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Birkhoff's theorem in f(R) theory of gravity

P.J. Ravindranath[1], Y. Aditya[2], D.R.K. Reddy[3,a], and M.V. Subba Rao[2]

[1] Department of Mathematics, Rajeev Gandhi Memorial College of Engineering and Technology, Nandyal, India
[2] Department of Mathematics, ANITS (A), Visakhapatnam - 531162, India
[3] Department of Applied Mathematics, Andhra University, Visakhapatnam - 530003, India

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Abstract. Birkhoff's theorem in general relativity states that every spherically symmetric solution of Einstein field equations in vacuum is either static or Schwarzschild. This theorem has been established in the scalar-tensor theories of gravity by Reddy (J. Phys. A **6**, 1867 (1973)). In this paper, we prove this theorem in $f(R)$ theory of gravity (where R Ricci scalar) when the scalaron $\Phi(R)$ of the theory is time-independent only. However, no attention is given to the order of the $f(R)$ theory. Here it is important to note that the Birkhoff theorem in $f(R)$ gravity states only that if R is time-independent, the spherical solution is static (not Schwarzschild).

1 Introduction

Recent astronomical observations, like type-Ia supernovae [1,2], cosmic microwave background (CMB) [3] and large-scale structure [4] indicate that our universe is expanding at an accelerating rate in its present stage. It has been conjectured that this cosmic acceleration is linked with an unusual anti-gravitational force having huge negative pressure dubbed as *dark energy*. In order to study the effects of dark energy, one of the best approaches is the modified theory of gravity. Here, the Einstein-Hilbert action has been generalized leading to various modified (alternative) theories of gravitation, such as scalar-tensor theories (including Brans-Dicke [5] and Saez-Ballester [6] theories), $f(R)$ [7, 8], $f(R,T)$ [9] theories (where R is the Ricci scalar and T is the trace of the energy-momentum tensor), braneworld scenarios (such as DGP and RSI), $f(G)$ theory (G is the Gauss-Bonett term), etc. Among these theories, the one which has received more and more attention in recent times is the $f(R)$ theory, obtained by replacing the Ricci scalar R by an arbitrary function $f(R)$ in the Einstein-Hilbert action. This theory of gravity provides a very natural unification of the early time inflation and late-time acceleration [10].

2 Formulation of f(R) theory of gravity

There are three formalisms which are applied to derive the $f(R)$ gravity field equations. One is the standard metric formalism, while others are the Palatini and metric-affine formalisms. Most of the work in the $f(R)$ modified theory has been done by using the standard metric formalism, where the action is given as follows:

$$S = \int \sqrt{-g}\left(\frac{1}{2\kappa^2}f(R) + \mathscr{L}_m\right)\mathrm{d}^4x, \tag{1}$$

where $\kappa^2 = 8\pi G$ and $f(R)$ is a general function of the Ricci scalar and $\mathscr{L}_m$ is the matter Lagrangian depending upon the metric g_{ij}.

The variation of action (1) with respect to the metric coefficients gives the following field equation:

$$F(R)R_{ij} - \frac{1}{2}f(R)g_{ij} - \nabla_i\nabla_j F(R) + g_{ij}\Box F(R) = \kappa^2 T_{ij}, \tag{2}$$

where $F(R) = \frac{\mathrm{d}f(R)}{\mathrm{d}R}$ and $\Box F(R) = \frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}\, g^{ij}\,\partial_j)$ and T_{ij} is the energy-momentum tensor of matter.

[a] e-mail: reddy_einstein@yahoo.com

By multiplying eq. (2) with g^{ij}, we get

$$F(R)R_i^i - \frac{1}{2}f(R)\delta_i^i - \nabla^i\nabla_i F(R) + \delta_i^i\Box F(R) = \kappa^2 T_i^i, \tag{3}$$

using the fact that $\delta_i^i = 4$ and that $\nabla^i\nabla_i = \Box$. The trace of the field equations is

$$R\,F(R) - 2f(R) + 3\Box F(R) = \kappa^2 T. \tag{4}$$

From eq. (2), we retrieve the Einstein's field equations when $f(R) = R$ and $F(R) = 1$. Also, from eq. (4), we obtain $R = \kappa^2 T$ which shows that the Ricci scalar is determined by matter. From eqs. (2) and (4), it is clear that in $f(R)$ gravity, the term $\Box F(R)$ does not vanish, and has an extra propagating degree of freedom $\Phi \equiv F(R)$, which is usually called *scalaron*. The dynamics of this is determined by eq. (4), which can also be written as $2\,f(R) = R\,F(R) + 3\Box F(R) - \kappa^2 T$. This gives a very important relation between $F(R)$ and $f(R)$, which can be used to simplify the field equations. Many researchers [11–20] have investigated various aspects in $f(R)$ gravity.

Every spherically symmetric vacuum solution of Einstein field equations is static according to the Birkhoff's theorem [21]. The verification of Birkhoff's theorem in alternative theories of gravitation is not new in the literature. Initially, Schücking [22] proved that Birkhoff's theorem is valid in Jordan's [23] extended theory of gravity when the gravitational invariant of the theory is time-independent. Other works in this direction are done by many researchers, like Reddy [24] in BD theory of gravity, Krori and Nandi [25] in both Sen-Dunn [26] and Ross theory [27], Singh [28] in Barber [29] theory, Reddy and Venkateswarlu [30] in scale-covariant theory of Canuto [31] and Reddy [32] in a conformally invariant scalar field theory proposed by Callan *et al.* [33]. On other hand, the Birkhoff's theorem has been extended by Das [34] to the combined electromagnetic and gravitational fields in general relativity. This so-called generalization of Birkhoff's theorem was found in Sen and Dunn [26], Nordtvedt [35], Callan *et al.* [33] and Barber [29] scalar-tensor theories of gravitation [36,37], when the scalar field in the above-said theories is independent of time. The following theorems refer specifically to black holes which could host a wormhole throat or a naked singularity. Hawking [38] has investigated black holes in the Brans-Dicke scalar-tensor theory of gravitation. Mayo and Bekenstein [39] have studied no hair for spherical black holes: charged and nonminimally coupled scalar field with self-interaction. In their work, they have proved three theorems in general relativity which rule out classical scalar hair of static, spherically symmetric, possibly electrically charged black holes. Bekenstein [40] discussed the black hole hair twenty-five years later. Sotiriou and Faraoni [41] studied black holes in Brans-Dicke scalar tensor and $f(R)$ gravity without assuming any symmetries apart from stationarity. Bhattacharya *et al.* [42] discussed Brans-Dicke theory with $\Lambda > 0$ black holes and large scale structures.

The above inspires us to establish Birkhoff's theorem of general relativity in $f(R)$ gravity. This has not been done so far in the literature.

3 Birkhoff's theorem in f(R) theory of gravity

Here, we prove that Birkhoff's theorem exists in the metric formalism of $f(R)$ theory when the scalaron $F(R)$ in this theory is time-independent. It is worth mentioning, here, that Faraoni [43] discussed the validity of Birkhoff's theorem in alternative theories of gravity at length and observed that this theorem is true in the metric formalism of $f(R)$ gravity when R is constant. This fact was observed by him by identifying the equivalence of Brans-Dicke theory with $f(R)$ gravity. He did this without the full set of explicit $f(R)$ gravity field equations for a general spherically symmetric space-time. Here we establish the validity of Birkhoff's theorem in $f(R)$ gravity using the complete set of field equations.

We assume the spherically symmetric metric in the form

$$\mathrm{d}s^2 = \exp(\nu)\,\mathrm{d}t^2 - \exp(\mu)\,\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{5}$$

where μ and ν are functions of both radial coordinate r and time t. The corresponding Ricci scalar is

$$R = \left(\frac{2\mu'}{r} - \nu'' - \frac{\nu'^2}{2} + \frac{\mu'\nu'}{2} - \frac{2\nu'}{r} - \frac{2}{r^2}\right)\exp(-\mu) + \left(\ddot{\mu} - \frac{\dot{\mu}\dot{\nu}}{2} + \frac{\dot{\mu}^2}{2}\right)\exp(-\nu) + \frac{2}{r^2}\,. \tag{6}$$

The $f(R)$ gravity vacuum field equations for metric (5) become (here the scalaron $F(r,t) \equiv \Phi(r,t)$)

$$\left(\frac{\ddot{\Phi}}{\Phi} - \frac{\nu'\Phi'}{2\Phi}\exp(\nu-\mu) - \frac{\dot{\nu}\dot{\Phi}}{2\Phi} + \frac{\ddot{\mu}}{2} - \frac{\dot{\mu}\dot{\nu}}{4} + \frac{\dot{\mu}^2}{4}\right)\exp(-\nu) - \frac{f(R)}{2\Phi} + \left(\frac{\mu'}{r} - \frac{\nu''}{2} - \frac{\nu'^2}{4} + \frac{\mu'\nu'}{4} - \frac{2\Phi'}{r\Phi}\right)\exp(-\mu) = 0, \tag{7}$$

$$\left(\frac{\mu'}{2r}-\frac{1}{r^2}-\frac{\nu'}{2r}-\frac{\Phi''}{\Phi}+\frac{\Phi'}{\Phi}\left(\frac{\mu'}{2}-\frac{1}{r}\right)+\frac{\dot{\mu}\dot{\Phi}}{2\Phi}\exp(\mu-\nu)\right)\exp(-\mu)$$

$$+\frac{1}{r^2}-\frac{f(R)}{2\Phi}+\left(\frac{\ddot{\Phi}}{\Phi}-\frac{\nu'\Phi'}{2\Phi}\exp(\nu-\mu)-\frac{\dot{\nu}\dot{\Phi}}{2\Phi}\right)\exp(-\nu)=0, \tag{8}$$

$$\left(-\frac{\nu'}{r}-\frac{\nu''}{2}-\frac{\nu'^2}{4}+\frac{\mu'\nu'}{4}-\frac{\Phi''}{\Phi}+\frac{\Phi'}{\Phi}\left(\frac{\mu'}{2}-\frac{2}{r}\right)+\frac{\dot{\mu}\dot{\Phi}}{2\Phi}\exp(\mu-\nu)\right)\exp(-\mu)$$

$$\left(\frac{\ddot{\mu}}{2}+\frac{\dot{\mu}^2}{4}-\frac{\dot{\mu}\dot{\nu}}{4}\right)\exp(-\nu)-\frac{f(R)}{2\Phi}=0, \tag{9}$$

$$\exp(-\mu)\left(\frac{\dot{\mu}}{r}+\frac{\dot{\Phi}'}{\Phi}-\frac{\dot{\mu}\Phi'}{2\Phi}-\frac{\nu'\dot{\Phi}}{2\Phi}\right)=0, \tag{10}$$

where prime ($'$) denotes partial differentiation with respect to radial coordinate r and dot ($\cdot$) denotes partial differentiation with respect to time t. When $f(R)=R$ and $\Phi=1$ the above system of equations reduces to the Einstein vacuum field equations in the spherically symmetric case and hence Birkhoff's theorem holds.

When the scalaron Φ is a function of r alone, *i.e.*, $\dot{\Phi}=0$, from eq. (10), we get

$$\dot{\mu}\left(\frac{1}{r}-\frac{\Phi'}{\Phi}\right)\exp(-\mu)=0, \tag{11}$$

which leads to either

$$\dot{\mu}=0, \tag{12}$$

or

$$\Phi=\Phi_0\,r^2, \tag{13}$$

where $\Phi_0>0$ is an integration constant.

Now, subtracting eq. (7) from (8), we get

$$(\mu'+\nu')\left(\frac{1}{r}-\frac{\Phi'}{2\Phi}\right)+\frac{\Phi''}{\Phi}=0. \tag{14}$$

Partially differentiating this equation with respect to time t, we obtain

$$(\dot{\mu}'+\dot{\nu}')\left(\frac{1}{r}-\frac{\Phi'}{2\Phi}\right)+(\mu'+\nu')\left(\frac{\dot{\Phi}\Phi'-\Phi\dot{\Phi}'}{2\Phi^2}\right)+\left(\frac{\Phi\dot{\Phi}''-\Phi''\dot{\Phi}}{\Phi^2}\right)=0. \tag{15}$$

In view of $\dot{\Phi}=0$ and eq. (12), from eq. (15) we obtain

$$\dot{\nu}'=0. \tag{16}$$

This implies that ν is linearly separable in t and r, so that

$$\nu=g_1(t)+g_2(r), \tag{17}$$

where g_1 and g_2 are arbitrary functions of t and r, respectively.

The above equation can also be written as

$$e^{\nu}\,\mathrm{d}t^2=e^{g_2(r)}\left(e^{\frac{g_1(t)}{2}}\,\mathrm{d}t\right)^2. \tag{18}$$

Now, introducing a time transformation [34,36],

$$\mathrm{d}\bar{t}=e^{\frac{g_1(t)}{2}}\,\mathrm{d}t, \tag{19}$$

and dropping over line afterwards, it follows, in view of of (12), that the metric (5) is static. Also, from eqs. (13) and (15), we can conclude that no solution exists in this case. Hence Birkhoff's theorem is proved in the metric formalism of $f(R)$ gravity when the scalaron is time-independent.

Discussion

The possible validity of Birkhoff's theorem has been discussed by many authors. It is also true that they all concluded that this theorem holds in $f(R)$ theory of gravity for particular cases, like $f(R) = R^{1+\delta}$ (where $\delta \neq 0$) [44–49] as well as those derived from $R + \frac{\alpha}{R^n}$ (where $n > 0$) [50, 51]. Also, Barraco and Hamity [52] showed that Birkhoff's theorem is valid for $f(R) = aR + bR^2$.

In refs. [53–56] it was found that Birkhoff's theorem holds upto second-order $f(R)$ theory and there exist spherically symmetric vacuum solutions to fourth-order theories of gravity. Also, Capozziello *et al.* [56] mentioned that the validity of Birkhoff's theorem is demonstrated for $f(R)$ theories only when the Ricci scalar is time-independent. They have also stated that Birkhoff's theorem does not represent a general feature of fourth-order gravity. Here we have established Birkhoff's theorem of general relativity through the explicit field equations of $f(R)$ gravity (without any reference to the order of the theory) considering the scalaron $\Phi(R)$ to be completely time-independent only. In fact, this is what was done in the Brans-Dicke theory to establish Birkhoff's theorem by Reddy [24].

4 Conclusion

Modified gravity theories have become very important in modern cosmology in view of the recent scenario of the accelerated expansion of the universe. In this context, $f(R)$ gravity is another alternative to the Brans-Dicke theory which has become very significant. In this paper, we have discussed in detail the validity of Birkhoff's theorem in the metric formalism of $f(R)$ theory. We have shown that this theorem is true in this theory when the scalaron is independent of the time coordinate. This fact physically means that the time-dependent gravitational radiation does not exist in vacuum when the scalaron of the theory is time-independent. This result is in accordance with the observations made by Faraouni [43] and Capozziello and Gomez [57].

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