar{x}^2 + \bar{y}^2 + \bar{z}^2 \} \right] ,
\end{aligned} \tag{15}$$

where we have introduced the abbreviation

$$Q \equiv \left(\frac{2N_b r_e}{\gamma \sigma_x^*} \sqrt{\frac{2}{\pi}} \right). \quad (16)$$

If we further neglect the hourglass effect and use the approximation (14), we obtain

$$\int ds \left\langle \frac{1}{\rho} \right\rangle_{x,y,z} \approx \frac{1}{(\sqrt{2\pi})^3} Q \int d\bar{x} d\bar{y} d\bar{z} \sqrt{\bar{x}^2 + \bar{y}^2} \exp \left[-\frac{1}{2} \bar{x}^2 \right] \exp \left[-\frac{1}{2} \{ (2\bar{z} - \bar{z})^2 + \bar{x}^2 + \bar{y}^2 + \bar{z}^2 \} \right]. \quad (17)$$

As shown in Appendix A, the four integrals can be solved analytically, with the result

$$\int ds \left\langle \frac{1}{\rho} \right\rangle_{x,y,z} = \frac{Q}{\sqrt{2}} E \left(\frac{1}{2} \right) \approx Q \times 0.955, \quad (18)$$

where $E(m)$ is the complete elliptic integral of the second kind. Equation (18) allows expressing the average number of photons emitted per beam particle in a head-on collision of flat beams with negligible hourglass effect as

$$\begin{aligned} n_\gamma &= \frac{5}{2\sqrt{3}} \alpha \gamma \int ds \left\langle \frac{1}{\rho} \right\rangle_{x,y,z} \approx \alpha \gamma \frac{5}{2\sqrt{3}} \left(\frac{2N_b r_e}{\gamma \sigma_x^*} \sqrt{\frac{2}{\pi}} \right) \times 0.955 \\ &\approx \alpha \gamma \frac{N_b r_e}{\gamma \sigma_x^*} \frac{5\sqrt{2}}{\sqrt{3}\pi} \times 0.955 \sim 2.199 \frac{\alpha N_b r_e}{\sigma_x^*}. \end{aligned} \quad (19)$$

If we do not use the approximation (13), but compute the integral (15) numerically using the full function $\mathcal{F}_{\text{FB}}$ for a flat Gaussian beam, (11), we obtain a coefficient of 2.145. We can compare our approximate coefficient 2.199 or the more precise 2.145 with a value of 2.12 quoted in the literature, e.g. Ref. [32], for similar conditions (that is, flat beams $\sigma_x^* \gg \sigma_y^*$, either no crossing angle or crab crossing, and negligible hourglass). The numbers 2.199 or 2.145 and 2.12 of Ref. [32] agree to within 4% or 1%, respectively.

Concerning the value of 2.12, we remark that for the case of round Gaussian beams ($\sigma_x^* = \sigma_y^*$), earlier a factor 1.06 was derived in Ref. [33]. The factor 2.12 for flat Gaussian beams given in Ref. [32] may have been obtained by multiplying the factor 1.06 for a round Gaussian beam with the aspect ratio form factor $2/(1 + \sigma_y^*/\sigma_x^*)$, that was derived for a homogeneous “elliptical cylinder” [33]. It is interesting to note that the rough estimates for the effective interaction length and bending radius of Ref. [13] would yield a coefficient $5/(2\sqrt{3}) \approx 1.44$, while those of Ref. [14] (considering the initial expression for $1/\rho_x$ therein) correspond to $\sim 5\sqrt{\pi}\sqrt{1.64}/(2\sqrt{6}) \approx 2.32$.

If, following [12, 34], we consider an effective bunch length of $2\sqrt{3}\sigma_z$, the corresponding mean Upsilon parameter $\Upsilon = (r_e/\alpha)\gamma^2\langle 1/\rho \rangle$ can be inferred from (19) as $\langle \Upsilon \rangle \approx (r_e/\alpha)\gamma^2(2\sqrt{3}/5)n_\gamma/(\sqrt{3}\sigma_z\alpha\gamma) \approx 2.199(2/5)N_b r_e^2\gamma/(\alpha\sigma_x^*\sigma_z)$. The coefficient $2.199 \times 2/5 \approx 0.8796$ is about 5.5% higher than the value $5/6 \approx 0.833$ quoted in Ref. [34]; the numerically calculated, more exact coefficient $2.145 \times 2/5$ is only about 1% larger.

6 Approximate flat-beam solution for $\int ds \langle 1/\rho^3 \rangle$ including crossing angle

For the quantum excitation, the 4-dimensional integral of interest is

$$\int ds \left\langle \frac{1}{\rho^3} \right\rangle_{x,y,z} = \frac{\sigma_z}{(\sqrt{2\pi})^3} \int d\bar{x} d\bar{y} d\bar{s} d\bar{z} \frac{1}{\rho^3} \exp \left[-\frac{1}{2} \{ (\bar{x} + \Phi \bar{z})^2 + \bar{y}^2 + \bar{z}^2 \} \right]. \quad (20)$$

Using our approximation (14) in the expression for $1/\rho$, Eq. (7), and also introducing the new variable $u \equiv (\bar{x} - 2\Phi \bar{s} + \Phi \bar{z})$ and the abbreviation

$$Q_3 \equiv \frac{1}{\sqrt{2\pi}^3} \frac{1}{\sigma_z^2} \left(\frac{2N_{bre}}{\gamma \sigma_x^*} \sqrt{\frac{2}{\pi}} \right)^3, \quad (21)$$

this can be simplified as

$$\begin{aligned} \int ds \left\langle \frac{1}{\rho^3} \right\rangle_{x,y,z} &\approx \frac{1}{(\sqrt{2\pi})^3} Q_3 \int d\bar{u} d\bar{y} d\bar{s} d\bar{z} \left[\sqrt{u^2 + \bar{y}^2} \right]^3 \\ &\exp \left[-\frac{1}{2} \{ 3u^2 + 3(2\bar{s} - \bar{z})^2 \} \right] \exp \left[-\frac{1}{2} \{ (u + 2\Phi \bar{s})^2 + \bar{y}^2 + \bar{z}^2 \} \right]. \end{aligned} \quad (22)$$

As shown in Appendix B, also these four integrals can be computed analytically, yielding

$$\int ds \left\langle \frac{1}{\rho^3} \right\rangle_{x,y,z} = Q_3 \left[\frac{(10 - 2\varphi)E\left(\frac{3-\varphi}{4-\varphi}\right) - K\left(\frac{3-\varphi}{4-\varphi}\right)}{\sqrt{(3 + 4\Phi^2)(4 - \varphi)^3}} \right], \quad (23)$$

where, $K(m)$ and $E(m)$ denote the complete elliptical integrals of the first and second kind, respectively.

Two limiting cases are of interest. For head-on collisions without a crossing angle, $\Phi = 0$, the solution (23) reduces to

$$\int ds \left\langle \frac{1}{\rho^3} \right\rangle_{x,y,z} \stackrel{\Phi \rightarrow 0}{\approx} Q_3 \left[\frac{10E(3/4) - K(3/4)}{\sqrt{192}} \right] \approx 0.7183 \times Q_3. \quad (24)$$

In the opposite limit, collisions with a large Piwinski angle, $\Phi \gg 1$ or $\varphi \rightarrow 1$, the solution (23) simplifies as

$$\int ds \left\langle \frac{1}{\rho^3} \right\rangle_{x,y,z} \stackrel{\Phi \gg 1}{\approx} Q_3 \left[\frac{8E(2/3) - K(2/3)}{2\sqrt{27}\Phi} \right] \approx Q_3 \times \frac{0.77562}{\Phi}. \quad (25)$$

7 Beamstrahlung quantum excitation and new equilibrium

According to Eq. (3), the quantum excitation [1] from beamstrahlung is proportional to the average of the inverse third power of the local bending radius.

Balancing the sum of the excitation due to beamstrahlung and due to arc synchrotron radiation against the radiation damping from the arcs alone (the average energy loss and, hence, the damping effect due to beamstrahlung is negligible [17]) yields the total equilibrium emittance $\varepsilon_{x,\text{tot}}$

$$\varepsilon_{x,\text{tot}} = \varepsilon_{x,\text{SR}} + \frac{n_{\text{IP}} \tau_{x,\text{SR}}}{4T_{\text{rev}}} \{ n_\gamma < u^2 > \}_{\text{BS}} \mathcal{H}_x^*, \quad (26)$$

and relative rms momentum spread δ_{tot} as

$$\sigma_{\delta, \text{tot}}^2 = \sigma_{\delta, \text{SR}}^2 + \frac{n_{\text{IP}} \tau_{E, \text{SR}}}{4T_{\text{rev}}} \{n_{\gamma} < u^2 >\}_{\text{BS}}, \quad (27)$$

where τ_x and τ_E denote the usual horizontal and longitudinal radiation damping times [1], respectively, T_{rev} the revolution period, and n_{IP} the number of interaction points. The terms with subindex SR refer to the standard equilibrium parameters without beamstrahlung. The dispersion invariant H_x^* is defined as [1]

$$H_x^* \equiv (\beta_x^* D_x'^* + \alpha_x^* D_x^*)^2 / \beta_x^* + D_x^{*2} / \beta_x^*, \quad (28)$$

where β_x^* , α_x^* , D_x^* and $D_x'^*$ denote optical beta and alpha function (Twiss parameters), the dispersion and slope of the dispersion at the collision point, respectively. In (26) and (27) we have ignored possible correlations between a particle's transverse or longitudinal amplitudes and the field it experiences in successive collisions during τ_{SR} .

The beamstrahlung parameters (Υ , δ_B , $< u >$) strongly depend on the bunch length, as does the local bending radius ρ . The “total” (equilibrium) bunch length is related to the total energy spread via the classical relation [1]

$$\sigma_z = \frac{\alpha_p C}{2\pi Q_s} \sigma_{\delta}, \quad (29)$$

where Q_s denotes the synchrotron tune, C the circumference, and α_p the momentum compaction factor.

If the IP dispersion is zero, beamstrahlung excites the beam particles only longitudinally, and the total energy spread follows from a self-consistency condition [16, 17]

$$\sigma_{\delta}^2 = \sigma_{\delta, \text{SR}}^2 + \frac{V}{\sigma_{\delta}^2 \beta_x^{*3/2} \epsilon_x^{3/2}}, \quad (30)$$

where we have introduced the coefficient

$$V \equiv \frac{n_{\text{IP}} \tau_{E, \text{SR}}}{4T_{\text{rev}}} \{n_{\gamma} < u^2 >\}_{\text{BS}} \sigma_{\delta}^2 \sigma_x^{*3}, \quad (31)$$

which has the dimension of a volume.

Inserting the expression $\{n_{\gamma} \langle u^2 \rangle\}_{\text{BS}}$ from Eq. (3), we obtain

$$V(\Phi) = \frac{55}{24\sqrt{3}} \frac{r_e^2 \gamma^5}{\alpha} \sigma_{\delta}^2 \sigma_x^{*3} \frac{n_{\text{IP}} \tau_{E, \text{SR}}}{4T_{\text{rev}}} \int ds \left\langle \frac{1}{\rho^3} \right\rangle_{x, y, z}, \quad (32)$$

and, substituting $\int ds \langle 1/\rho^3 \rangle_{x, y, z}$ from Eq. (10), and σ_z from (29), this becomes

$$\begin{aligned} V = & \frac{55}{24\sqrt{3}} \frac{1}{(2\pi)^{3/2}} \frac{r_e^2 \gamma^5}{\alpha} \left(\frac{2\pi Q_s}{\alpha_p C} \right)^2 \left(\frac{2N_b r_e}{\gamma} \right)^3 \frac{n_{\text{IP}} \tau_{E, \text{SR}}}{4T_{\text{rev}}} \sigma_x^{*3} \\ & \int_{\bar{s}, \bar{x}, \bar{y}, \bar{z}} \frac{d\bar{s} d\bar{x} d\bar{y} d\bar{z}}{G_y(\bar{s}, -\bar{x}; \Phi)} |\mathcal{F}(\bar{x} - 2\Phi\bar{s} + \Phi\bar{z}, \bar{y}, \bar{s}; \Phi)|^3 \\ & \exp\left(-3\frac{(2\bar{s} - \bar{z})^2}{2}\right) \exp\left(-\frac{(\bar{x} + \bar{z}\Phi)^2}{2} - \frac{\bar{y}^2}{2G_y^2(\bar{s}, -\bar{x}; \Phi)} - \frac{\bar{z}^2}{2}\right), \end{aligned} \quad (33)$$

which can be computed numerically, using the full expression (8) for $\mathcal{F}$.

If instead we substitute $\int ds \langle 1/\rho \rangle_{x,y,z}$ our approximate expression from Eq. (23), valid for flat beams and negligible hourglass effect, then Eq. (33) simplifies to

$$V(\Phi)_{\text{approx}} \approx \frac{55\pi^2}{3\sqrt{3}} \left[\sqrt{\frac{2}{\pi}} \right]^3 \frac{n_{\text{IP}} \tau_{\text{E,SR}} Q_s^2 r_e^5 \gamma^2 N_b^3}{a_c^2 C^2 \alpha} \left[\frac{(10 - 2\varphi) E \left(2\pi \mid \frac{3-\varphi}{4-\varphi} \right) - F \left(2\pi \mid \frac{3-\varphi}{4-\varphi} \right)}{4\sqrt{(3+4\Phi^2)(4-\varphi)^3}} \right], \quad (34)$$

which explicitly demonstrates that, $V(\Phi)$, or $V(\Phi)_{\text{approx}}$, was defined so as to be independent of σ_x^* , σ_z and σ_δ . However, this coefficient V strongly depends on Φ . In the limit $\Phi \gg 1$, we can further simplify the expression for V by evoking the additional approximation (25), which yields

$$V(\Phi)_{\text{approx}, \Phi \gg 1} \approx \frac{55\pi^2}{3\sqrt{3}} \left[\sqrt{\frac{2}{\pi}} \right]^3 \frac{n_{\text{IP}} \tau_{\text{E,SR}} Q_s^2 r_e^5 \gamma^2 N_b^3}{a_c^2 C^2 \alpha} \left[\frac{8E \left(2\pi \mid \frac{2}{3} \right) - F \left(2\pi \mid \frac{2}{3} \right)}{8\sqrt{27}\Phi} \right]. \quad (35)$$

8 Equilibrium beam parameters for FCC and CEPC

Table 1 presents example parameters from the FCC-ee and CEPC designs. For each of the operation modes shown, we can use the above formulae to derive the equilibrium parameters σ_z and σ_δ including the effect of beamstrahlung.

The solution is found by iteration: Starting from $\sigma_\delta^{(0)} = \sigma_{\delta,\text{SR}}$ and $\sigma_z^{(1)} = \sigma_{z,\text{SR}}$, we first compute $\Phi^{(1)}$, then $V^{(1)}$ — from one of the three equations (33), (34) or (35) —, and, finally, obtain the new estimates $\sigma_\delta^{(2)}$ and $\sigma_z^{(2)}$ from (30) and (29). Typically, the result converges after the third iteration, meaning that for $n > 3$ the values of $\sigma_\delta^{(n)}$ and $\sigma_z^{(n)}$ differ from $\sigma_\delta^{(3)}$ and $\sigma_z^{(3)}$ by less than 1%, as is illustrated in Figs. 3, 4 and 5. For the ZH running mode, the analytical solution and its asymptotic limit both agree nearly perfectly with the result of the multi-particle multi-turn simulation, while the results of the full numerical solution is a few percent higher. For the WW and Z running modes, where the relative blow up is stronger, the prediction from the multi-particle multi-turn tracking simulation is bracketed between our iterative full numerical integration, using (33) with (8), and our iterative analytical solutions based on (34) and (35). The small differences are on the order of 5%. In all cases, the asymptotic solution for $\Phi \gg 1$, (35) stays close to the analytical solution (34). Equilibrium beam parameters obtained from this iteration process, for the complete analytical solution (33), are highlighted through bold font in Table 1.

The strong impact of beamstrahlung is evident when comparing the rms bunch length and momentum spread due to standard arc synchrotron radiation, $\sigma_{z,\text{SR}}$ and $\sigma_{\delta,\text{SR}}$, and the corresponding values in collision, $\sigma_{z,\text{BS}}$ and $\sigma_{\delta,\text{BS}}$. Beamstrahlung increases the bunch length and momentum spread by a factor ranging from about 1.5 to 4, depending on the beam energy. In particular, for the initial FCC-ee low-energy run on the Z pole, the beamstrahlung greatly blows up the bunch length and energy spread. The strong effect of beamstrahlung in this mode of operation is explained by the weak radiation damping at the low energy, where each electron, or positron, emits multiple beamstrahlung photons during one radiation damping time. At higher beam energies the radiation damping gains importance and, as a result, the energy-spread increase due to beamstrahlung becomes less pronounced. By contrast, the effect of beamstrahlung on the beam lifetime considered in Ref. [13] (particle loss due to single photon emission) is significant only at the highest beam energy, namely at the ZH production peak (120 GeV per beam) and above.

Table 1 Example beam parameters for CEPC Higgs production [3] and three operation modes of FCC-ee [2], illustrating the effect of beamstrahlung on the rms relative momentum spread, $\sigma_{\delta,BS}$, and on the rms bunch length, $\sigma_{z,BS}$, according to Eqs. (29) and (30), for $n_{IP} = 2$ identical IPs. The analytically computed values can be compared with the result of beam-beam tracking simulations, namely the values $\sigma_{\delta,SIM}$ and $\sigma_{z,SIM}$ shown underneath. Displayed in bold are parameters computed by the iterative full numerical integration of the complete solution (33), using the expression (8) for $\mathcal{F}$. The last two rows indicate the tolerances on spurious IP dispersion for a transverse emittance growth of less than 10%, based on Eq. (38). The FCC-ee simulation values (subindex “SIM”) are from D. Shatilov [2], obtained with the beam-beam code LIFETRAC, while the CEPC simulation values [19] were provided by the code IBB

Machine Mode	CEPC ZH	FCC Z	FCC WW	FCC ZH
beam energy E_b [GeV]	120	45.6	80	120
circumference C [km]	100.0	97.76	97.76	97.76
crossing angle θ_c [mrad]	33	30	30	30
bunches per beam n_b	242	16640	2000	328
bunch population N_b [10^{10}]	15	17	15	18
horizontal emittance ε_x [nm]	1.21	0.27	0.84	0.63
vertical emittance ε_y [pm]	2.40	1.00	1.70	1.30
momentum compaction α_p [10^{-6}]	11.10	14.80	14.80	7.30
horizontal IP beta function β_x^* [m]	0.36	0.15	0.20	0.30
vertical IP beta function β_y^* [mm]	1.5	0.8	1.0	1.0
bunch length $\sigma_{z,SR}$ [mm]	2.72	3.50	3.00	3.14
rms bunch length $\sigma_{z,BS}$ [mm]	3.94	12.45	6.07	5.39
rms bunch length $\sigma_{z,SIM}$ [mm]	4.0	12.1	6.0	5.3
rms relative momentum spread $\sigma_{\delta,SR}$ [%]	0.10	0.038	0.066	0.099
rms relative momentum spread $\sigma_{\delta,BS}$ [%]	0.144	0.137	0.135	0.172
rms relative momentum spread $\sigma_{\delta,SIM}$ [%]	0.146	0.132	0.131	0.165
Piwiński angle Φ_{BS}	3.11	29.35	7.02	5.88
energy loss / turn U_0 [GeV]	1.73	0.036	0.34	1.72
revolution frequency f_{rev} [Hz]	3003	3000	3000	3000
RF frequency f_{RF} [MHz]	650	400	400	400
RF voltage V_{rf} [GV]	2.17	0.10	0.75	2.0
synchrotron tune Q_s	0.065	0.025	0.051	0.036
longitudinal damping time τ_E [ms]	23.4	418.3	77.5	23.0
revolution period T_{rev} [ms]	0.33	0.33	0.33	0.33
number of IPs n_{IP}	2	2	2	2
L_{CDR} [$10^{35} \text{ cm}^{-2} \text{ s}^{-1}$]	0.30	23.00	2.80	0.85
Υ_{max} [10^{-4}]	13.4	14.9	13.2	21.5
Υ_{ave} [10^{-4}]	5.6	6.2	5.5	9.0
undulator parameter K_{max}	6.5	2.6	4.6	6.3
$ H_x^* _{max}$ [μm] ($\Delta\varepsilon_x < 0.1\varepsilon_x$)	56	8	30	16
$ H_y^* _{max}$ [nm] ($\Delta\varepsilon_y < 0.1\varepsilon_y$)	112	58	61	33

Fig. 3 Relative rms momentum spread increase versus the number of iterations for the FCC-ee ZH production mode at 240 GeV centre-of-mass energy; shown are results for the full analytical solution, based on (33) and (8) [green markers], for the analytical solution (34) [blue markers] and for asymptotic analytical approximation (35) in the case of large Piwinski angle [red markers]. For comparison, the black line is the rms momentum spread increase from a many-turn (quasi-)strong-strong beam-beam simulation with the code LIFETRAC, as obtained by D. Shatilov for the FCC-ee Conceptual Design Report [2]

Fig. 4 Relative rms momentum spread increase versus the number of iterations for the FCC-ee WW pair production mode at 160 GeV centre-of-mass energy; shown are results for the full analytical solution, based on (33) and (8) [green markers], for the analytical solution (34) [blue markers] and for asymptotic analytical approximation (35) in the case of large Piwinski angle [red markers]. For comparison, the black line the rms momentum spread increase obtained from a many-turn (quasi-)strong-strong beam-beam simulation with the code LIFETRAC, as obtained by D. Shatilov for the FCC-ee Conceptual Design Report [2]

The comparison between our full analytical result and the simulated values from the IBB code reveals an excellent agreement, at the per cent level, for the case of CEPC (ZH running). However, for the three FCC-ee scenarios, we observe a slightly larger discrepancy, namely numbers for the relative momentum spread σ_δ and bunch length σ_s that are 3–4% higher than those simulated by LIFETRAC. Finally, the difference between the momentum spread, or

Fig. 5 Relative rms momentum spread increase versus the number of iterations for the FCC-ee Z pole running mode at 91 GeV centre-of-mass energy; shown are results for the full analytical solution, based on (33) and (8) [green markers], for the analytical solution (34) [blue markers] and for asymptotic analytical approximation (35) in the case of large Piwinski angle [red markers]. For comparison, the black line the rms momentum spread increase obtained from a many-turn (quasi-)strong-strong beam-beam simulation with the code LIFETRAC, as obtained by D. Shatilov for the FCC-ee Conceptual Design Report [2]

bunch length, for FCC-ee, obtained from our full solution and the approximation of [19] is about 1.4% on the Z pole, and $3.7\% \pm 0.1\%$ at the WW and ZH energies (where the results of [19] are close to the LIFETRAC values), respectively, while for the CEPC ZH (where our value matches the IBB simulation) it is 5.1%.

9 Spurious dispersion at the interaction point

In the presence of nonzero IP dispersion, the transverse emittance also increases due to the beamstrahlung. To determine the size of this effect, we consider a spurious nonzero horizontal dispersion at the interaction point (IP), limited in magnitude to $|D_x^*|\sigma_{\delta, tot} \ll \sqrt{\beta_x^* \varepsilon_{x, SR}}$, so that the additional dispersive beam size component remains small compared with the value expected from the IP beta function and ε , the horizontal emittance defined by the arc synchrotron radiation without beamstrahlung. In presence of nonzero horizontal dispersion at the IP, the transverse horizontal emittance $\varepsilon_{x, tot} = (\varepsilon_{x, SR} + \Delta\varepsilon_x)$ is determined by the additional equation

$$\varepsilon_{x, tot} \approx \varepsilon_{x, SR} + \frac{2V H_x^*}{\sigma_{\delta, tot}^2 \beta_x^{*3/2} \varepsilon_{x, tot}^{3/2}}, \quad (36)$$

which needs to be solved self-consistently together with (30). An equivalent equation, for the case of spurious vertical dispersion, applies to the vertical emittance:

$$\varepsilon_{y, tot} \approx \varepsilon_{y, SR} + \frac{2V H_y^*}{\sigma_{\delta, tot}^2 \beta_x^{*3/2} \varepsilon_{x, tot}^{3/2}}, \quad (37)$$

where the last term on the right-hand side represents the additional vertical emittance blow up $\Delta\varepsilon_y$.

We typically require that the spurious dispersion at the IP should be sufficiently well corrected not to result in a significant emittance blow up $\Delta\epsilon_{x(y)}/\epsilon_{x(y)}$. From Eqs. (36) and (37) we can then derive the maximum acceptable values for the IP dispersion, namely

$$|H_{x,y}^*|_{\max} = \frac{\sigma_{\delta,\text{tot}}^2 \beta_x^{*3/2} \epsilon_{x,\text{SR}}^{3/2} \epsilon_{x,y,\text{SR}}}{2V} \left[\frac{\Delta\epsilon_{x,y}}{\epsilon_{x,y}} \right]_{\max}. \quad (38)$$

The resulting tolerances on the two dispersion invariants, for a maximum blow up $\Delta\epsilon_{x(y)}/\epsilon_{x(y)}$ of 10%, are shown in the last two rows of Table 1. The tolerances $|H_{x,y}^*|_{\max}$ impose tolerances on the IP dispersion of

$$|D_{x,y}^*|_{\max,\text{BS}} < \sqrt{\beta_{x,y}^* |H_{x,y}^*|_{\max}}. \quad (39)$$

Taking the parameters of the FCC-ee Z running mode in Table 1 as an example, the residual IP dispersion should be limited to $|D_x^*|_{\max,\text{BS}} < 1$ mm and $|D_y^*|_{\max,\text{BS}} < 5$ μm .

We can compare these values with two other traditional tolerances on the residual IP dispersion. Firstly, a classical tolerance can be derived from the requirement that the geometric IP beam size does not increase by more than 5% due to the residual IP dispersion and synchrotron-radiation energy spread, namely

$$|D_{x,y}^*|_{\max,\text{classical}} < \frac{\sigma_{x,y}^*}{\sqrt{10} \sigma_{\delta,\text{SR}}}, \quad (40)$$

which amounts to 5 mm and 24 μm , respectively, for the FCC-ee Z case, and is about 5 times looser than the tolerance in presence of beamstrahlung.

We note that if on the right-hand side we considered the rms energy spread due to beamstrahlung, $\sigma_{\delta,\text{BS}}$, instead of $\sigma_{\delta,\text{SR}}$, the vertical IP dispersion tolerance due to the resulting geometric beam size increase would be about 7 μm , or only $\sim 50\%$ larger than the tolerance from the emittance blow up. Taking into account both of these effects plus a possible additional contribution to the emittance blow up from residual IP dispersion slope, D_y^* , the net vertical IP dispersion tolerance is about 3 μm . A similar tolerance on the residual IP dispersion, of about 1 μm , was inferred from a quasi weak-strong beam-beam simulation [35].

Secondly, we may also require that the excitation of synchrotron (SB) resonances due to the residual IP dispersion be small compared with the synchrotron coupling induced by the horizontal crossing angle, e.g. [36], or

$$|D_{x,y}^*|_{\max,\text{SB}} < \frac{\sigma_{z,\text{SR}} \theta_c}{2 \sigma_{\delta,\text{SR}}}. \quad (41)$$

For the FCC-ee Z example, the right-hand side evaluates to 140 mm, which is easily achieved.

10 Conclusions

In this article, we have derived analytical expressions characterizing the multi-turn impact of beamstrahlung on the beam parameters of future circular high-energy electron-positron colliders, including hourglass effect, crossing angle, crab-waist collision scheme and interaction-point dispersion. Confirming earlier rough estimates and strong-strong beam-beam particle-tracking simulation results, our analytical approach predicts that beamstrahlung will greatly affect the equilibrium beam distribution of future circular Higgs (or electroweak) factories, in particular the momentum spread and bunch length. Specifically, the beamstrahlung increases

the FCC-ee rms bunch length compared with the natural value expected from standard synchrotron radiation alone, by a factor 3.4 on the Z pole, a factor 1.9 at the WW threshold, and a factor 1.6 at the Higgs (ZH) production peak, and the CEPC rms bunch length by about 40% at the ZH energy. Therefore, this effect must be taken into account for the designs of FCC-ee, or CEPC, in addition to the constraints reported in [13, 15]. Beamstrahlung also introduces new tolerances on the IP optics parameters, in particular on both the residual dispersion and slope of dispersion at the IP, which are significantly tighter than the traditional tolerances imposed by the geometric beam size increase without beamstrahlung or by the resulting synchrotron coupling. As an example, due to the effect of beamstrahlung the residual vertical IP dispersion for Z pole operation should not exceed a few micron.

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A Computing $\int ds \langle 1/\rho \rangle$ for the head-on collision of flat beams

In Section 5, after neglecting the hourglass effect and using the approximation (14), we had obtained Eq. (17):

$$\int ds \left\langle \frac{1}{\rho} \right\rangle_{x,y,z} \approx \frac{1}{(\sqrt{2\pi})^3} Q \int d\bar{x} d\bar{y} d\bar{z} \sqrt{\bar{x}^2 + \bar{y}^2} \exp \left[-\frac{1}{2} \bar{x}^2 \right] \exp \left[-\frac{1}{2} \{ (2\bar{s} - \bar{z})^2 + \bar{x}^2 + \bar{y}^2 + \bar{z}^2 \} \right]. \quad (42)$$

To perform the integration the coordinates x and y are transformed to polar coordinates r and φ , as $\bar{x} = r \cos \varphi$, and $\bar{y} = r \sin \varphi$, which leads to

$$\int ds \left\langle \frac{1}{\rho} \right\rangle_{x,y,z} \approx \frac{Q}{(\sqrt{2\pi})^3} \int dr d\varphi r^2 \exp \left[-\frac{1}{2} \{ r^2 (1 + \cos^2 \varphi) \} \right] \int d\bar{s} d\bar{z} r^2 \exp \left[-\frac{1}{2} \{ 4\bar{s}^2 - 4\bar{s}\bar{z} + 2\bar{z}^2 \} \right], \quad (43)$$

where the integrals over r and φ can be solved as

$$\int dr d\varphi r^2 \exp \left[-\frac{1}{2} \{ r^2 (1 + \cos^2 \varphi) \} \right] = \left[\frac{\sqrt{2\pi}}{2} \right] \left[\int d\varphi \frac{1}{(1 + \cos^2 \varphi)^{3/2}} \right] = 2\sqrt{\pi} E \left(\frac{1}{2} \right), \quad (44)$$

where $E(m)$ is the complete elliptic integral of the second kind and the identity

$$\int_0^\infty d\tau \tau^2 e^{-a\tau^2} = \frac{1}{4} \sqrt{\frac{\pi}{a^3}} \quad (45)$$

has been used.

The integrals over $\bar{s}$ and $\bar{z}$ can be computed as

$$\int d\bar{s} d\bar{z} \exp \left[-\frac{1}{2} \{4s^2 - 4sz + 2z^2\} \right] = \left[\frac{\sqrt{2\pi}}{2} \right] \int d\bar{z} \exp \left[-\frac{1}{2} \{2\bar{z}^2\} \right] \exp \left[\frac{1}{2} \{\bar{z}^2\} \right] = \pi \quad (46)$$

with the help of the identity

$$\int_{-\infty}^{\infty} e^{-(a\tau^2 + b\tau)} d\tau = \sqrt{\frac{\pi}{a}} \exp \left[\frac{b^2}{4a} \right]. \quad (47)$$

Putting everything together, and inserting (46) and (44) into (43), we obtain

$$\int ds \left\langle \frac{1}{\rho} \right\rangle_{x,y,z} = \frac{Q}{\sqrt{2}} E \left(\frac{1}{2} \right) \approx Q \times 0.955 \quad (48)$$

which is Eq. (18).

B Computing $\int ds \langle 1/\rho^3 \rangle$ for the collisions of flat beams with a crossing angle

In Section 6, analysing the quantum excitation for the collision of flat beams with a crossing angle, we found the 4-dimensional integral of Eq. (49),

$$\int ds \left\langle \frac{1}{\rho^3} \right\rangle_{x,y,z} \approx \frac{1}{(\sqrt{2\pi})^3} Q_3 \int d\bar{u} d\bar{y} d\bar{s} d\bar{z} \left[\sqrt{u^2 + \bar{y}^2} \right]^3 \exp \left[-\frac{1}{2} \{3u^2 + 3(2\bar{s} - \bar{z})^2\} \right] \exp \left[-\frac{1}{2} \{(u + 2\Phi\bar{s})^2 + \bar{y}^2 + \bar{z}^2\} \right], \quad (49)$$

with Q_3 defined in Eq. (21).

The argument of the exponential can be rearranged as

$$\begin{aligned} 3u^2 + 3(2\bar{s} - \bar{z})^2 + (u + 2\Phi\bar{s})^2 + \bar{y}^2 + \bar{z}^2 &= 4u^2 + \bar{y}^2 + 12\bar{s}^2 + 4\Phi\bar{s}u + 4\bar{z}^2 - 12\bar{s}\bar{z} + 4\Phi^2\bar{s}^2 \\ &= 4u^2 + \bar{y}^2 + 4(3 + \Phi^2)\bar{s}^2 + 4\Phi\bar{s}u + \underbrace{4\bar{z}^2 - 12\bar{s}\bar{z}}. \end{aligned}$$

Then integrating over $\bar{z}$

$$\int_{-\infty}^{+\infty} d\bar{z} \exp \left[-\frac{1}{2} \{4\bar{z}^2 - [12\bar{s}]\bar{z}\} \right] = \left[\frac{1}{2} \sqrt{2\pi} \right] \left[\exp \left[\frac{1}{2} 9\bar{s}^2 \right] \right]$$

adds to the arguments as

$$4u^2 + \bar{y}^2 + \underbrace{[12 - 9 + \Phi^2]\bar{s}^2 + [4\Phi u]\bar{s}},$$

and integrating over $\bar{s}$

$$\int_{-\infty}^{\infty} d\bar{s} \exp \left[-\left\{ \left[\frac{3 + 4\Phi^2}{2} \right] \bar{s}^2 + [2\Phi u]\bar{s} \right\} \right] = \left[\frac{\sqrt{2\pi}}{\sqrt{3 + 4\Phi^2}} \right] \exp \left[\frac{1}{2} \frac{4\Phi^2 u^2}{(3 + 4\Phi^2)} \right] \quad (50)$$

adds to the arguments as

$$= \left[4 - \frac{4\Phi^2}{3 + 4\Phi^2} \right] u^2 + \bar{y}^2 = u^2 + \bar{y}^2 + [3 - \varphi]u^2, = r^2(1 + [3 - \varphi] \cos^2 \theta),$$

where we have introduced $\varphi \equiv (4\Phi^2)/(3 + 4\Phi^2)$.

Now, the integration over r and θ can be carried out to yield

$$\begin{aligned} \int dr d\theta r^4 \exp \left[-\frac{1}{2} \{4\bar{x}^2 + \bar{y}^2\} \right] &= \int dr d\theta r^4 \exp \left[-\frac{1}{2} \{r^2 (1 + (3 - \varphi) \cos^2 \theta)\} \right] \\ &= \left[\frac{3}{2} \sqrt{2\pi} \right] \int \frac{d\theta}{(1 + (3 - \varphi) \cos^2 \theta)^{5/2}} \\ &= \left[\frac{3}{2} \sqrt{2\pi} \right] \left[\frac{(40 - 8\varphi) E \left(\frac{3-\varphi}{4-\varphi} \right) - 4K \left(\frac{3-\varphi}{4-\varphi} \right)}{3(4 - \varphi)^{3/2}} \right] \end{aligned} \quad (51)$$

where we made use of the identity

$$\int_0^\infty \tau^4 \exp \left[-\frac{1}{2} a \tau^2 \right] = \frac{3}{2} \frac{\sqrt{2\pi}}{a^{5/2}}.$$

Finally, including all numerical factors, the full integral can be expressed as

$$\begin{aligned} \int ds \left\langle \frac{1}{\rho^3} \right\rangle_{x,y,z} &\approx \frac{Q_3}{(\sqrt{2\pi})^3} \left[\frac{3}{2} \sqrt{2\pi} \right] \left[\frac{(40 - 8\varphi) E \left(\frac{3-\varphi}{4-\varphi} \right) - 4K \left(\frac{3-\varphi}{4-\varphi} \right)}{3(4 - \varphi)^{3/2}} \right] \left[\frac{\sqrt{2\pi}}{2} \right] \left[\frac{\sqrt{2\pi}}{\sqrt{3 + 4\varphi^2}} \right] \\ &= Q_3 \left[\frac{(10 - 2\varphi) E \left(\frac{3-\varphi}{4-\varphi} \right) - K \left(\frac{3-\varphi}{4-\varphi} \right)}{\sqrt{(3 + 4\varphi^2)(4 - \varphi)^3}} \right], \end{aligned} \quad (52)$$

with $K(m)$ and $E(m)$ denoting the complete elliptical integrals of the first and second kind, respectively. This result is Eq. (23).

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