EDITORIAL: “Solitons, Integrability, Nonlinear Waves: Theory and Applications”

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Nonlinear waves have long been at the research focus of both physicists and mathematicians, in diverse settings ranging from electromagnetic waves in nonlinear optics to matter waves in Bose–Einstein condensates, from Langmuir waves in plasma to internal and rogue waves in hydrodynamics. The study of physical phenomena by means of mathematical models often leads to nonlinear evolution equations known as integrable systems. One of the distinguished features of integrable systems is that they admit soliton solutions, i.e., stable, localized traveling waves which preserve their shape and velocity in the interaction. Other fundamental properties of integrable systems are their universal nature, and the fact that they can be effectively linearized, e.g., via the Inverse Scattering Transform (IST), or reduced to appropriate Riemann–Hilbert problems. Moreover, explicit solutions can often be derived by the Zakharov–Shabat dressing method, by Bäcklund or Darboux transformations, or by Hirota’s bilinear method. Prototypical examples of such integrable equations in 1+1 dimensions are the nonlinear Schrödinger (NLS) equation and its multicomponent generalizations, the sine-Gordon equation, the Korteweg–de Vries (KdV) and the modified KdV equations, the Kulish–Sklyanin model, etc. The most notable examples of integrable systems in 2+1-dimensions are the Kadomtsev–Petviashvili (KP) equations, and the Davey–Stewartson equations.

The aim of this special issue is to present the latest developments in the theory of nonlinear waves and integrable systems, and some of their applications. Below, we briefly outline the contributions to the present Focus Point (FP) summarizing their achievements in nonlinear wave phenomena and integrable systems. Some open problems and questions are also identified.

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A. General theory [1–5]

The first two papers [1,2] are parts of a bigger project that will be developed in the future. The first part [1] combines the well-known interpretation of the IST as a generalized Fourier transform with the ideas of the τ -function.

The generalized Fourier transforms (GFT) for the hierarchies of multi-component models of Manakov type are revisited. The aim of the authors is to adapt GFT so that one could treat the whole hierarchy of nonlinear evolution equations simultaneously. To this end, they consider the potential Q of the Lax operator L as local coordinate on some symmetric space depending on an infinite number of variables t , and z_k , $k = 1, 2, \dots$. The dependence on z_k is determined by the k th higher flow (conserved density) of the hierarchy. Thus, they have an infinite set of commuting operators with common fundamental analytic solution. Then, the authors analyze the properties of the resolvent and thus determine the spectral properties of L . Next, they derive the generalized Fourier transforms that linearize this hierarchy of nonlinear evolution equations and establish their fundamental properties, as well as the dynamical compatibility of each pair of such flows. Using the classical R -matrix approach, the Poisson brackets between all conserved quantities are derived, first assuming that Q is a quasi-periodic function of t . Next, taking the limit when the period tends to ∞ , the Poisson brackets between the scattering data of L are obtained. In addition, possible relation of this approach to the one based on the τ -function that could lead to multi-dimensional integrable equations is analyzed. The authors were also able to relate the τ -functions with the sewing function of the corresponding Riemann–Hilbert problem. This may explain why equations in $2 + 1$ dimensions like KP are not only integrable, but may allow the generalized Fourier transforms.

The paper [2] is a continuation of [1] in the sense that it also treats the hierarchies of Manakov type. It starts by stressing their important applications to nonlinear optics and other branches of physics. Next, a method for calculating the spectral curves for the multi-phase solutions of these equations is presented. The authors develop an approach which allows one to derive explicitly new solutions to the Manakov model in terms of Weierstrass elliptic functions, as well as in hyperbolic functions. In addition, it has been shown that the density of the first integral of motion of the Manakov model satisfies the Kadomtsev–Petviashvili equation. Other multi-component generalization of the Manakov system to which this method can be extended are discussed.

In [3], a generalized nonlocal NLS equation with distributed coefficients and with linear and harmonic oscillator potentials is considered analytically. In the first part of his analysis, the author provides exact solutions of the nonlocal system with constant coefficients in closed form. Exact solutions are found for different values of the parameters, both in localized or soliton form and in more general elliptic functions as well. For a specific value of the parameter q , self-similar arguments can reduce the nonlocal partial differential equations (PDEs) to a set of ordinary differential equations (ODEs). A solution is found also for the defocusing case. The second part of the analysis deals with the general nonlocal system. The goal is to utilize the findings of the first part to construct solution for this case. To this end, the author uses a special self-similar transformation which eliminates the distance-dependent coefficients. This transformation is also related to the physical problem as it involves the amplitude, width, center and phase of the solution. Undistorted propagation may still be possible, but under strict conditions for the varying coefficients of the system. Finally, a brief comparison is made to the relative analysis for the local, cubic NLS equation. These findings suggest that general solutions of nonlinear evolution equations may also exist well beyond the integrability limit.

The paper [4] presents several integrable systems arising as negative flows of the well-known Heisenberg ferromagnet hierarchy, which is gauge equivalent to the hierarchy of

the NLS equation. The author has outlined the application of the dressing method for the calculation of the soliton solutions. He has also found that there are interesting integrable examples of such models with Lax representation, containing both positive and negative flows that may lead to new types of soliton interactions.

In [5], the Hamilton principle applied previously to non-conservative systems is explored for complex PDEs of the NLS type, examining the dynamics of the coherent solitary wave structures. The formalism of the non-conservative variational approximation (NCVA) is compared to two other variational techniques used in dissipative systems. All three variational techniques produce equivalent equations of motion for the perturbed NLS models. Then, the authors showed the relevance of the NCVA method by exploring test case examples within the NLS setting including combinations of linear and density-dependent loss and gain.

B. Integrable systems and physical application [6–10]

The short-pulse equation (SPE), originally introduced as an equation for pseudospherical surfaces, was later derived as a model for the propagation of ultrashort optical pulses in nonlinear media, and more recently also in metamaterials. In 2015, a complex-valued version of the SPE, i.e., a complex short-pulse (CSP) equation, was derived from Maxwell's equations. Like its real counterpart, the complex SPE is also a completely integrable system, and, in addition to standard soliton solutions, the CSP equation admits loop solitons, which are not single-valued, as well as solutions that “breathe” between single-valued and multi-valued states. Also, the interaction of single-valued solitons can result in a multi-valued solution. In the work [6], a Riemann–Hilbert approach for the IST for the CSP equation is developed. As a byproduct of the IST, soliton solutions are also obtained. Unlike the real-valued SPE, in the complex case discrete eigenvalues are not necessarily restricted to the imaginary axis, and, as consequence, smooth 1-soliton solutions exist for any choice of discrete eigenvalue with imaginary part less than its real part.

The work [7] in this FP deals with a class of square matrix nonlinear Schrödinger (MNLS) systems whose reductions include two equations that model hyperfine spin $F = 1$ spinor Bose–Einstein condensates in the focusing and defocusing dispersion regimes, and two novel (mixed sign) equations that were recently shown to be integrable. The main goal of the paper is to discuss the bright soliton solutions and their interactions for the focusing MNLS and for the two mixed sign systems within the framework of the IST. The nature of the solitons and their interactions depends on whether the associated norming constants (polarization matrices) are rank-one matrices (giving rise to ferromagnetic solitons) or full rank (corresponding to polar solitons). By computing the long-time asymptotics of the 2-soliton solutions, in [7] the changes in the polarization matrix of each soliton due to the interaction are determined. Correspondingly, explicit formulas for the soliton interactions are given for all possible types of interacting solitons, and for all three inequivalent reductions of the MNLS systems that admit regular bright soliton solutions.

In recent years, the study of nonlinear waves in media that are governed by nonlinear semi-discrete evolution systems (discrete in space and continuous in time) has attracted significant interest. The Salerno model is a discrete variant of the celebrated NLS equation interpolating between the discrete NLS (DNLS) equation and the completely integrable Ablowitz–Ladik (AL) model by appropriately tuning the relevant homotopy parameter. Although the AL model possesses an explicit time-periodic solution known as the Kuznetsov–Ma (KM) breather, the existence of time-periodic solutions away from the integrable limit had not yet been studied. The paper [8] in this FP sheds light on the existence and stability of time-

periodic solutions of the Salerno model. In this work, the homotopy parameter of the model is varied by employing a pseudo-arclength continuation algorithm where time-periodic solutions are identified via fixed-point iterations. Correspondingly, the solutions are shown to transform into time-periodic patterns featuring small, yet non-decaying far-field oscillations. Numerical simulations are also performed that suggest the existence of previously unknown time-periodic solutions even in the integrable case, and the stability of the solutions is studied via Floquet theory. A continuation of these patterns toward the DNLS limit is also discussed in the paper.

In the paper [9] included in this FP, the generation and propagation of solitary waves is studied in the context of the Hertz chain and the Toda lattice. The analysis of the kinetic and potential energy of a solitary wave shows that under certain circumstances the energy profiles in these systems (i.e., their spatial distribution) look reasonably close to each other. On the other hand, the width of the wave is notably different in the two systems. The dynamical behavior of these systems in response to an initial velocity impulse is also investigated. For the Toda lattice, this is achieved by employing the IST, and the ratio between the energy of the resulting solitary wave and the energy of the impulse, as a function of the impulse velocity, is obtained. The dynamics of the Toda system is then compared to that of the Hertz system, for which the corresponding quantities are obtained through numerical simulations. For the Hertz system, the study reveals a universality in the fraction of the energy stored in the resulting solitary traveling wave irrespectively of the size of the impulse. This fraction turns out to only depend on the nonlinear exponent. Finally, the relation between the velocity of the resulting solitary wave and the velocity of the impulse is investigated, and an alternative proof for the numerical scaling rule of Hertz type systems is provided.

Inspired by the theory of scale relativity, an NLS equation has been recently proposed to model dark matter halos. The equation involves a logarithmic nonlinearity associated with an effective temperature and a source of dissipation. The paper [10] in this FP studies the Benjamin–Feir-type modulational instability exhibited by this model. The analysis is then extended to further generalizations of the equation in the presence of short-range interactions, giving rise to Gross–Pitaevskii-like and Cahn–Hilliard-like equations, as well as to a generalization emerging from the Lynden–Bell distribution. In each case, a criterion leading to modulational instability and the corresponding growth rate is provided. Interestingly, the analysis reveals that the gravitational potential does not influence the growth rate of this type of instabilities.

C. Applications to hydrodynamics [11–15]

In [11], the authors analyze steady two-dimensional surface waves on an ideal irrotational fluid over a complex multi-bumped topography. They study analytically the case when the far upstream flow is slightly supercritical. Fully nonlinear equations are formulated via the von Mises variables that parametrize the bundle of streamlines in the flow over obstacles. For a given small-height topography (without assumption of its symmetry with respect to the y axis), they construct approximate two-parametric solution sets which approach the branches of solitary waves as the typical height of the obstacle vanishes.

The paper [12] is devoted to the derivation of a new hyperbolic model describing the propagation of internal waves in stratified shallow water with a non-hydrostatic pressure distribution. The construction of the hyperbolic model is based on the use of additional ‘instantaneous’ variables. This allows one to reduce the dispersive multi-layer Green–Naghdi model to a first-order system of evolution equations. Great attention is paid to the study of

three-layer flows over uneven bottom in the Boussinesq approximation with the additional assumption of hydrostatic pressure in the intermediate layer. The hyperbolicity conditions of the obtained equations of three-layer flows are formulated, and solutions in the class of traveling waves are studied. Based on the proposed hyperbolic and dispersive models, numerical calculations of the generation and propagation of internal solitary waves are carried out and their comparison with experimental data is given. Within the framework of the proposed three-layer hyperbolic model, a numerical study of the propagation and interaction of symmetric and non-symmetric soliton-like waves is performed.

In [13], the authors consider two problems with a free boundary for the Navier–Stokes equations. In the first problem, the fluid occupies a horizontal strip whose lower boundary is a motionless wall and whose upper boundary is a straight-line free boundary parallel to the wall. In the second problem, the fluid motion is rotationally symmetric. Here, the flow domain is a horizontal layer bounded by a solid plane and a parallel flat free surface. In both problems, the vertical velocity and pressure are independent of the longitudinal coordinates. In the first problem, there are three modes of motion: stabilization to a quiescent state with increasing time, blow up of the solution within a finite time, and an intermediate self-similar mode in which the layer thickness unlimitedly increases with time. Blow up regime has a specific character: free boundary goes to infinity in a finite time. The same situation occurs in the second problem if the solid surface bounding the layer does not move. However, its rotation can prevent the solution collapse. Both solutions considered in [13] have a group-theoretical nature: they are partially invariant (in the sense of Ovsiannikov) solutions to the Navier–Stokes equations.

The authors of the paper [14] consider two Boussinesq models that describe propagation of small-amplitude long water waves. Exact solutions of the classical Boussinesq equation that represent the interaction of wave packets and waves on solitons are found. Indeed, some one-dimensional Boussinesq models have a variety of solutions. Besides solitons, this may be also wave packets, waves on solitons, wave fusion and decay. The authors are using only the Hirota representation and computer algebra systems to find solutions. They also find various solutions for one of the variants of the Boussinesq system. In particular, these solutions can be interpreted as the fusion and decay of solitary waves, as well as the interaction of more complex structures. It should be noticed that examples of both elastic and non-elastic solitons interaction are found. However, it is unclear whether the Hirota representation exists for the specific equation and how to find it. On the other hand, one can try to look for solutions in the form of a rational function of exponentials. Preliminary calculations show that this may work. As a result, the authors obtain a highly overdetermined systems of nonlinear algebraic equations, which turned out to be compatible.

In [15], solutions of the generalized KdV–Burgers equation are analyzed in the case when the dissipation coefficient depends on the spatial coordinate and time. Solutions in the form of traveling waves describing the structure of discontinuities, including special discontinuities, are studied. In the frame of this problem, nonstationary solutions of the generalized KdV–Burgers equation included the special discontinuity are numerically found. The number of instabilities the background Stokes wave possesses and the damping strength are varied. The perturbed flow is analyzed by appealing to the spectral theory of the NLS equation. For a constant dissipation parameter, only one of the states behind the discontinuity admitted by the conservation law corresponds to a traveling wave solution (to a discontinuity with a structure). It was established that if the flux function has two points of inflection and the same velocity value corresponds to three states behind the discontinuity that satisfy the conservation law, then traveling-wave solutions exist for all three states. An example was considered in which three solutions exist for under-compressive shocks and infinitely many solutions exist for

classical discontinuities. For these discontinuities to be regarded as admissible, they have to be examined for linear stability.

D. Global existence, stability and rogue waves [16–18]

The authors of [16] apply the Darboux matrix method and construct a generalized Darboux transformation for a generalized NLS equation possessing higher-order terms which refer to a femtosecond pulse propagation in a nonlinear fiber optic. This NLS equation models also the propagation of soliton in a Heisenberg spin chain. Using the above generalized Darboux transformation, the authors calculated the first-, second- and third-order rogue wave solutions and provided general formulas for generating N th order ones. Then, they discussed the effect of higher-order terms contained in the above equation. Starting from a seed solution and a generalized Darboux transformation, the authors constructed rogue wave solutions from first to third order. A general formula for generating N th order rogue wave solution has been proposed.

Paper [17] studies the Cauchy problem for a class of nonlinear wave equations with a linear pseudo-differential operator. In the framework of variational arguments, the existence and non-existence of global solution to this problem are established when the total initial energy is less than or equal to the mountain pass level. Further, a finite-time blow up result for arbitrarily high initial energy is established. One of the most challenging open problems seems to be the global existence of the solution at arbitrarily positive initial energy level.

In [18], the authors address spatially periodic breather solutions (SPBs) of the NLS equation. It is known that SPBs are related to the heteroclinic orbits of unstable Stokes waves and are typically unstable. In this paper, the authors study the effects of dissipation on single-mode and multi-mode SPBs using a linearly damped NLS equation. The number of instabilities the background Stokes wave possesses and the damping strength are varied. Viewing the damped dynamics as near integrable, the perturbed flow is analyzed by appealing to the spectral theory of the NLS equation. A broad categorization of the routes to stability of the SPBs and how the route depends on the mode structure of the SPBs and the instabilities of the Stokes wave is obtained, as well as the distinguishing features of the damped flow. The three distinct routes to stability for non-maximal SPBs in $N = 2$ and 3 unstable modes regimes have been broadly categorized.

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