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A generalization of the Einstein–Maxwell equations

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Abstract The proposed modifications of the Einstein–Maxwell equations include: (1) the addition of a scalar term to the electromagnetic side of the equation rather than to the gravitational side, (2) the introduction of a four-dimensional, nonlinear electromagnetic constitutive tensor, (3) the addition of curvature terms arising from the non-metric components of a general symmetric connection and (4) the addition of a non-isotropic pressure tensor. The scalar term is defined by the condition that a spherically symmetric particle be force-free and mathematically well behaved everywhere. The constitutive tensor introduces two structure fields: One contributes to the mass and the other contributes to the angular momentum. The additional curvature terms couple both to particle solutions and to localized electromagnetic and gravitational wave solutions. The pressure term is needed for the most general spherically symmetric, static metric. It results in a distinction between the Schwarzschild mass and the inertial mass.

1 Introduction

This approach to the construction of a classical unified field theory depends on modifying the Einstein–Maxwell equations in four ways. The first modification is to move the scalar term, which has been conjectured since the early days of Einstein's cosmological constant, to the electromagnetic side of the equations and to require that it be defined by the condition that a spherically symmetric particle be force-free and mathematically well behaved everywhere. This simplifies the calculations. The second modification is to introduce a four-dimensional electromagnetic constitutive tensor which has two auxiliary structure fields. One of the fields contributes to the mass, and the other contributes to the angular momentum. The second field is due to a direct coupling between electric and magnetic fields. The third modification is to introduce additional curvature terms on the gravitational side of the equations. These terms arise from the non-metric components of a general symmetric connection and are essential to all of the four-dimensional solutions. The fourth modification is to add a non-isotropic pressure term in order to satisfy the generalized equation for the most general spherically symmetric, static metric. Section 2 begins by looking at Maxwell's equations in flat space in a three-dimensional notation in order to develop a physical understanding of the modifications to the electromagnetic side of the equation. Section 3 reviews non-Riemannian geometry and gives the form of the generalized Einstein–Maxwell equations used in this paper. Sections 4 and 5 solve the particle equations in three and four dimensions.

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Section 5 concludes with showing the distinction between the Schwarzschild mass and the inertial mass. Section 6 solves the equations for the electromagnetic and gravitational waves. The two types of waves are independent of each other and couple only to components of the non-Riemannian connection. The conclusions are in Sect. 7.

A supplemental file has been uploaded. It contains Mathematica® [1] notebooks which verify the derivations in Sects. 4–6. The MathTensor™ [2] Application Package is required.

2 Maxwell's equations

If there are no material media, Maxwell's equations can be written in 3 dimensions, using SI units, as:

$$E_i = -\phi_{;i} - \partial_t A_i \qquad B^i = \varepsilon^{ijk} A_{k;j} \tag{2.1a}$$

$$D_i = \epsilon_{ij} E^j - \gamma_{ji} B^j \qquad H_i = \alpha_{ij} B^j + \gamma_{ij} E^j \tag{2.1b}$$

$$\rho = D^i{}_{;i} \qquad j^i = \varepsilon^{ijk} H_{k;j} - \partial_t D^i \tag{2.1c}$$

where ε^{ijk} is the Levi-Civita tensor and α_{ij} is the inverse permeability. In free space with metric g_{ij},

$$\epsilon_{ij} = \epsilon_0 g_{ij} \qquad \alpha_{ij} = \mu_0^{-1} g_{ij} \qquad c^2 \epsilon_0 \mu_0 = 1 \tag{2.2}$$

In vector–dyadic notation,

$$\boldsymbol{D} = \underline{\boldsymbol{\epsilon}} \cdot \boldsymbol{E} - \boldsymbol{B} \cdot \underline{\boldsymbol{\gamma}} \qquad \boldsymbol{H} = \underline{\boldsymbol{\alpha}} \cdot \boldsymbol{B} + \underline{\boldsymbol{\gamma}} \cdot \boldsymbol{E} \tag{2.3}$$

Mathematically, the γ_{ij} terms arise from the fact that, in the four-dimensional formulation (e.g., Post [3, pp. 127–134]), the constitutive relations are described by a fourth rank tensor. Physically, they represent a direct coupling between the electric and magnetic fields which traditionally has been thought to be of interest only in material media. The particular form of the coupling used in this paper assumes that there is no optical activity. In this paper, we will show that solutions for which $\boldsymbol{B} = 0$ and $\underline{\boldsymbol{\gamma}} \neq 0$ can be used to represent particles with spin.

We will generalize the traditional definitions of the energy density, the stress tensor and the Poynting vector in three ways. The first generalization is to make the definitions fully symmetric. The second generalization is to introduce a scalar term Q which is motivated by long history of adding scalar fields to general relativity beginning with Einstein's cosmological constant as the simplest case. In a sense, it can be regarded as simply moving a generalized cosmological term from the gravitational side of the Einstein–Maxwell equations to the electromagnetic side. However, adding a scalar term to the electromagnetic stress-energy tensor turns out to make solving the equations much simpler. The third generalization is to add a non-isotropic pressure tensor which will be needed only for particles in curved space.

$$\mathcal{E} = \tfrac{1}{2}(B^i H_i + E^i D_i) - Q = \tfrac{1}{2}(\alpha_{ij} B^i B^j + \epsilon_{ij} E^i E^j) - Q \tag{2.4a}$$

$$\begin{aligned} T^{ij} &= -\tfrac{1}{2}(E^i D^j + E^j D^i + H^i B^j + H^j B^i) \\ &\quad + \tfrac{1}{2} g^{ij} (\alpha_{mn} B^m B^n + \epsilon_{mn} E^m E^n) + g^{ij} Q + P^{ij} \end{aligned} \tag{2.4b}$$

$$N^i = \tfrac{1}{2} \varepsilon^{ijk} (E_j H_k + c^2 D_j B_k) \tag{2.4c}$$

T^{ij} is defined with the opposite sign from what is usually used in 3 dimensions. It is useful because it lets T^{ij} be the spatial part of $T^{\mu\nu}$, which will be defined so that $T^4{}_4 = -\mathcal{E}$.

Note that the symmetry in (2.3) ensures that there are no γ_{ij} terms in the energy density (2.4a). Thus, there is no reason from an energy point of view for the γ_{ij} terms not to exist. The function Q will be chosen so that particle solutions are force-free and have finite self-energies. They can be constructed to have an exponential decay to the far field values rather than a polynomial decay, thus avoiding conflict with experimental results. The force density and the power loss density are

$$F_i = -T_i{}^j{}_{;j} - c^{-2}\partial_t N_i \qquad\qquad \mathcal{P} = -N^i{}_{;i} - \partial_t \mathcal{E} \tag{2.5}$$

In 4 dimensions, the electromagnetic fields and the current density are defined by

$$A_\mu = c(\boldsymbol{A}, -\phi) \tag{2.6a}$$

$$f_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \tag{2.6b}$$

$$p^{\mu\nu} = \tfrac{1}{2}\chi^{\mu\nu\rho\sigma} f_{\rho\sigma} \tag{2.6c}$$

$$j^\mu = p^{\mu\nu}{}_{;\nu} \tag{2.6d}$$

where $f_{\mu\nu}$ and $p^{\mu\nu}$ are antisymmetric and the constitutive tensor $\chi^{\mu\nu\rho\sigma}$ has the symmetries

$$\chi^{\mu\nu\rho\sigma} = -\chi^{\nu\mu\rho\sigma} \qquad \chi^{\mu\nu\rho\sigma} = -\chi^{\mu\nu\sigma\rho} \qquad \chi^{\mu\nu\rho\sigma} = \chi^{\rho\sigma\mu\nu} \tag{2.7}$$

The last of these conditions is the assumption of no optical activity. Post [3, p. 130] argues for additional symmetries which have not been assumed here. The stress-energy tensor and the force density are

$$T^{\mu\nu} = \tfrac{1}{2}(f^\mu{}_\tau p^{\nu\tau} + f^\nu{}_\tau p^{\mu\tau}) - g^{\mu\nu}(\tfrac{1}{4} f_{\kappa\tau} p^{\kappa\tau} - Q) + P^{\mu\nu} \tag{2.8a}$$

$$f_\mu = -T_\mu{}^\nu{}_{;\nu} \tag{2.8b}$$

In spherical coordinates in flat space, the metric is given by

$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2(\theta)\mathrm{d}\varphi^2 - c^2\mathrm{d}t^2 \\ \sqrt{-g} &= cr^2\sin(\theta) \end{aligned} \tag{2.9}$$

3 Non-Riemannian geometry and the Einstein–Maxwell equations

Eisenhart [4] shows that the most general asymmetric connection can be written in the form

$$\begin{aligned} &L^\mu_{\alpha\beta} = \Omega^\mu_{\alpha\beta} + \tilde{\Gamma}^\mu_{\alpha\beta} && \tilde{\Gamma}^\mu_{\alpha\beta} = a^\mu_{\alpha\beta} + \Gamma^\mu_{\alpha\beta} \\ &\Omega^\mu_{\alpha\beta} = -\Omega^\mu_{\beta\alpha} && a^\mu_{\alpha\beta} = a^\mu_{\beta\alpha} && \Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha} \end{aligned} \tag{3.1}$$

where $\Omega^\mu_{\alpha\beta}$ and $a^\mu_{\alpha\beta}$ are tensors and $\Gamma^\mu_{\alpha\beta}$ is the metric connection (Christoffel symbols). A solidus ("|") will denote covariant differentiation with respect to the asymmetric connection $L^\mu_{\alpha\beta}$, a semicolon will denote covariant differentiation with respect to the metric connection $\Gamma^\mu_{\alpha\beta}$ and a comma will denote partial differentiation with respect to the coordinates. (Eisenhart uses the Christoffel symbols for the metric connection, $\Gamma^\mu_{\alpha\beta}$ for the general symmetric connection and a comma to denote covariant differentiation with respect to the general symmetric connection.)

Since $g_{\mu\nu;\tau} = 0$, covariant differentiation with respect to the metric connection is more convenient than covariant differentiation with respect to the asymmetric connection which has the additional complications:

$$g_{\mu\nu|\tau} = -g_{\alpha\nu}(\Omega^\alpha_{\mu\tau} + a^\alpha_{\mu\tau}) - g_{\mu\alpha}(\Omega^\alpha_{\nu\tau} + a^\alpha_{\nu\tau}) \neq 0 \tag{3.2a}$$

$$g^{\mu\nu}{}_{|\tau} = g^{\alpha\nu}(\Omega^{\mu}_{\alpha\tau} + a^{\mu}_{\alpha\tau}) + g^{\mu\alpha}(\Omega^{\nu}_{\alpha\tau} + a^{\nu}_{\alpha\tau}) \neq 0 \tag{3.2b}$$

The curvature tensor for $L^{\mu}_{\alpha\beta}$ can be written as the sum of the curvature tensors for the antisymmetric part of the connection, $\Omega^{\mu}_{\alpha\beta}$, and the symmetric part of the connection, $\tilde{\Gamma}^{\mu}_{\alpha\beta}$, [4, (5.3)].

$$\begin{aligned} &L^{\mu}{}_{\nu\rho\sigma} = \Omega^{\mu}{}_{\nu\rho\sigma} + B^{\mu}{}_{\nu\rho\sigma} \\ &L^{\mu}{}_{\nu\rho\sigma} = -L^{\mu}{}_{\nu\sigma\rho} \qquad \Omega^{\mu}{}_{\nu\rho\sigma} = -\Omega^{\mu}{}_{\nu\sigma\rho} \qquad B^{\mu}{}_{\nu\rho\sigma} = -B^{\mu}{}_{\nu\sigma\rho} \end{aligned} \tag{3.3}$$

From [4, (5.5)] and (3.1),

$$\begin{aligned} \Omega^{\mu}{}_{\nu\rho\sigma} &= \Omega^{\mu}_{\nu\sigma|\rho} - \Omega^{\mu}_{\nu\rho|\sigma} + \Omega^{\mu}_{\alpha\sigma}\Omega^{\alpha}_{\nu\rho} - \Omega^{\mu}_{\alpha\rho}\Omega^{\alpha}_{\nu\sigma} - 2\Omega^{\mu}_{\nu\alpha}\Omega^{\alpha}_{\rho\sigma} \\ &= \Omega^{\mu}_{\nu\sigma;\rho} - \Omega^{\mu}_{\nu\rho;\sigma} + \Omega^{\mu}_{\alpha\rho}\Omega^{\alpha}_{\nu\sigma} - \Omega^{\mu}_{\alpha\sigma}\Omega^{\alpha}_{\nu\rho} + 2\Omega^{\mu}_{\alpha\rho}a^{\alpha}_{\nu\sigma} - 2\Omega^{\mu}_{\alpha\sigma}a^{\alpha}_{\nu\rho} \end{aligned} \tag{3.4}$$

From [4, (5.15)],

$$B^{\mu}{}_{\nu\rho\sigma} = R^{\mu}{}_{\nu\rho\sigma} + a^{\mu}_{\nu\sigma;\rho} - a^{\mu}_{\nu\rho;\sigma} + a^{\alpha}_{\nu\sigma}a^{\mu}_{\alpha\rho} - a^{\alpha}_{\nu\rho}a^{\mu}_{\alpha\sigma} \tag{3.5}$$

where $R^{\mu}{}_{\nu\rho\sigma}$ is the Riemann curvature tensor for the metric $g_{\mu\nu}$.

The spin is described by the non-Riemannian part of the connection. In Einstein–Cartan theory, the assumption is that $a^{\mu}_{\alpha\beta} = 0$ and $\Omega^{\mu}_{\alpha\beta} \neq 0$. In Weyl theory, the assumption is that $a^{\mu}_{\alpha\mu} \neq 0$ and $\Omega^{\mu}_{\alpha\beta} = 0$. [4, §30]. In this paper, we have assumed

$$\Omega^{\mu}_{\alpha\beta} = 0 \qquad a^{\mu}_{\alpha\mu} = 0 \qquad g^{\alpha\beta}a^{\mu}_{\alpha\beta} = 0 \qquad a^{\alpha}_{\beta\mu}a^{\beta}_{\alpha\nu} = 0 \tag{3.6}$$

The gauge invariance of $f_{\mu\nu}$ (2.6b) is preserved. Then

$$B_{\mu\nu} = B^{\alpha}{}_{\mu\alpha\nu} = R_{\mu\nu} + a^{\alpha}_{\mu\nu;\alpha} \qquad B = g^{\mu\nu}B_{\mu\nu} = R \tag{3.7}$$

Since $B = R$ and since the divergence of the Einstein tensor $G^{\mu\nu}{}_{;\nu} = 0$ can be derived either from direct calculation or from the variation $\delta(\sqrt{-g}R)$, we will write the generalized form of the Einstein–Maxwell equations as

$$\begin{aligned} G_{\mu\nu} + S_{\mu\nu} &= 8\pi G c^{-4} T_{\mu\nu} \\ G_{\mu\nu} &= R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R \\ S_{\mu\nu} &= a^{\alpha}_{\mu\nu;\alpha} \end{aligned} \tag{3.8}$$

where G is Newton's gravitational constant.

In every coordinate system,

$$a^{\mu}_{\alpha\mu} = a^{1}_{\alpha 1} + a^{2}_{\alpha 2} + a^{3}_{\alpha 3} + a^{4}_{\alpha 4} = 0 \tag{3.9}$$

The solutions presented in this paper are such that for each one there exists a coordinate system in which

$$a^{1}_{\alpha 1} = a^{2}_{\alpha 2} = a^{3}_{\alpha 3} = a^{4}_{\alpha 4} = 0 \tag{3.10}$$

4 Particle equations in three-dimensional flat space

In spherical coordinates (r, θ, φ) in flat space, let the particles be defined by

$$\boldsymbol{E} = f_e(r)\boldsymbol{e}_r = -\phi'_e(r)\boldsymbol{e}_r \tag{4.1a}$$

$$\boldsymbol{A} = 0 \tag{4.1b}$$

$$\underline{\boldsymbol{\alpha}} = c^2\underline{\boldsymbol{\epsilon}} = c^2\epsilon_0 f_\epsilon(r)(\boldsymbol{e}_r\boldsymbol{e}_r + \boldsymbol{e}_\theta\boldsymbol{e}_\theta + \boldsymbol{e}_\varphi\boldsymbol{e}_\varphi) \tag{4.1c}$$

$$\underline{\boldsymbol{\gamma}} = h(r)[(2\boldsymbol{e}_r\boldsymbol{e}_r - \boldsymbol{e}_\theta\boldsymbol{e}_\theta - \boldsymbol{e}_\varphi\boldsymbol{e}_\varphi)\cos(\theta) + (\boldsymbol{e}_r\boldsymbol{e}_\theta + \boldsymbol{e}_\theta\boldsymbol{e}_r)\sin(\theta)] \tag{4.1d}$$

$$\boldsymbol{F} = \left\{\epsilon_0(2r^4)^{-1}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^4 f_e^2(r) f_\epsilon(r)\right] - Q'(r)\right\}\boldsymbol{e}_r = 0 \tag{4.1e}$$

where the form of $\underline{\boldsymbol{\gamma}}$ has been chosen by trial and error so that in weak external fields there are no singularities in the various volume integrals for force, momentum, etc.

Then

$$Q(r) = \tfrac{1}{2}\epsilon_0 f_e^2(r) f_\epsilon(r) - 2\epsilon_0\int_r^\infty \mathrm{d}r'\,(r')^{-1} f_e^2(r') f_\epsilon(r') \tag{4.2a}$$

$$\mathcal{E}(r) = \tfrac{1}{2}\epsilon_0 f_e^2(r) f_\epsilon(r) - Q(r) = 2\epsilon_0\int_r^\infty \mathrm{d}r'\,(r')^{-1} f_e^2(r') f_\epsilon(r') \tag{4.2b}$$

$$\boldsymbol{D} = \epsilon_0 f_\epsilon(r) f_e(r)\boldsymbol{e}_r \tag{4.2c}$$

$$\boldsymbol{H} = f_e(r)h(r)[2\cos(\theta)\boldsymbol{e}_r + \sin(\theta)\boldsymbol{e}_\theta] \tag{4.2d}$$

$$\boldsymbol{N} = \tfrac{1}{2}h(r) f_e^2(r)\sin(\theta)\boldsymbol{e}_\varphi \tag{4.2e}$$

For finite, continuously differentiable functions, these solutions are force-free and radiationless. As $r \to 0$, we will assume that $f_e(r)$ is finite; and that $f_e(r) \to 0$ and $h(r) \to 0$ fast enough that there are no singularities and no directional discontinuities. In order to minimize any disagreement with experimental results in the far field, we will assume that as $r \to \infty$, the limits are approached exponentially rather than polynomially. For example:

$$\begin{aligned}
&\lim_{r\to\infty} f_e(r) = \begin{cases} q(4\pi\epsilon_0 r^2)^{-1}\{1 - \exp[-(r/r_0)^3]\} \to q(4\pi\epsilon_0 r^2)^{-1} & \text{charged particle} \\ q(4\pi\epsilon_0 r_0^2)^{-1}(r/r_0)^3\exp[-(r/r_0)^3] \to 0 & \text{neutral particle}\end{cases}\\
&\lim_{r\to\infty} f_\epsilon(r) = 1 - \exp[-(r/r_0)^3] \to 1\\
&\lim_{r\to\infty} h(r) = \mu_\gamma\epsilon_0(qr)^{-1}\{1 - \exp[-(r/r_0)^3]\} \to \mu_\gamma\epsilon_0(qr)^{-1}
\end{aligned} \tag{4.3}$$

where r_0 is the nominal radius of the particle and μ_γ is the magnetic moment arising from the action of $\underline{\boldsymbol{\gamma}}$. Note that the expression for $Q(r)$ in (4.2a) is an integral expression in the electromagnetic field rather than a local expression. The limits of the integral have been chosen to insure the correct asymptotic behavior as $r \to \infty$. We will show later that $Q(r)$ is local in terms of the electromagnetic field and the curved metric.

The rest mass

$$m_0 = c^{-2}\int_0^\infty \mathrm{d}r\,r^2\int_0^\pi \mathrm{d}\theta\,\sin(\theta)\int_0^{2\pi}\mathrm{d}\varphi\,\mathcal{E}(r) \tag{4.4a}$$

$$= 8\pi\epsilon_0 c^{-2}\int_0^\infty \mathrm{d}r\,r^2\int_r^\infty \mathrm{d}r'\,(r')^{-1} f_e^2(r') f_\epsilon(r') \tag{4.4b}$$

$$= \tfrac{8}{3}\pi\epsilon_0 c^{-2}\int_0^\infty \mathrm{d}r\,r^2 f_e^2(r) f_\epsilon(r) \tag{4.4c}$$

In Sect. 5, (4.4b) is proved to be the mass term in the far field of the Schwarzschild metric. From (4.2a) and (4.2b), we can prove that without the $Q(r)$ term, the factor $\frac{8}{3}$ in (4.4c) would have been 2. The additional mass is due to the force-free condition.

Since

$$\begin{aligned} \boldsymbol{e}_r &= \sin(\theta)\cos(\varphi)\boldsymbol{e}_x + \sin(\theta)\sin(\varphi)\boldsymbol{e}_y + \cos(\theta)\boldsymbol{e}_z \\ \boldsymbol{e}_\theta &= \cos(\theta)\cos(\varphi)\boldsymbol{e}_x + \cos(\theta)\sin(\varphi)\boldsymbol{e}_y - \sin(\theta)\boldsymbol{e}_z \\ \boldsymbol{e}_\varphi &= -\sin(\varphi)\boldsymbol{e}_x + \cos(\varphi)\boldsymbol{e}_y \end{aligned} \tag{4.5}$$

the total momentum and angular momentum in the rest frame are given by

$$\boldsymbol{P}_\mathrm{T} = c^{-2}\int_0^\infty \mathrm{d}r\, r^2 \int_0^\pi \mathrm{d}\theta\ \sin(\theta) \int_0^{2\pi} \mathrm{d}\varphi\ \boldsymbol{N} = 0 \tag{4.6a}$$

$$\begin{aligned} \boldsymbol{J}_\mathrm{T} &= c^{-2}\int_0^\infty \mathrm{d}r\, r^2 \int_0^\pi \mathrm{d}\theta\ \sin(\theta) \int_0^{2\pi} \mathrm{d}\varphi\ \boldsymbol{r}\times\boldsymbol{N} \\ &= \tfrac{4}{3}\pi c^{-2}\int_0^\infty \mathrm{d}r\, r^3 h(r) f_e^2(r)\boldsymbol{e}_z \end{aligned} \tag{4.6b}$$

If there are constant external fields

$$\boldsymbol{E}_0 = E_{0x}\boldsymbol{e_x} + E_{0y}\boldsymbol{e_y} + E_{0z}\boldsymbol{e_z} \qquad \boldsymbol{B}_0 = B_{0x}\boldsymbol{e_x} + B_{0y}\boldsymbol{e_y} + B_{0z}\boldsymbol{e_z} \tag{4.7}$$

which do not, to a first approximation, modify $f_\epsilon(r)$, $h(r)$ and $Q(r)$ and if the accelerations are low so that radiation reaction effects can be ignored, then the total force and the total torque in the rest frame are

$$\begin{aligned} \boldsymbol{F}_\mathrm{T} &= \int_0^\infty \mathrm{d}r\, r^2 \int_0^\pi \mathrm{d}\theta\ \sin(\theta) \int_0^{2\pi} \mathrm{d}\varphi\ \boldsymbol{F}_\mathrm{ext} \\ &= 4\pi\epsilon_0 r^2 f_\epsilon(r) f_e(r)\ \big|_{r=0}^\infty\ \boldsymbol{E}_0 \\ &= \begin{cases} q\boldsymbol{E}_0 & \text{charged particle} \\ 0 & \text{neutral particle} \end{cases} \end{aligned} \tag{4.8a}$$

$$\begin{aligned} \boldsymbol{W}_\mathrm{T} &= \int_0^\infty \mathrm{d}r\, r^2 \int_0^\pi \mathrm{d}\theta\ \sin(\theta) \int_0^{2\pi} \mathrm{d}\varphi\, \boldsymbol{r}\times\boldsymbol{F}_\mathrm{ext} \\ &= 2\pi r^3 h(r) f_e(r)\Big|_{r=0}^\infty \boldsymbol{e}_z\times\boldsymbol{B}_0 \\ &= \begin{cases} \frac{1}{2}\mu_\gamma\boldsymbol{e}_z\times\boldsymbol{B}_0 & \text{charged particle} \\ 0 & \text{neutral particle} \end{cases} \end{aligned} \tag{4.8b}$$

where $\boldsymbol{F}_\mathrm{ext}$ is calculated from (2.5). The factor of $\frac{1}{2}$ in $\boldsymbol{W}_\mathrm{T}$ distinguishes this result from the normal magnetic dipole, $\boldsymbol{W}_\mathrm{T} = \mu_\mathrm{m}\boldsymbol{e}_z\times\boldsymbol{B}$. Thus the numerical values for μ_γ are related to the numerical values reported for μ_m by

$$\mu_\gamma = 2\mu_\mathrm{m} \tag{4.9}$$

This corresponds to the quantum spin factor $g_s = 2$. If we consider the special case

$$h(r) = \mu_\gamma\epsilon_0(qr)^{-1} f_\epsilon(r) \tag{4.10}$$

then (4.6b) and (4.4c) give

$$\boldsymbol{J}_\mathrm{T} = \mu_\gamma m_0(2q)^{-1}\boldsymbol{e}_z \tag{4.11}$$

If we set the z-component of $\boldsymbol{J}_\mathrm{T}$ to $\frac{1}{2}\hbar$ for spin $\frac{1}{2}$ particles, then we obtain the standard magneton result

$$2\mu_\mathrm{m} = \mu_\gamma = \frac{q\hbar}{m_0} \tag{4.12}$$

where the quantum spin factor $g_s = 2$ appears automatically. This suggests that $f_\epsilon(r)$ and $h(r)$ are not independent.

5 Particle equations in four-dimensional non-Riemannian space

In the rest frame of a particle, the most general spherically symmetric, static metric (e.g., see Misner et al. [5, §23]) can be written in the form

$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = f_{g1}^{-1}(r)\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2(\theta)\mathrm{d}\varphi^2 - c^2 f_{g1}(r) f_{g2}(r)\mathrm{d}t^2 \\ \sqrt{-g} &= cr^2 f_{g2}^{1/2}(r)\sin(\theta) \end{aligned} \tag{5.1}$$

Since $\sqrt{-g}$ includes a factor of $f_{g2}^{1/2}(r)$, four-dimensional volume integrals must be modified accordingly. It also requires $f_{g2}(r) > 0$. The requirement that the metric be flat as $r \to \infty$, forces $f_{g2}(r) \to 1$ in that limit. The advantage of this form is that it will clearly distinguish the separate roles of $f_{g1}(r)$ and $f_{g2}(r)$.

In accordance with (3.6), let the only nonzero components of $a^\mu_{\nu\sigma}$ be

$$a^1_{43} = a^1_{34} = -c\, r\zeta(r) f_{g1}(r) f_{g2}^{1/2}(r)\sin^2(\theta) \tag{5.2}$$

Let the non-isotropic pressure tensor have the diagonal form

$$P^1{}_1 = p_{g1}(r) \quad P^2{}_2 = P^3{}_3 = p_{g2}(r) \quad P^4{}_4 = 0 \tag{5.3}$$

Let

$$\begin{aligned} &A_\mu = (0,\ 0,\ 0,\ -c\phi_e(r)) \qquad\qquad A^\mu = \left(0,\ 0,\ 0,\ c^{-1} f_{g1}^{-1}(r) f_{g2}^{-1}(r)\phi_e(r)\right) \\ &\phi_e(r) = \int_r^\infty \mathrm{d}r'\, f_e(r') f_{g2}^{1/2}(r') \end{aligned} \tag{5.4}$$

The metric and non-metric components of the constitutive tensor are specified by

$$\begin{aligned} \chi_{\mu\nu\rho\sigma} &= \epsilon_0 f_\epsilon(r)(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}) \\ \chi_{3241} &= -2cr^2 h(r) f_{g2}^{1/2}(r)\sin(\theta)\cos(\theta) \\ \chi_{3242} &= -r^2 f_{g1}(r)\chi_{3141} = -cr^3 h(r) f_{g1}^{1/2}(r) f_{g2}^{1/2}(r)\sin^2(\theta) \\ \chi_{2143} &= -\chi_{3142} = cr^2 h(r) f_{g2}^{1/2}(r)\sin(\theta)\cos(\theta) \end{aligned} \tag{5.5}$$

Then the nonzero components of $T_{\mu\nu}$, $G_{\mu\nu}$ and $S_{\mu\nu}$ are

$$T^1{}_1 = -\tfrac{1}{2}\epsilon_0 f_\epsilon(r) f_e^2(r) + Q(r) + p_{g1}(r) \tag{5.6a}$$

$$T^4{}_4 = -\tfrac{1}{2}\epsilon_0 f_\epsilon(r) f_e^2(r) + Q(r) \tag{5.6b}$$

$$T^2{}_2 = T^3{}_3 = \tfrac{1}{2}\epsilon_0 f_\epsilon(r) f_e^2(r) + Q(r) + p_{g2}(r) \tag{5.6c}$$

$$T_{34} = -\tfrac{1}{2}crh(r) f_e^2(r) f_{g1}^{1/2}(r) f_{g2}^{1/2}(r)\sin^2(\theta) \tag{5.6d}$$

$$G^1{}_1 = r^{-2}[-1 + f_{g1}(r) + rf'_{g1}(r)] + f_{g1}(r) f'_{g2}(r)[rf_{g2}(r)]^{-1} \tag{5.6e}$$

$$G^4{}_4 = r^{-2}[-1 + f_{g1}(r) + rf'_{g1}(r)] \tag{5.6f}$$

$$\begin{aligned} G^2{}_2 = G^3{}_3 &= \tfrac{1}{4}r^{-1}[2f'_{g1}(r) + rf''_{g1}(r)] - \tfrac{1}{4}f_{g1}(r)[f'_{g2}(r) f_{g2}^{-1}(r)]^2 \\ &\quad + \tfrac{1}{4}[rf_{g2}(r)]^{-1}\{3rf'_{g1}(r) f'_{g2}(r) + 2f_{g1}(r)[f'_{g2}(r) + rf''_{g2}(r)]\} \end{aligned} \tag{5.6g}$$

$$S_{34} = -cr^{-1} f_{g1}^{1/2}(r) \frac{d}{dr}\left[r^2 f_{g1}^{1/2}(r) f_{g2}^{1/2}(r)\zeta(r)\right] \sin^2(\theta) \tag{5.6h}$$

The solutions to the generalized Einstein–Maxwell equations (3.8) are

$$p_{g1}(r) = c^4(8\pi G)^{-1} f_{g1}(r) f_{g2}'(r)[r f_{g2}(r)]^{-1} \tag{5.7a}$$

$$p_{g2}(r) = p_{g1}(r)\left\{1 + \tfrac{1}{4}r[f_{g1}'(r) f_{g1}^{-1}(r) + f_{g2}'(r) f_{g2}^{-1}(r)]\right\} + \tfrac{1}{2}r p_{g1}'(r) \tag{5.7b}$$

$$Q(r) = \tfrac{1}{2}\epsilon_0 f_e^2(r) f_\epsilon(r) - 2\epsilon_0 \int_r^\infty \mathrm{d}r'\, (r')^{-1} f_e^2(r') f_\epsilon(r') \qquad \text{same as (4.2a)} \tag{5.7c}$$

$$\mathcal{E}(r) = -T^4{}_4 = 2\epsilon_0 \int_r^\infty dr'\, (r')^{-1} f_e^2(r') f_\epsilon(r') \qquad \text{same as (4.2b)} \tag{5.7d}$$

$$f_{g1}(r) = 1 - \frac{16\pi G\epsilon_0}{c^4 r}\int_0^r \mathrm{d}r'\,(r')^2 \int_{r'}^\infty \mathrm{d}r''\,(r'')^{-1} f_e^2(r'') f_\epsilon(r'') \tag{5.7e}$$

$$\zeta(r) = \frac{4\pi G}{c^4 r^2} f_{g1}^{-1/2}(r) f_{g2}^{-1/2}(r) \int_0^r \mathrm{d}r'(r')^2\, h(r') f_{g2}^{1/2}(r') f_e^2(r') \tag{5.7f}$$

Comparison of (5.7e) with the Schwarzschild metric, for which $f_{g1}(r) = 1 - 2Gm_0c^{-2}r^{-1}$, shows that

$$m_{0s} = 8\pi\epsilon_0 c^{-2} \int_0^\infty \mathrm{d}r\, r^2 \int_r^\infty \mathrm{d}r'\,(r')^{-1} f_e^2(r') f_\epsilon(r') \tag{5.8}$$

which we define to be the Schwarzschild mass. It agrees with the result in flat space (4.4b). An alternative definition of the mass is to modify the integral in (4.4b) to include the factor of $f_{g2}^{1/2}(r)$ in $\sqrt{-g}$. The inertial mass is defined to be

$$m_{0i} = 8\pi\epsilon_0 c^{-2} \int_0^\infty \mathrm{d}r\, r^2 f_{g2}^{1/2}(r) \int_r^\infty \mathrm{d}r'\,(r')^{-1} f_e^2(r') f_\epsilon(r') \tag{5.9}$$

If $f_{g2}(r) = 1$, then $p_{g1}(r) = 0$, $p_{g2}(r) = 0$ and $m_{0i} = m_{0s}$. Newton's third law and law of gravitation are an argument for setting $f_{g2}(r) = 1$ whenever action at a distance is a valid approximation. However at present, there is no theory that determines $f_{g2}(r)$ whenever field effects are expected to be significant on either an astronomical scale or a microscopic scale.

For particles with an asymptotic form similar to (4.3), the asymptotic limit of (5.7e) is

$$\lim_{r\to\infty} f_{g1}(r) = 1 - \frac{2Gm_{0s}}{c^2 r} + \frac{Gq_\mathrm{T}^2}{4\pi\epsilon_0 c^4 r^2} + \cdots, \begin{cases} q_\mathrm{T} = q & \text{charged particle} \\ q_\mathrm{T} = 0 & \text{neutral particle} \end{cases} \tag{5.10}$$

The first three terms agree with the Reissner–Nordstöm metric. We can construct solutions such that higher-order terms decrease exponentially in order to ensure agreement with experimental results in the far field.

Note that (5.7e) can be inverted to give

$$16\pi G\epsilon_0 c^{-4} f_e^2(r) f_\epsilon(r) = f_{g1}''(r) + 2r^{-2}[1 - f_{g1}(r)] \tag{5.11}$$

thus showing that any well-behaved $f_{g1}(r)$ can be expressed in terms of $f_e^2(r) f_\epsilon(r)$. Note also that $Q(r)$ can be expressed as

$$Q(r) = \tfrac{1}{2}\epsilon_0 f_e^2(r) f_\epsilon(r) + c^4(8\pi G r^2)^{-1}[f_{g1}(r) + r f_{g1}'(r) - 1] \tag{5.12}$$

This expresses $Q(r)$ as the difference between the traditional form of Maxwell's energy density and Einstein's gravitational energy density. It is a local function in terms of the electromagnetic field and the curved metric. The advantage of (5.7c) is that it is explicitly derived from the force-free condition (4.1e).

6 Electromagnetic and gravitational waves

In a curved space with a Peres [6,7] type of cylindrically symmetric metric

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}r^2 + r^2\mathrm{d}\varphi^2 + \mathrm{d}z^2 - c^2\mathrm{d}t^2 - f_{gw}(z-ct)f_g(r)(\mathrm{d}z - c\mathrm{d}t)^2 \\ \sqrt{-g} &= cr \end{aligned} \tag{6.1}$$

there exist electromagnetic waves and gravitational waves. Let

$$A_\mu = cf_{\mathrm{emw}}(z-ct)f_{\mathrm{em}}(r)(1,0,0,0) \tag{6.2a}$$

$$\chi_{\mu\nu\rho\sigma} = \epsilon_0(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}) \tag{6.2b}$$

$$c^2a^1_{33} = -ca^1_{43} = a^1_{44} = f_{\mathrm{aw}}(z-ct)f_a(r) \tag{6.2c}$$

where $a^\mu_{\nu\sigma}$ obeys the constraints (3.6). The asymptotic conditions must include

$$\lim_{r\to 0} f_{\mathrm{em}}(r) = 0 \qquad \lim_{r\to\infty} f_{\mathrm{em}}(r) = 0 \qquad \lim_{r\to\infty} f_g(r) = 0 \qquad \lim_{r\to\infty} f_a(r) = 0 \tag{6.3}$$

For this type of electromagnetic wave,

$$j^\mu = \epsilon_0 r^{-1}\frac{\mathrm{d}}{\mathrm{d}r}[rf_{\mathrm{em}}(r)]f'_{\mathrm{emw}}(z-ct)(0,\ 0,\ c,\ 1) \tag{6.4a}$$

$$j_\mu = \epsilon_0 r^{-1}\frac{\mathrm{d}}{\mathrm{d}r}[rf_{\mathrm{em}}(r)]f'_{\mathrm{emw}}(z-ct)(0,\ 0,\ c,\ -c^2) \tag{6.4b}$$

In free space, electromagnetic waves are usually assumed to have zero current, $j_\mu = 0$. However, if we admit the possibility of nonzero field currents such that $j_\mu j^\mu = 0$, then we have a class of force-free wave solutions that have a spatial variation in the plane perpendicular to the direction of propagation. These null-vector field currents are intrinsic to the structure of the wave; they are not an external source.

The nonzero components of $T_{\mu\nu}$, $G_{\mu\nu}$ and $S_{\mu\nu}$ are

$$T_{44} = -cT_{34} = c^2T_{33} = c^4\epsilon_0[f'_{\mathrm{emw}}(z-ct)]^2 f^2_{\mathrm{em}}(r) \tag{6.5a}$$

$$G_{44} = -cG_{34} = c^2G_{33} = c^2(2r)^{-1}f_{\mathrm{gw}}(z-ct)\frac{\mathrm{d}}{\mathrm{d}r}\left[rf'_g(r)\right] \tag{6.5b}$$

$$S_{44} = -cS_{34} = c^2S_{33} = r^{-1}f_{\mathrm{aw}}(z-ct)\frac{\mathrm{d}}{\mathrm{d}r}\left[rf_a(r)\right] \tag{6.5c}$$

Thus $T^{\mu\nu}{}_{;\nu} = 0$ and $S^{\mu\nu}{}_{;\nu} = 0$. From (3.8),

$$\begin{aligned} &c^2(2r)^{-1}\frac{\mathrm{d}}{\mathrm{d}r}\left[rf'_g(r)\right]f_{\mathrm{gw}}(z-ct) + r^{-1}\frac{\mathrm{d}}{\mathrm{d}r}\left[rf_a(r)\right]f_{\mathrm{aw}}(z-ct) \\ &\quad = 8\pi G\epsilon_0 f^2_{\mathrm{em}}(r)[f'_{\mathrm{emw}}(z-ct)]^2 \ge 0 \end{aligned} \tag{6.6}$$

Integration gives

$$\begin{aligned} &\tfrac{1}{2}c^2 f'_g(r)f_{\mathrm{gw}}(z-ct) + f_a(r)f_{\mathrm{aw}}(z-ct) \\ &\quad = 8\pi G\epsilon_0 r^{-1}\int_0^r \mathrm{d}r'r'f^2_{\mathrm{em}}(r')[f'_{\mathrm{emw}}(z-ct)]^2 \ge 0 \end{aligned} \tag{6.7}$$

In general, $f_g'(r)$ has both positive and negative regions so the gravitational wave cannot couple to the electromagnetic wave for all values of r. Therefore, $f_a(r) f_{\text{aw}}(z - ct)$ has to have two pieces: one that couples to the electromagnetic wave and one that couples to the gravitational wave. The electromagnetic and gravitational waves exist independently. This type of electromagnetic wave is independent of the Riemannian part of the curvature tensor.

7 Conclusions

We have modified the Einstein–Maxwell equations by adding four types of terms and have derived solutions for static, spherical particles and for localized electromagnetic and gravitational cylindrical waves. The solutions are force-free and mathematically well behaved. We have introduced a four-dimensional constitutive tensor with two structure fields that appear to be related by a Bohr magneton condition. We have shown that there is a distinction between the Schwarzschild mass and the inertial mass. We have also shown that the curvature terms arising from the non-metric components of a general symmetric connection couple in various ways to the particle solutions and to the localized electromagnetic and gravitational wave solutions.

Data Availability Statement This manuscript has associated data in a data repository. [Authors' comment: The zip file containing the Mathematica notebooks that were used to do the derivations is available from the author upon request.]

Compliance with ethical standards

Conflict of interest The author states that there is no conflict of interest. The research was entirely funded by the author.

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