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Construction of the energy matrix for complex atoms Part II: Explicit formulae for inter-configuration interactions

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Abstract. Formulae for the first-order electrostatic interaction between configurations up to four open shells (see Part I in Eur. Phys. J. Plus **130**, 14 (2015)) are presented. They form the basis for the design of an efficient computer package allowing large-scale calculations of fine and hyperfine structure and predict spectral line strengths of complex atoms.

1 Introduction

We have developed a method which allows to analyse complex electronic systems composed of a configuration of up to four open shells, taking into account all electromagnetic interactions expected in an atom in accordance with the second-order perturbation theory. Within this theory all possible combinations following the excitation of one or two electrons from closed shells to particular open shells are considered [1]. Appropriate formulae and computer codes have been developed for many years by our research group.

Recently, we have presented a method for determining oscillator strengths that is an alternative to the commonly used, and purely theoretical calculations, or to the semi-empirical approach combined with theoretically calculated transition integrals [2–4]. Accurate oscillator strengths (gf-values) are among the most important kinds of atomic data. They are of particular importance in astronomy, for reliable determination of chemical abundances in stellar atmospheres, in plasma physics, and for the comparison with theoretical works. It would be interesting to perform a similar analysis for rare-earths elements spectra, for which a large number of experimental data has been provided in recent years.

In the analysis of the spectra of complex atoms there appear electrostatic interactions between many types of configurations involving up to four open electronic shells. In this case constructing the energy matrix is more complicated than in the case previously described in [5,6]. In this work those missing interactions were added. These formulae were used by us for the interpretation of the spectra of europium, praseodymium and tantalum atoms [7–9].

Our earlier analysis for europium and praseodymium atoms concerned only the configuration system of one parity [7, 8]. In the even configuration system of Pr I, the configurations $4f^2 5d6s^2$, $4f^2 5d^2 6s$, $4f^3 6s6p$, $4f^3 5d6p$ and $4f^2 5d^3$ were taken into account. In the odd system, besides the configuration with two and three open shells, we must take into account also 4f5d6s6p configuration. This required the introduction of new formulae, which we present in this work.

The correctness of the presented formulae has been verified by comparing the energy eigenvalues obtained in our computer package and the values from Cowan-code [10,11]. In order to verify the phase relationship, these comparisons were made for systems of at least three mutually interacting configurations. Direct comparison of matrix elements was not possible because of the different coupling schemes.

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2 Explanation of used symbols

In all the formulae given below, the symbol $\mathbf{G}^t$ stands for a particular term of Coulomb repulsion represented by irreducible tensors of rank t: $\sum_{i>j} r_<^t/r_>^{t+1}(\mathbf{C}_i^t \cdot \mathbf{C}_j^t)$, where $r_<$ and $r_>$ indicate the distances from the nucleus to the nearer and the further away electron, respectively. The summation over t is omitted. The expressions describing $\mathbf{G}^t$ element contain coupling schemes used for the derivation of the formula.

For nj-coefficients, one- and two-particle fractional parentage coefficients, the generally accepted notations were used.

The expression $[x, y]$ represents $(2x+1)(2y+1)$. Reduced matrix elements $\mathbf{C}^t$ and $\mathbf{U}^t$ represent

$$\left(l_1 \left\|\mathbf{C}^t\right\| l_2\right) = (-1)^{l_1}\left[(2l_1+1)(2l_2+1)\right]^{1/2}\begin{pmatrix} l_1 & t & l_2 \\ 0 & 0 & 0 \end{pmatrix} \tag{1}$$

$$\left\langle nl^N\alpha_0 S_0 L_0 \left\|\mathbf{U}^t\right\| nl^N\alpha_0' S_0' L_0'\right\rangle = \delta(S_0, S_0')\ N\ (-1)^{L_0+l+t}\ [L_0, L_0']^{1/2} \times \sum_{\bar\alpha\bar S\bar L}(-1)^{\bar L}\left(nl^N\alpha_0 S_0 L_0\{|nl^{N-1}\bar\alpha\bar S\bar L\right)\left(nl^N\alpha_0' S_0' L_0'\{|nl^{N-1}\bar\alpha\bar S\bar L\right)\begin{Bmatrix} l & l & t \\ L_0 & L_0' & \bar L \end{Bmatrix}. \tag{2}$$

The consideration of the configuration with three open shells, where the second and third shell contain up to three electrons, requires the coupling of the four angular momenta. It is therefore necessary to use $12j$-coefficients, introduced by Jahn and Hoppe [12] and studied by Ord-Smith [13], who found 16 symmetry relations together with a convenient new notation. In the comprehensive work of Jucys *et al.* [14], the $12j$-coefficients of this form were referred to as symbols of the first kind. The more convenient in use, symbols of the second kind, with 24 symmetry properties, were introduced. A number of useful sum rules on nj-coefficients, which we used to derive our formulae, were also presented.

The consideration of the configuration with four open shells requires coupling of up to five angular momenta. It is therefore necessary to use nj-coefficients up to $15j$-symbols. The definition of $15j$-coefficients, their symmetry relations and the sum rules were introduced by Jucys *et al.* [14].

3 Explicit formulae for inter-configuration interactions

The formulae for the first-order electrostatic interaction between the configurations containing up to four open shells are presented below.

Some of formulae were used for the fine structure analysis of the odd configurations system of the europium atom: $4f^7 6s^2$, $4f^7 5d6s$, $4f^7 5d7s$, $4f^7 6snd$ (n=6,...,9), $4f^7 5d^2$, $4f^7 6p^2$, $4f^7 6sns$ (n=7,...,10), $4f^6 6s^2 np$ (n=6,7), $4f^6 5d6s6p$. The wave functions obtained from our analysis of the fine structure (*fs*) were used in an analysis of the hyperfine structure splittings. For most of the levels with known hyperfine structure (*hfs*), a satisfactory interpretation of *hfs* are obtained [7].

3.1 Configurations with two open shells

– Interaction $l^N l_1 \leftrightarrow l^{N-2} l_1^3$:

$$\left\langle nl^N\alpha_0 S_0 L_0, n_1 l_1; SL \left|\mathbf{G^k}\right| nl^{N-2}\alpha_0' S_0' L_0', n_1 l_1^3\alpha_1' S_1' L_1'; SL\right\rangle = \sqrt{\frac{3N(N-1)}{2}}\sum_{\hat\alpha\hat S\hat L}\left(nl^N\alpha_0 S_0 L_0\{|nl^{N-2}\alpha_0' S_0' L_0', nl^2\hat\alpha\hat S\hat L\right)\left(n_1 l_1^3\alpha_1' S_1' L_1'\{|n_1 l_1^2\hat\alpha\hat S\hat L\right)[S_0, L_0, S_1', L_1']^{1/2} \times (-1)^{2S_1'+S_0'+L_0'+S+L+\hat L+l+3/2}\begin{Bmatrix} 1/2 & S & S_0 \\ S_0' & \hat S & S_1' \end{Bmatrix}\begin{Bmatrix} l_1 & L & L_0 \\ L_0' & \hat L & L_1' \end{Bmatrix}\begin{Bmatrix} l & l_1 & t \\ l_1 & l & \hat L \end{Bmatrix}\left(l\left\|\mathbf{C}^t\right\| l_1\right)^2 R^t\left(nlnl, n_1 l_1 n_1 l_1\right). \tag{3}$$

3.2 Configurations with three open shells

– Interaction $l^N l_1^3 \leftrightarrow l^N l_2^2 l_1$:

$$\left\langle nl^N\alpha_0 S_0 L_0, n_1 l_1^3\alpha_2 S_2 L_2; SL\left|\mathbf{G^k}\right| nl^N\alpha_0' S_0' L_0', (n_2 l_2^2\alpha_1' S_1' L_1', n_1 l_1)S_2' L_2'; SL\right\rangle = \sqrt{3}\ \delta(\alpha_0 S_0 L_0, \alpha_0' S_0' L_0')\ \delta(S_2 L_2, S_2' L_2')\left(n_1 l_1^3\alpha_2 S_2 L_2\{|n_1 l_1^2\alpha_1' S_1' L_1'\right)(-1)^{l_1+l_2+L_1'} \times\begin{Bmatrix} l_1 & l_2 & t \\ l_2 & l_1 & L_1' \end{Bmatrix}\left(l_1\left\|\mathbf{C}^t\right\| l_2\right)^2 R^t\left(n_1 l_1 n_1 l_1, n_2 l_2 n_2 l_2\right). \tag{4}$$

– Interaction $l^N l_1 l_2^2 \leftrightarrow l^N l_1^3$:

$$\begin{aligned}
&\left\langle (nl^N\alpha_0 S_0 L_0, n_1 l_1) S_1 L_1, n_2 l_2^2 \alpha_2 S_2 L_2; SL \left| \mathbf{G^k} \right| nl^N \alpha_0' S_0' L_0', n_1 l_1^3 \alpha_2' S_2' L_2'; SL \right\rangle = \\
&\sqrt{3}\ \delta(\alpha_0 S_0 L_0, \alpha_0' S_0' L_0') \left(n_1 l_1^3 \alpha_2' S_2' L_2' \{| n_1 l_1^2 \alpha_2 S_2 L_2\right)\ (-1)^{S_0'+L_0'+S+L+3S_2'+L_2'+L_2+l_1+l_2+1} [S_1, L_1, S_2', L_2']^{1/2} \\
&\times \begin{Bmatrix} S_2 & S & S_1 \\ S_0' & 1/2 & S_2' \end{Bmatrix} \begin{Bmatrix} L_2 & L & L_1 \\ L_0' & l_1 & L_2' \end{Bmatrix} \begin{Bmatrix} l_2 & l_1 & t \\ l_1 & l_2 & L_2 \end{Bmatrix} \left(l_1 \left\| \mathbf{C}^t \right\| l_2\right)^2 R^t\left(n_1 l_1 n_1 l_1, n_2 l_2 n_2 l_2\right). \qquad (5)
\end{aligned}$$

– Interaction $l^N l_1 l_3^2 \leftrightarrow l^N l_2 l_3^2$:

$$\begin{aligned}
&\left\langle (nl^N\alpha_0 S_0 L_0, n_1 l_1) S_1 L_1, n_3 l_3^2 \alpha_2 S_2 L_2; SL \left| \mathbf{G^k} \right| (nl^N \alpha_0' S_0' L_0', n_2 l_2) S_1' L_1', n_3 l_3^2 \alpha_2' S_2' L_2'; SL \right\rangle = \\
&2\delta(\alpha_0 S_0 L_0, \alpha_0' S_0' L_0')\ [L_1, L_1', L_2, L_2']^{1/2} \\
&\times \left[\delta(S_1, S_1')\ \delta(S_2, S_2')\ (-1)^{l_2+L+L_0+L_1+L_1'} \begin{Bmatrix} L_2' & L_2 & t \\ l_3 & l_3 & l_3 \end{Bmatrix} \begin{Bmatrix} L_2' & L_2 & t \\ L_1 & L_1' & L \end{Bmatrix} \begin{Bmatrix} l_2 & l_1 & t \\ L_1 & L_1' & L_0 \end{Bmatrix} \right. \\
&\times \left(l_1 \left\| \mathbf{C}^t \right\| l_2\right) \left(l_3 \left\| \mathbf{C}^t \right\| l_3\right) R^t\left(n_1 l_1 n_3 l_3, n_2 l_2 n_3 l_3\right) \\
&+ (-1)^{S_2+L_2+3S_2'+L_2'+l_1+l_3+1}\ [S_1, S_1', S_2, S_2']^{1/2} \begin{Bmatrix} S_0 & 1/2 & S_1 \\ 1/2 & 1/2 & S_2 \\ S_1' & S_2' & S \end{Bmatrix} \begin{Bmatrix} L_0 & l_1 & L_1 & L_2 \\ l_2 & t & l_3 & L_2' \\ L_1' & l_3 & L & l_3 \end{Bmatrix} \\
&\left. \times \left(l_1 \left\| \mathbf{C}^t \right\| l_3\right) \left(l_3 \left\| \mathbf{C}^t \right\| l_2\right) R^t\left(n_1 l_1 n_3 l_3, n_3 l_3 n_2 l_2\right) \right]. \qquad (6)
\end{aligned}$$

– Interaction $l^N l_1 l_2^2 \leftrightarrow l^N l_3^2 l_1$:

$$\begin{aligned}
&\left\langle (nl^N\alpha_0 S_0 L_0, n_1 l_1) S_1 L_1, n_2 l_2^2 \alpha_2 S_2 L_2; SL \left| \mathbf{G^k} \right| nl^N \alpha_0' S_0' L_0', (n_3 l_3^2 \alpha_1' S_1' L_1', n_1 l_1) S_2' L_2'; SL \right\rangle = \\
&\delta(\alpha_0 S_0 L_0, \alpha_0' S_0' L_0')\ \delta(\alpha_2 S_2 L_2, \alpha_1' S_1' L_1')\ \ (-1)^{S+L+S_0+L_0+L_2+3S_2'+L_2'+l_2+l_3+1}\ [S_1, L_1, S_2', L_2']^{1/2} \\
&\times \begin{Bmatrix} S_2 & S & S_1 \\ S_0 & 1/2 & S_2' \end{Bmatrix} \begin{Bmatrix} L_2 & L & L_1 \\ L_0 & l_1 & L_2' \end{Bmatrix} \begin{Bmatrix} l_2 & l_3 & t \\ l_3 & l_2 & L_2 \end{Bmatrix} \left(l_2 \left\| \mathbf{C}^t \right\| l_3\right)^2 R^t\left(n_2 l_2 n_2 l_2, n_3 l_3 n_3 l_3\right). \qquad (7)
\end{aligned}$$

– Interaction $l^N l_1^2 l_2 \leftrightarrow l^N l_1^3$:

$$\begin{aligned}
&\left\langle nl^N\alpha_0 S_0 L_0, (n_1 l_1^2 \alpha_1 S_1 L_1, n_2 l_2) S_2 L_2; SL \left| \mathbf{G^k} \right| nl^N \alpha_0' S_0' L_0', n_1 l_1^3 \alpha_2' S_2' L_2'; SL \right\rangle = \\
&2\sqrt{3}\ \delta(\alpha_0 S_0 L_0, \alpha_0' S_0' L_0')\ \delta(S_2 L_2, S_2' L_2') \sum_{\hat\alpha \hat S \hat L} \left(n_1 l_1^3 \alpha_2' S_2' L_2' \{| n_1 l_1^2 \hat\alpha \hat S \hat L\right)\ (-1)^{2S_0+2S+\hat L+l_1+l_2+1} \\
&\times [S_1, L_1, \hat S, \hat L]^{1/2}\ \begin{Bmatrix} 1/2 & 1/2 & S_1 \\ 1/2 & S_2 & \hat S \end{Bmatrix} \begin{Bmatrix} l_1 & l_1 & L_1 \\ l_2 & L_2 & \hat L \end{Bmatrix} \begin{Bmatrix} l_1 & l_1 & t \\ l_2 & l_1 & \hat L \end{Bmatrix} \left(l_1 \left\| \mathbf{C}^t \right\| l_1\right) \left(l_1 \left\| \mathbf{C}^t \right\| l_2\right) R^t\left(n_1 l_1 n_1 l_1, n_1 l_1 n_2 l_2\right) \\
&+ \sqrt{3}\ \left(n_1 l_1^3 \alpha_2' S_2' L_2' \{| n_1 l_1^2 \alpha_1 S_1 L_1\right)\ (-1)^{l_2}\ [L_2, L_2']^{1/2} \\
&\times \left[\delta(S_0, S_0') \delta(S_2, S_2')\ (-1)^{L_1+L_0'+L+\hat L} \left\langle nl^N \alpha_0 S_0 L_0 \left\| \mathbf{U}^t \right\| nl^N \alpha_0' S_0' L_0' \right\rangle \begin{Bmatrix} L_2' & L_2 & t \\ l_2 & l_1 & L_1 \end{Bmatrix} \right. \\
&\times \begin{Bmatrix} L_2' & L_2 & t \\ L_0 & L_0' & \hat L \end{Bmatrix} \left(l \left\| \mathbf{C}^t \right\| l\right) \left(l_2 \left\| \mathbf{C}^t \right\| l_1\right) R^t\left(nl n_2 l_2, nl n_1 l_1\right) \\
&+ [S_0, L_0, S_0', L_0', S_2, S_2']^{1/2}\ N \sum_{\bar\alpha \bar S \bar L} \left(nl^N \alpha_0 S_0 L_0 \{| nl^{N-1} \bar\alpha \bar S \bar L\right) \left(nl^N \alpha_0' S_0' L_0' \{| nl^{N-1} \bar\alpha \bar S \bar L\right) \\
&\left. \times (-1)^{S_2+L_2+3S_2'+L_2'+l+1} \begin{Bmatrix} \bar S & 1/2 & S_0 \\ 1/2 & S_1 & S_2 \\ S_0' & S_2' & S \end{Bmatrix} \begin{Bmatrix} \bar L & l & L_0 & L_2 \\ l & t & l_2 & L_2' \\ L_0' & l_1 & L & L_1 \end{Bmatrix} \left(l \left\| \mathbf{C}^t \right\| l_1\right) \left(l_2 \left\| \mathbf{C}^t \right\| l\right) R^t\left(nl n_2 l_2, n_1 l_1 nl\right) \right]. \qquad (8)
\end{aligned}$$

– Interaction $l^{N-1}l_1^2l_3 \leftrightarrow l^N l_2 l_1$:

$$\begin{aligned}
&\left\langle nl^{N-1}\alpha_0 S_0 L_0,(n_1 l_1^2\alpha_2 S_2 L_2, n_3 l_3)S_3 L_3; SL \left|\mathbf{G^k}\right| nl^N\alpha_1' S_1' L_1',(n_2 l_2 n_1 l_1)S_2' L_2'; SL\right\rangle = \\
&\sqrt{2N}\ \left(nl^N\alpha_1' S_1' L_1' \{|nl^{N-1}\alpha_0 S_0 L_0\right)\ (-1)^{S_0+L_0+S+L+S_3+L_2+l}\ [S_3,L_3,S_1',L_1',L_2,L_2']^{1/2} \\
&\times \begin{Bmatrix} S_0 & 1/2 & S_1' \\ S_2' & S & S_3 \end{Bmatrix} \begin{Bmatrix} L_0 & l & L_1' \\ L_2' & L & L_3 \end{Bmatrix} \\
&\times \Bigg[(-1)^{L_2'+L_3+l_2}[S_2,S_2']^{1/2} \begin{Bmatrix} 1/2 & 1/2 & S_2 \\ S_3 & 1/2 & S_2' \end{Bmatrix} \begin{Bmatrix} l_1 & l_2 & L_2' \\ l_1 & t & l \\ L_2 & l_3 & L_3 \end{Bmatrix} \left(l_1 \left\|\mathbf{C}^t\right\| l\right) \left(l_3 \left\|\mathbf{C}^t\right\| l_2\right) R^t\left(n_1 l_1 n_3 l_3, nl n_2 l_2\right) \\
&+ \delta(S_2,S_2')\ (-1)^{S_2'} \begin{Bmatrix} t & l_1 & l_2 \\ l_1 & L_2' & L_2 \end{Bmatrix} \begin{Bmatrix} L_2 & t & L_2' \\ l & L_3 & l_3 \end{Bmatrix} \left(l_1 \left\|\mathbf{C}^t\right\| l_2\right) \left(l_3 \left\|\mathbf{C}^t\right\| l\right) R^t\left(n_1 l_1 n_3 l_3, n_2 l_2 nl\right) \Bigg].
\end{aligned} \tag{9}$$

– Interaction $l^N l_1^2 l_2 \leftrightarrow l^N l_1 l_3^2$:

$$\begin{aligned}
&\left\langle nl^N\alpha_0 S_0 L_0,(n_1 l_1^2\alpha_1 S_1 L_1, n_2 l_2)S_2 L_2; SL \left|\mathbf{G^k}\right| (nl^N\alpha_0' S_0' L_0', n_1 l_1)S_1' L_1', n_3 l_3^2\alpha_2' S_2' L_2'; SL\right\rangle = \\
&2\ \delta(\alpha_0 S_0 L_0,\alpha_0' S_0' L_0')\ (-1)^{S+L+S_0+L_0+S_2+L_2+L_2'+l_2+l_3}\ [S_1,L_1,S_1',L_1',S_2,L_2,S_2',L_2']^{1/2} \\
&\times \begin{Bmatrix} 1/2 & 1/2 & S_1 \\ 1/2 & S_2 & S_2' \end{Bmatrix} \begin{Bmatrix} l_1 & l_1 & L_1 \\ l_2 & L_2 & L_2' \end{Bmatrix} \begin{Bmatrix} S_0 & 1/2 & S_1' \\ S_2' & S & S_2 \end{Bmatrix} \begin{Bmatrix} L_0 & l_1 & L_1' \\ L_2' & L & L_2 \end{Bmatrix} \\
&\times \begin{Bmatrix} l_3 & l_1 & t \\ l_2 & l_3 & L_2' \end{Bmatrix} \left(l_3 \left\|\mathbf{C}^t\right\| l_1\right) \left(l_3 \left\|\mathbf{C}^t\right\| l_2\right) R^t\left(n_1 l_1 n_2 l_2, n_3 l_3 n_3 l_3\right).
\end{aligned} \tag{10}$$

– Interaction $l^N l_1^2 l_2 \leftrightarrow l^N l_3^2 l_2$:

$$\begin{aligned}
&\left\langle nl^N\alpha_0 S_0 L_0,(n_1 l_1^2\alpha_1 S_1 L_1, n_2 l_2)S_2 L_2; SL \left|\mathbf{G^k}\right| nl^N\alpha_0' S_0' L_0',(n_3 l_3^2\alpha_1' S_1' L_1', n_2 l_2)S_2' L_2'; SL\right\rangle = \\
&\delta(\alpha_0 S_0 L_0,\alpha_0' S_0' L_0')\ \delta(\alpha_1 S_1 L_1,\alpha_1' S_1' L_1')\ \delta(S_2 L_2, S_2' L_2')\ (-1)^{2S_0+2S+2S_2+l_1+l_3+L_1} \begin{Bmatrix} l_1 & l_3 & t \\ l_3 & l_1 & L_1 \end{Bmatrix} \\
&\times \left(l_1 \left\|\mathbf{C}^t\right\| l_3\right)^2 R^t\left(n_1 l_1 n_1 l_1, n_3 l_3 n_3 l_3\right).
\end{aligned} \tag{11}$$

– Interaction $l^N l_3 l_1^2 \leftrightarrow l^N l_1^2 l_2$:

$$\begin{aligned}
&\left\langle (nl^N\alpha_0 S_0 L_0, n_3 l_3)S_1 L_1, n_1 l_1^2\alpha_2 S_2 L_2; SL \left|\mathbf{G^k}\right| nl^N\alpha_0' S_0' L_0',(n_1 l_1^2\alpha_1' S_1' L_1', n_2 l_2)S_2' L_2'; SL\right\rangle = \\
&2\ \delta(\alpha_0 S_0 L_0,\alpha_0' S_0' L_0')\ (-1)^{S_0+L_0+S+L+S_2'+L_1'+L_2+l_2}\ [S_1,L_1,L_1',L_2,S_2',L_2']^{1/2} \begin{Bmatrix} S_0 & 1/2 & S_1 \\ S_2 & S & S_2' \end{Bmatrix} \begin{Bmatrix} L_0 & l_3 & L_1 \\ L_2 & L & L_2' \end{Bmatrix} \\
&\times \Bigg[\delta(S_2,S_1') \begin{Bmatrix} L_1' & l_1 & l_1 \\ l_1 & t & L_2 \end{Bmatrix} \begin{Bmatrix} L_2 & L_1' & t \\ l_2 & l_3 & L_2' \end{Bmatrix} \left(l_3 \left\|\mathbf{C}^t\right\| l_2\right) \left(l_1 \left\|\mathbf{C}^t\right\| l_1\right) R^t\left(n_3 l_3 n_1 l_1, n_2 l_2 n_1 l_1\right) \\
&+ (-1)^{L_2'+l_1}[S_1',S_2]^{1/2} \begin{Bmatrix} 1/2 & 1/2 & S_2 \\ S_2' & 1/2 & S_1' \end{Bmatrix} \begin{Bmatrix} L_2' & l_2 & L_1' \\ l_3 & t & l_1 \\ L_2 & l_1 & l_1 \end{Bmatrix} \left(l_3 \left\|\mathbf{C}^t\right\| l_1\right) \left(l_1 \left\|\mathbf{C}^t\right\| l_2\right) R^t\left(n_3 l_3 n_1 l_1, n_1 l_1 n_2 l_2\right) \Bigg] \\
&+ \delta(\alpha_2 S_2 L_2,\alpha_1' S_1' L_1')\ (-1)^{1+3S_2'+L_2'+S_0+S+L+l_3}\ [S_1,L_1,S_2',L_2']^{1/2} \begin{Bmatrix} S_1' & S & S_1 \\ S_0' & 1/2 & S_2' \end{Bmatrix} \begin{Bmatrix} L_1' & L & L_1 \\ L_0' & l_2 & L_2' \end{Bmatrix} \\
&\times \Bigg[\delta(S_0,S_0')(-1)^{L_1} \left\langle nl^N\alpha_0 S_0 L_0 \left\|\mathbf{U}^t\right\| nl^N\alpha_0' S_0' L_0'\right\rangle \begin{Bmatrix} l_2 & t & l_3 \\ L_0 & L_1 & L_0' \end{Bmatrix} \left(l \left\|\mathbf{C}^t\right\| l\right) \left(l_3 \left\|\mathbf{C}^t\right\| l_2\right) R^t\left(nl n_3 l_3, nl n_2 l_2\right) \\
&+ [S_0,L_0,S_0',L_0']^{1/2}\ (-1)^{l+2S_0'+L_0'}\ N \sum_{\bar\alpha\bar S\bar L} \left(nl^N\alpha_0 S_0 L_0\{|nl^{N-1}\bar\alpha\bar S\bar L\right) \left(nl^N\alpha_0' S_0' L_0'\{|nl^{N-1}\bar\alpha\bar S\bar L\right) \\
&\times \begin{Bmatrix} 1/2 & \bar S & S_0' \\ 1/2 & S_1 & S_0 \end{Bmatrix} \begin{Bmatrix} \bar L & l & L_0' \\ l & t & l_2 \\ L_0 & l_3 & L_1 \end{Bmatrix} \left(l \left\|\mathbf{C}^t\right\| l_2\right) \left(l_3 \left\|\mathbf{C}^t\right\| l\right) R^t\left(nl n_3 l_3, n_2 l_2 nl\right) \Bigg].
\end{aligned} \tag{12}$$

3.3 Configurations with four open shells

– Interaction $l^N l_1 \leftrightarrow l^{N-2} l_1 l_2 l_3$:

$$\begin{aligned}&\left\langle nl^N \alpha_0 S_0 L_0, n_1 l_1; SL \left| \mathbf{G^k} \right| (nl^{N-2} \alpha_0' S_0' L_0', n_1 l_1) S_1' L_1', (n_2 l_2 n_3 l_3) S_2' L_2'; SL \right\rangle = \\ &\sqrt{N(N-1)} \left(nl^N \alpha_0 S_0 L_0 \{ | nl^{N-2} \alpha_0' S_0' L_0', nl^2 \alpha_2' S_2' L_2' \right) \ (-1)^{S_0+L_0+S_1'+L_1'+L_2'+l+l_1+l_2+1/2} \\ &\times [S_0, L_0, S_1', L_1']^{1/2} \ \begin{Bmatrix} S_2' & S_0' & S_0 \\ 1/2 & S & S_1' \end{Bmatrix} \begin{Bmatrix} L_2' & L_0' & L_0 \\ l_1 & L & L_1' \end{Bmatrix} \begin{Bmatrix} l & l_2 & t \\ l_3 & l & L_2' \end{Bmatrix} \left(l \left\| \mathbf{C}^t \right\| l_2 \right) \left(l \left\| \mathbf{C}^t \right\| l_3 \right) R^t \left(nlnl, n_2 l_2 n_3 l_3 \right). \end{aligned} \tag{13}$$

– Interaction $l^N l_2 \leftrightarrow l^{N-2} l_1 l_2 l_3$:

$$\begin{aligned}&\left\langle nl^N \alpha_0 S_0 L_0, n_2 l_2; SL \left| \mathbf{G^k} \right| (nl^{N-2} \alpha_0' S_0' L_0', n_1 l_1) S_1' L_1', (n_2 l_2 n_3 l_3) S_2' L_2'; SL \right\rangle = \\ &\sqrt{N(N-1)} \sum_{\hat\alpha \hat S \hat L} \left(nl^N \alpha_0 S_0 L_0 \{ | nl^{N-2} \alpha_0' S_0' L_0', nl^2 \hat\alpha \hat S \hat L \right) \begin{Bmatrix} S_1' & S_2' & S \\ 1/2 & S_0 & 1/2 \end{Bmatrix} \begin{Bmatrix} 1/2 & 1/2 & \hat S \\ S_0 & S_0' & S_1' \end{Bmatrix} \\ &\times [S_1', L_1', S_2', L_2', \hat S, \hat L, S_0, L_0]^{1/2} \ (-1)^{S_0'+\hat S+S_1'+S_2'+3S+3S_0+L_0+L_0'+L_1'+L_2'+L+l+l_3+1} \begin{Bmatrix} l & l_1 & t \\ l_3 & l & \hat L \end{Bmatrix} \\ &\times \begin{Bmatrix} l_1 & l_3 & \hat L \\ L_0 & L_0' & L_1' \end{Bmatrix} \begin{Bmatrix} L_1' & L_2' & L \\ l_2 & L_0 & l_3 \end{Bmatrix} \left(l \left\| \mathbf{C}^t \right\| l_1 \right) \left(l \left\| \mathbf{C}^t \right\| l_3 \right) R^t \left(nlnl, n_1 l_1 n_3 l_3 \right). \end{aligned} \tag{14}$$

– Interaction $l^{N+1} l_1^2 \leftrightarrow l^N l_1 l_2 l_3$:

$$\begin{aligned}&\left\langle nl^{N+1} \alpha_1 S_1 L_1, n_1 l_1^2 \alpha_2 S_2 L_2; SL \left| \mathbf{G^k} \right| (nl^N \alpha_0' S_0' L_0', n_1 l_1) S_1' L_1', (n_2 l_2 n_3 l_3) S_2' L_2'; SL \right\rangle = \\ &\sqrt{2(N+1)} \left(nl^{N+1} \alpha_1 S_1 L_1 \{ | nl^N \alpha_0' S_0' L_0' \right) [S_1, L_1, S_2, L_2, S_1', L_1', S_2', L_2']^{1/2} \\ &\times (-1)^{l_1+l_2+1} \ \begin{Bmatrix} S_0' & 1/2 & S_1 \\ 1/2 & 1/2 & S_2 \\ S_1' & S_2' & S \end{Bmatrix} \begin{Bmatrix} L_0' & l & L_1 \\ l_1 & l_1 & L_2 \\ L_1' & L_2' & L \end{Bmatrix} \\ &\times \left[(-1)^{L_2'} \begin{Bmatrix} l & l_2 & t \\ l_3 & l_1 & L_2' \end{Bmatrix} \left(l \left\| \mathbf{C}^t \right\| l_2 \right) \left(l_1 \left\| \mathbf{C}^t \right\| l_3 \right) R^t \left(nln_1 l_1, n_2 l_2 n_3 l_3 \right) \right. \\ &\left. + (-1)^{S_2'} \begin{Bmatrix} l & l_3 & t \\ l_2 & l_1 & L_2' \end{Bmatrix} \left(l \left\| \mathbf{C}^t \right\| l_3 \right) \left(l_1 \left\| \mathbf{C}^t \right\| l_2 \right) R^t \left(nln_1 l_1, n_3 l_3 n_2 l_2 \right) \right]. \end{aligned} \tag{15}$$

– Interaction $l^{N+1} l_2^2 \leftrightarrow l^N l_1 l_2 l_3$ or $l^{N+1} l_2^2 \leftrightarrow l^N l_1 l_3 l_2$:

$$\begin{aligned}&\left\langle nl^{N+1} \alpha_1 S_1 L_1, n_2 l_2^2 \alpha_2 S_2 L_2; SL \left| \mathbf{G^k} \right| (nl^N \alpha_0' S_0' L_0', n_1 l_1) S_1' L_1', (n_2 l_2 n_3 l_3) S_2' L_2'; SL \right\rangle = \\ &\sqrt{2(N+1)} \left(nl^{N+1} \alpha_1 S_1 L_1 \{ | nl^N \alpha_0' S_0' L_0' \right) [L_1, L_2, L_1', L_2']^{1/2} \ (-1)^{l_2} \\ &\times \left[\delta(S_1, S_1') \ \delta(S_2, S_2') \ (-1)^{L_1+L_1'+L+L_0'+l_1+l_3} \begin{Bmatrix} L_2' & L_2 & t \\ l_2 & l_3 & l_2 \end{Bmatrix} \begin{Bmatrix} L_2' & L_2 & t \\ L_1 & L_1' & L \end{Bmatrix} \begin{Bmatrix} l_1 & l & t \\ L_1 & L_1' & L_0' \end{Bmatrix} \right. \\ &\times \left(l \left\| \mathbf{C}^t \right\| l_1 \right) \left(l_2 \left\| \mathbf{C}^t \right\| l_3 \right) R^t \left(nln_2 l_2, n_1 l_1 n_3 l_3 \right) \\ &+ (-1)^{S_2+3S_2'+L_2+L_2'+l+1} [S_1, S_2, S_1', S_2']^{1/2} \ \begin{Bmatrix} S_0' & 1/2 & S_1 \\ 1/2 & 1/2 & S_2 \\ S_1' & S_2' & S \end{Bmatrix} \begin{Bmatrix} L_0' & l & L_1 & L_2 \\ l_1 & t & l_2 & L_2' \\ L_1' & l_3 & L & l_2 \end{Bmatrix} \\ &\left. \times \left(l \left\| \mathbf{C}^t \right\| l_3 \right) \left(l_2 \left\| \mathbf{C}^t \right\| l_1 \right) R^t \left(nln_2 l_2, n_3 l_3 n_1 l_1 \right) \right], \end{aligned} \tag{16}$$

$$\begin{aligned}&\left\langle nl^{N+1} \alpha_1 S_1 L_1, n_2 l_2^2 \alpha_2 S_2 L_2; SL \left| \mathbf{G^k} \right| (nl^N \alpha_0' S_0' L_0', n_1 l_1) S_1' L_1', (n_3 l_3 n_2 l_2) S_2' L_2'; SL \right\rangle = \\ &(-1)^{l_2+l_3-S_2'-L_2'} \\ &\times \left\langle nl^{N+1} \alpha_1 S_1 L_1, n_2 l_2^2 \alpha_2 S_2 L_2; SL \left| \mathbf{G^k} \right| (nl^N \alpha_0' S_0' L_0', n_1 l_1) S_1' L_1', (n_2 l_2 n_3 l_3) S_2' L_2'; SL \right\rangle. \end{aligned} \tag{17}$$

– Interaction $l^N l_1^3 \leftrightarrow l^N l_1 l_2 l_3$:

$$\begin{aligned}&\left\langle nl^N\alpha_0 S_0 L_0, n_1 l_1^3\alpha_2 S_2 L_2; SL\left|\mathbf{G^k}\right|(nl^N\alpha_0' S_0' L_0', n_1 l_1)S_1' L_1', (n_2 l_2 n_3 l_3)S_2' L_2'; SL\right\rangle = \\ &\sqrt{6}\ \delta(\alpha_0 S_0 L_0, \alpha_0' S_0' L_0')\left(n_1 l_1^3\alpha_2 S_2 L_2\{|n_1 l_1^2\alpha_2' S_2' L_2'\right)(-1)^{S+L+S_0+L_0+3S_2+L_2+L_2'+l_1+l_2+1}\ [S_1', L_1', S_2, L_2]^{1/2} \\ &\times\begin{Bmatrix} S_2' & S & S_1' \\ S_0 & 1/2 & S_2\end{Bmatrix}\begin{Bmatrix} L_2' & L & L_1' \\ L_0 & l_1 & L_2\end{Bmatrix}\begin{Bmatrix} l_1 & l_2 & t \\ l_3 & l_1 & L_2'\end{Bmatrix}\left(l_1\left\|\mathbf{C}^t\right\|l_2\right)\left(l_1\left\|\mathbf{C}^t\right\|l_3\right)R^t\left(n_1 l_1 n_1 l_1, n_2 l_2 n_3 l_3\right).\end{aligned} \tag{18}$$

– Interaction $l^N l_1^3 \leftrightarrow l^N l_2 l_1 l_3$:

$$\begin{aligned}&\left\langle nl^N\alpha_0 S_0 L_0, n_1 l_1^3\alpha_2 S_2 L_2; SL\left|\mathbf{G^k}\right|(nl^N\alpha_0' S_0' L_0', n_2 l_2)S_1' L_1', (n_1 l_1 n_3 l_3)S_2' L_2'; SL\right\rangle = \\ &\sqrt{6}\ \delta(\alpha_0 S_0 L_0, \alpha_0' S_0' L_0')(-1)^{S+L+S_0+L_0+3S_2+L_2+l_1+l_3+1}\ [S_1', L_1', S_2, L_2, S_2', L_2']^{1/2} \\ &\times\sum_{\bar\alpha\bar S\bar L}(-1)^{\bar L}\ \left(n_1 l_1^3\alpha_2 S_2 L_2\{|l_1^2\bar\alpha\bar S\bar L\right)\ [\bar S, \bar L]^{1/2}\ \begin{Bmatrix} 1/2 & 1/2 & S_2' \\ 1/2 & S_2 & \bar S\end{Bmatrix}\begin{Bmatrix} l_1 & l_3 & L_2' \\ l_2 & L_2 & \bar L\end{Bmatrix}\begin{Bmatrix} l_1 & l_2 & t \\ l_3 & l_1 & \bar L\end{Bmatrix} \\ &\times\begin{Bmatrix} S_0 & 1/2 & S_1' \\ S_2' & S & S_2\end{Bmatrix}\begin{Bmatrix} L_0 & l_2 & L_1' \\ L_2' & L & L_2\end{Bmatrix}\left(l_1\left\|\mathbf{C}^t\right\|l_2\right)\left(l_1\left\|\mathbf{C}^t\right\|l_3\right)R^t\left(n_1 l_1 n_1 l_1, n_2 l_2 n_3 l_3\right).\end{aligned} \tag{19}$$

– Interaction $l^{N+1} l_1 l_2 \leftrightarrow l^N l_1 l_3 l_2$ or $l^{N+1} l_1 l_2 \leftrightarrow l^N l_1 l_2 l_3$:

$$\begin{aligned}&\left\langle nl^{N+1}\alpha_1 S_1 L_1, (n_1 l_1 n_2 l_2)S_2 L_2; SL\left|\mathbf{G^k}\right|(nl^N\alpha_0' S_0' L_0', n_1 l_1)S_1' L_1', (n_3 l_3 n_2 l_2)S_2' L_2'; SL\right\rangle = \\ &\sqrt{N+1}\left(nl^{N+1}\alpha_1 S_1 L_1\{|nl^N\alpha_0' S_0' L_0'\right)[L_1, L_2, L_1', L_2']^{1/2} \\ &\times\Bigg[\delta(S_1, S_1')\ \delta(S_2, S_2')\ (-1)^{L_0'+L_1+L_1'+L_2+L_2'+L+l_2}\begin{Bmatrix} L_1' & L_1 & t \\ l & l_1 & L_0'\end{Bmatrix}\begin{Bmatrix} L_2 & L_2' & t \\ L_1' & L_1 & L\end{Bmatrix}\begin{Bmatrix} l_3 & l_1 & t \\ L_2 & L_2' & l_2\end{Bmatrix} \\ &\times\left(l\left\|\mathbf{C}^t\right\|l_1\right)\left(l_1\left\|\mathbf{C}^t\right\|l_3\right)R^t\left(nln_1 l_1, n_1 l_1 n_3 l_3\right) \\ &+[S_1, S_2, S_1', S_2']^{1/2}\ (-1)^{l+l_3+2S_0'+2S}\begin{Bmatrix} 1/2 & 1/2 & S_2' \\ 1/2 & S_0' & S_1' \\ S_2 & S_1 & S\end{Bmatrix}\begin{Bmatrix} L & L_2 & L_1 & l \\ L_2' & l_2 & l_3 & l_1 \\ L_1' & l_1 & L_0' & t\end{Bmatrix}\left(l\left\|\mathbf{C}^t\right\|l_3\right)\left(l_1\left\|\mathbf{C}^t\right\|l_1\right)R^t\left(nln_1 l_1, n_3 l_3 n_1 l_1\right)\Bigg] \\ &+N\sqrt{N+1}\sum_{\bar\alpha\bar S\bar L,\hat\alpha\hat S\hat L}\left(nl^{N+1}\alpha_1 S_1 L_1\{|nl^{N-1}\bar\alpha\bar S\bar L, nl^2\hat\alpha\hat S\hat L\right)\left(nl^N\alpha_0' S_0' L_0'\{|nl^{N-1}\bar\alpha\bar S\bar L\right)\begin{Bmatrix} l & l & t \\ l_3 & l & \hat L\end{Bmatrix} \\ &\times[S_1, L_1, S_2, L_2, \hat S, \hat L, S_0', L_0', S_1', L_1', S_2', L_2']^{1/2}\ (-1)^{l+l_3+2S_2+S_1+L_1+\bar S+\bar L+\hat L}\begin{Bmatrix} 1/2 & S_1 & S_0' \\ \bar S & 1/2 & \hat S\end{Bmatrix}\begin{Bmatrix} 1/2 & S_1 & S_0' \\ 1/2 & S_2 & 1/2 \\ S_2' & S & S_1'\end{Bmatrix} \\ &\times\begin{Bmatrix} l_3 & L_1 & L_0' \\ \bar L & l & \hat L\end{Bmatrix}\begin{Bmatrix} l_3 & L_1 & L_0' \\ l_2 & L_2 & l_1 \\ L_2' & L & L_1'\end{Bmatrix}\left(l\left\|\mathbf{C}^t\right\|l\right)\left(l\left\|\mathbf{C}^t\right\|l_3\right)R^t\left(nlnl, nln_3 l_3\right) \\ &+\sqrt{N+1}\left(nl^{N+1}\alpha_1 S_1 L_1\{|nl^N\alpha_0' S_0' L_0'\right)(-1)^{l_2+l_3+1}[S_1, L_1, S_2, L_2, S_1', L_1', S_2', L_2']^{1/2}\begin{Bmatrix} S_0' & 1/2 & S_1 \\ 1/2 & 1/2 & S_2 \\ S_1' & S_2' & S\end{Bmatrix}\begin{Bmatrix} L_0' & l & L_1 \\ l_1 & l_2 & L_2 \\ L_1' & L_2 & L\end{Bmatrix} \\ &\times\Bigg[(-1)^{L_2'}\ \begin{Bmatrix} l & l_3 & t \\ l_2 & l_2 & L_2'\end{Bmatrix}\left(l\left\|\mathbf{C}^t\right\|l_3\right)\left(l_2\left\|\mathbf{C}^t\right\|l_2\right)R^t\left(nln_2 l_2, n_3 l_3 n_2 l_2\right) \\ &+(-1)^{S_2'}\begin{Bmatrix} l & l_2 & t \\ l_3 & l_2 & L_2'\end{Bmatrix}\left(l\left\|\mathbf{C}^t\right\|l_2\right)\left(l_2\left\|\mathbf{C}^t\right\|l_3\right)R^t\left(nln_2 l_2, n_2 l_2 n_3 l_3\right)\Bigg]\end{aligned} \tag{20}$$

$$\begin{aligned}&\left\langle nl^{N+1}\alpha_1 S_1 L_1, (n_1 l_1 n_2 l_2)S_2 L_2; SL\left|\mathbf{G^k}\right|(nl^N\alpha_0' S_0' L_0', n_1 l_1)S_1' L_1', (n_2 l_2 n_3 l_3)S_2' L_2'; SL\right\rangle = \\ &(-1)^{l_2+l_3-S_2'-L_2'} \\ &\times\left\langle nl^{N+1}\alpha_1 S_1 L_1, (n_1 l_1 n_2 l_2)S_2 L_2; SL\left|\mathbf{G^k}\right|(nl^N\alpha_0' S_0' L_0', n_1 l_1)S_1' L_1', (n_3 l_3 n_2 l_2)S_2' L_2'; SL\right\rangle.\end{aligned} \tag{21}$$

– Interaction $l^{N+1}l_2l_3 \leftrightarrow l^N l_2l_4l_5$:

$$\begin{aligned}&\left\langle nl^{N+1}\alpha_1S_1L_1,(n_2l_2n_3l_3)S_2L_2;SL\left|\mathbf{G^k}\right|(nl^N\alpha_0'S_0'L_0',n_2l_2)S_1'L_1',(n_4l_4n_5l_5)S_2'L_2';SL\right\rangle=\\&\sqrt{N+1}\left(nl^{N+1}\alpha_1S_1L_1\{|nl^N\alpha_0'S_0'L_0'\right)(-1)^{l_3+l_4+1}[S_1,L_1,S_2,L_2,S_1',L_1',S_2',L_2']^{1/2}\\&\times\begin{Bmatrix}S_0&1/2&S_1\\1/2&1/2&S_2\\S_1'&S_2'&S\end{Bmatrix}\begin{Bmatrix}L_0&l_1&L_1\\l_2&l_3&L_2\\L_1'&L_2'&L\end{Bmatrix}\left[(-1)^{L_2'}\begin{Bmatrix}l&l_4&t\\l_5&l_3&L_2'\end{Bmatrix}\left(l\left\|\mathbf{C}^t\right\|l_4\right)\left(l_3\left\|\mathbf{C}^t\right\|l_5\right)R^t\left(nln_3l_3,n_4l_4n_5l_5\right)\right.\\&\left.+(-1)^{S_2'}\begin{Bmatrix}l&l_5&t\\l_4&l_3&L_2'\end{Bmatrix}\left(l\left\|\mathbf{C}^t\right\|l_5\right)\left(l_3\left\|\mathbf{C}^t\right\|l_4\right)R^t\left(nln_3l_3,n_5l_5n_4l_4\right)\right].\end{aligned}\tag{22}$$

– Interaction $l^{N+1}l_2l_4 \leftrightarrow l^N l_1l_2l_3$:

$$\begin{aligned}&\left\langle nl^{N+1}\alpha_1S_1L_1,(n_2l_2n_4l_4)S_2L_2;SL\left|\mathbf{G^k}\right|(nl^N\alpha_0'S_0'L_0',n_1l_1)S_1'L_1',(n_2l_2n_3l_3)S_2'L_2';SL\right\rangle=\\&\sqrt{N+1}\left(nl^{N+1}\alpha_1S_1L_1\{|nl^N\alpha_0'S_0'L_0'\right)[L_1,L_2,L_1',L_2']^{1/2}\\&\times\left[\delta(S_1,S_1')\delta(S_2,S_2')\ (-1)^{l_1+l_2+l_3+L+L_1+L_1'+L_0'}\begin{Bmatrix}L_2'&L_2&t\\l_4&l_3&l_2\end{Bmatrix}\begin{Bmatrix}L_2'&L_2&t\\L_1&L_1'&L\end{Bmatrix}\begin{Bmatrix}l_1&l&t\\L_1&L_1'&L_0'\end{Bmatrix}\right.\\&\times\left(l\left\|\mathbf{C}^t\right\|l_1\right)\left(l_4\left\|\mathbf{C}^t\right\|l_3\right)R^t\left(nln_4l_4,n_1l_1n_3l_3\right)\\&+(-1)^{1+l+l_4+L_2+S_2+L_2'+3S_2'}\ [S_1,S_2,S_1',S_2']^{1/2}\begin{Bmatrix}S_0'&1/2&S_1\\1/2&1/2&S_2\\S_1'&S_2'&S\end{Bmatrix}\begin{Bmatrix}L_0'&l&L_1&L_2\\l_1&t&l_4&L_2'\\L_1'&l_3&L&l_2\end{Bmatrix}\\&\left.\times\left(l\left\|\mathbf{C}^t\right\|l_3\right)\left(l_4\left\|\mathbf{C}^t\right\|l_1\right)R^t\left(nln_4l_4,n_3l_3n_1l_1\right)\right]\\&+\delta(l_3,l_4)\sqrt{N+1}\left(nl^{N+1}\alpha_1S_1L_1\{|nl^N\alpha_0'S_0'L_0'\right)[L_1,L_2,L_1',L_2']^{1/2}\\&\times\left[\delta(S_1,S_1')\delta(S_2,S_2')\ (-1)^{l_1+l_2+l_3+L+L_1+L_1'+L_2+L_2'+L_0'}\begin{Bmatrix}L_2&L_2'&t\\l_2&l_2&l_3\end{Bmatrix}\begin{Bmatrix}L_2&L_2'&t\\L_1&L_1'&L\end{Bmatrix}\begin{Bmatrix}l_1&l&t\\L_1&L_1'&L_0'\end{Bmatrix}\right.\\&\times\left(l\left\|\mathbf{C}^t\right\|l_1\right)\left(l_2\left\|\mathbf{C}^t\right\|l_2\right)R^t\left(nln_2l_2,n_1l_1n_2l_2\right)\\&\left.+(-1)^{1+l+l_2}\ [S_1,S_2,S_1',S_2']^{1/2}\begin{Bmatrix}S_0'&1/2&S_1\\1/2&1/2&S_2\\S_1'&S_2'&S\end{Bmatrix}\begin{Bmatrix}L_0'&l&L_1&L_2\\l_1&t&l_2&L_2'\\L_1'&l_2&L&l_3\end{Bmatrix}\left(l\left\|\mathbf{C}^t\right\|l_2\right)\left(l_2\left\|\mathbf{C}^t\right\|l_1\right)R^t\left(nln_2l_2,n_2l_2n_1l_1\right)\right].\end{aligned}\tag{23}$$

– Interaction $l^{N+1}l_1l_2 \leftrightarrow l^N l_3l_1l_2$:

$$\begin{aligned}&\left\langle nl^{N+1}\alpha_1S_1L_1,(n_1l_1n_2l_2)S_2L_2;SL\left|\mathbf{G^k}\right|(nl^N\alpha_0'S_0'L_0',n_3l_3)S_1'L_1',(n_1l_1n_2l_2)S_2'L_2';SL\right\rangle=\\&\delta(S_1L_1,S_1'L_1')\delta(S_2L_2,S_2'L_2')\ \sqrt{N+1}\sum_{\bar\alpha\bar S\bar L}\left(nl^{N+1}\alpha_1S_1L_1\{|nl^N\bar\alpha\bar S\bar L\right)\delta(S_0',\bar S)(-1)^{l+L_1+L_0'}\\&\times\begin{Bmatrix}\bar L&t&L_0'\\l_3&L_1&l\end{Bmatrix}\left\langle nl^N\bar\alpha\bar S\bar L\left\|\mathbf{U}^t\right\|nl^N\alpha_0'S_0'L_0'\right\rangle\left(l\left\|\mathbf{C}^t\right\|l\right)\left(l\left\|\mathbf{C}^t\right\|l_3\right)R^t\left(nlnl,nln_3l_3\right)\\&+\sqrt{N+1}\left(nl^{N+1}\alpha_1S_1L_1\{|nl^N\alpha_0'S_0'L_0'\right)[L_1,L_2,L_1',L_2']^{1/2}\\&\times\left[\delta(S_1,S_1')\delta(S_2,S_2')\ (-1)^{l_1+l_2+l_3+L+L_1+L_1'+L_0'}\begin{Bmatrix}L_2&L_2'&t\\l_2&l_2&l_1\end{Bmatrix}\begin{Bmatrix}L_2&L_2'&t\\L_1&L_1'&L\end{Bmatrix}\begin{Bmatrix}l_3&l&t\\L_1&L_1'&L_0'\end{Bmatrix}\right.\\&\times\left(l\left\|\mathbf{C}^t\right\|l_3\right)\left(l_2\left\|\mathbf{C}^t\right\|l_2\right)R^t\left(nln_2l_2,n_3l_3n_2l_2\right)\\&+(-1)^{S_2+L_2+S_2'+L_2'+l+l_2+1}\ [S_1,S_2,S_1',S_2']^{1/2}\begin{Bmatrix}S_0'&1/2&S_1\\1/2&1/2&S_2\\S_1'&S_2'&S\end{Bmatrix}\begin{Bmatrix}L_0'&l&L_1&L_2\\l_3&t&l_2&L_2'\\L_1'&l_2&L&l_1\end{Bmatrix}\\&\left.\times\left(l\left\|\mathbf{C}^t\right\|l_2\right)\left(l_2\left\|\mathbf{C}^t\right\|l_3\right)R^t\left(nln_2l_2,n_2l_2n_3l_3\right)\right]\end{aligned}$$

$$+\sqrt{N+1}\left(nl^{N+1}\alpha_1 S_1 L_1\{|nl^N\alpha_0' S_0' L_0'\right)[L_1,L_2,L_1',L_2']^{1/2}$$
$$\times\left[\delta(S_1,S_1')\delta(S_2,S_2')\ (-1)^{l_1+l_2+l_3+L+L_1+L_1'+L_2+L_2'+L_0'}\begin{Bmatrix}L_2 & L_2' & t\\ l_1 & l_1 & l_2\end{Bmatrix}\begin{Bmatrix}L_2 & L_2' & t\\ L_1 & L_1' & L\end{Bmatrix}\begin{Bmatrix}l_3 & l & t\\ L_1 & L_1' & L_0'\end{Bmatrix}\right.$$
$$\times\left(l\left\|\mathbf{C}^t\right\|l_3\right)\left(l_1\left\|\mathbf{C}^t\right\|l_1\right)R^t\left(nln_1l_1,n_3l_3n_1l_1\right)$$
$$\left.+(-1)^{1+l+l_1}\ [S_1,S_2,S_1',S_2']^{1/2}\begin{Bmatrix}S_0' & 1/2 & S_1\\ 1/2 & 1/2 & S_2\\ S_1' & S_2' & S\end{Bmatrix}\begin{Bmatrix}L_0' & l & L_1 & L_2\\ l_3 & t & l_1 & L_2'\\ L_1' & l_1 & L & l_2\end{Bmatrix}\left(l\left\|\mathbf{C}^t\right\|l_1\right)\left(l_1\left\|\mathbf{C}^t\right\|l_3\right)R^t\left(nln_1l_1,n_1l_1n_3l_3\right)\right].\qquad(24)$$

– Interaction $l^N l_2 l_4^2 \leftrightarrow l^N l_1 l_2 l_3$ (valid also for $n_4l_4=n_3l_3$):

$$\left\langle(nl^N\alpha_0S_0L_0,n_2l_2)S_1L_1,n_4l_4^2\alpha_2S_2L_2;SL\left|\mathbf{G}^{\mathbf{k}}\right|(nl^N\alpha_0'S_0'L_0',n_1l_1)S_1'L_1',(n_2l_2n_3l_3)S_2'L_2';SL\right\rangle=$$
$$\sqrt{2}\ \delta(\alpha_0S_0L_0,\alpha_0'S_0'L_0')\ (-1)^{l_1+l_4+L_2+1}\ [S_1,L_1,S_2,L_2,S_1',L_1',S_2',L_2']^{1/2}$$
$$\times\begin{Bmatrix}S_0 & 1/2 & S_1'\\ 1/2 & 1/2 & S_2'\\ S_1 & S_2 & S\end{Bmatrix}\begin{Bmatrix}L_0 & l_1 & L_1'\\ l_2 & l_3 & L_2'\\ L_1 & L_2 & L\end{Bmatrix}\begin{Bmatrix}l_4 & l_1 & t\\ l_3 & l_4 & L_2\end{Bmatrix}\left(l_4\left\|\mathbf{C}^t\right\|l_1\right)\left(l_4\left\|\mathbf{C}^t\right\|l_3\right)R^t\left(n_4l_4n_4l_4,n_1l_1n_3l_3\right)$$
$$+\sqrt{2}\ (-1)^{l_3+1}\ [S_1,L_1,S_2,L_2,S_1',L_1',S_2',L_2']^{1/2}\begin{Bmatrix}S_0 & 1/2 & S_1\\ 1/2 & 1/2 & S_2\\ S_1' & S_2' & S\end{Bmatrix}\begin{Bmatrix}L_0 & l_2 & L_1\\ l_3 & l_3 & L_2\\ L_1' & L_2' & L\end{Bmatrix}$$
$$\times\left[\delta(S_0,S_0')(-1)^{L_0'+L_1'}\left\langle nl^N\alpha_0S_0L_0\left\|\mathbf{U}^t\right\|nl^N\alpha_0'S_0'L_0'\right\rangle\begin{Bmatrix}l_1 & t & l_3\\ L_0 & L_1' & L_0'\end{Bmatrix}\left(l\left\|\mathbf{C}^t\right\|l\right)\left(l_3\left\|\mathbf{C}^t\right\|l_1\right)R^t\left(nln_3l_3,nln_1l_1\right)\right.$$
$$+[S_0,L_0,S_0',L_0']^{1/2}\ (-1)^{l+S_0+S_0'}\ N\sum_{\bar\alpha\bar S\bar L}\left(nl^N\alpha_0S_0L_0\{|nl^{N-1}\bar\alpha\bar S\bar L\right)\left(nl^N\alpha_0'S_0'L_0'\{|nl^{N-1}\bar\alpha\bar S\bar L\right)$$
$$\left.\times\begin{Bmatrix}1/2 & \bar S & S_0'\\ 1/2 & S_1' & S_0\end{Bmatrix}\begin{Bmatrix}\bar L & l & L_0'\\ l & t & l_1\\ L_0 & l_3 & L_1'\end{Bmatrix}\left(l\left\|\mathbf{C}^t\right\|l_1\right)\left(l_3\left\|\mathbf{C}^t\right\|l\right)R^t\left(nln_3l_3,n_1l_1nl\right)\right].\qquad(25)$$

– Interaction $l^N l_1 l_2^2 \leftrightarrow l^N l_1 l_2 l_3$:

$$\left\langle(nl^N\alpha_0S_0L_0,n_1l_1)S_1L_1,n_2l_2^2\alpha_2S_2L_2;SL\left|\mathbf{G}^{\mathbf{k}}\right|(nl^N\alpha_0'S_0'L_0',n_1l_1)S_1'L_1',(n_2l_2n_3l_3)S_2'L_2';SL\right\rangle=$$
$$\sqrt{2}\ (-1)^{l_3+L_0}\ [L_1,L_2,L_1',L_2']^{1/2}$$
$$\times\left[\delta(S_0,S_0')\ \delta(S_1,S_1')\ \delta(S_2,S_2')\ (-1)^{l_1+l_2+L+L_2+S_2}\begin{Bmatrix}L_1' & L_1 & t\\ L_0 & L_0' & l_1\end{Bmatrix}\begin{Bmatrix}L_1' & L_1 & t\\ L_2 & L_2' & L\end{Bmatrix}\begin{Bmatrix}l_3 & l_2 & t\\ L_2 & L_2' & l_2\end{Bmatrix}\right.$$
$$\times\left\langle nl^N\alpha_0S_0L_0\left\|\mathbf{U}^t\right\|nl^N\alpha_0'S_0'L_0'\right\rangle\left(l\left\|\mathbf{C}^t\right\|l\right)\left(l_2\left\|\mathbf{C}^t\right\|l_3\right)R^t\left(nln_2l_2,nln_3l_3\right)$$
$$+[S_0,L_0,S_0',L_0',S_1,S_2,S_1',S_2']^{1/2}\ (-1)^{L_0'+L_1+L_1'+L_2'+S_0+3S_0'+3S_1+S_1'+3S_2'+l+1}$$
$$\times N\sum_{\bar\alpha\bar S\bar L}\left(nl^N\alpha_0S_0L_0\{|nl^{N-1}\bar\alpha\bar S\bar L\right)\left(nl^N\alpha_0'S_0'L_0'\{|nl^{N-1}\bar\alpha\bar S\bar L\right)\begin{Bmatrix}1/2 & 1/2 & S_2' & S_1'\\ 1/2 & \bar S & S_0' & S_1\\ S_2 & S_0 & S & 1/2\end{Bmatrix}$$
$$\left.\times\begin{Bmatrix}l_3 & & L_2' & & L_1' & & L_0' & & l\\ & l_2 & & L & & l_1 & & \bar L & & t\\ l_2 & & L_2 & & L_1 & & L_0 & & l\end{Bmatrix}\left(l\left\|\mathbf{C}^t\right\|l_3\right)\left(l_2\left\|\mathbf{C}^t\right\|l\right)R^t\left(nln_2l_2,n_3l_3nl\right)\right]$$
$$+\sqrt{2}\ \delta(\alpha_0S_0L_0,\alpha_0'S_0'L_0')\delta(S_1L_1,S_1'L_1')\delta(S_2L_2,S_2'L_2')\ (-1)^{L_2}\begin{Bmatrix}l_2 & l_2 & t\\ l_3 & l_2 & L_2\end{Bmatrix}$$
$$\times\left(l_2\left\|\mathbf{C}^t\right\|l_2\right)\left(l_2\left\|\mathbf{C}^t\right\|l_3\right)R^t\left(n_2l_2n_2l_2,n_2l_2n_3l_3\right)$$
$$+\sqrt{2}\ \delta(\alpha_0S_0L_0,\alpha_0'S_0'L_0')\ [L_1,L_2,L_1',L_2']^{1/2}$$

$$\times \Bigg[\delta(S_1,S_1')\delta(S_2,S_2')\;(-1)^{l_1+L_0+L+L_1+L_1'}\;\begin{Bmatrix} L_2 & L_2' & t \\ l_3 & l_2 & l_2 \end{Bmatrix}\begin{Bmatrix} L_2 & L_2' & t \\ L_1' & L_1 & L \end{Bmatrix}\begin{Bmatrix} l_1 & l & t \\ L_1 & L_1' & L_0 \end{Bmatrix}$$
$$\times \left(l_1 \left\|\mathbf{C}^t\right\| l_1\right)\left(l_3 \left\|\mathbf{C}^t\right\| l_2\right) R^t\left(n_1l_1n_2l_2, n_1l_1n_3l_3\right)$$
$$+(-1)^{3S_2+S_2'+L_2+L_2'+l+l_3+1}\;[S_1,S_2,S_1',S_2']^{1/2}\begin{Bmatrix} S_0 & 1/2 & S_1' \\ 1/2 & 1/2 & S_2' \\ S_1 & S_2 & S \end{Bmatrix}$$
$$\times \begin{Bmatrix} L_0 & l_1 & L_1' & L_2' \\ l_1 & t & l_3 & L_2 \\ L_1 & l_2 & L & l_2 \end{Bmatrix}\left(l_1 \left\|\mathbf{C}^t\right\| l_2\right)\left(l_3 \left\|\mathbf{C}^t\right\| l_1\right) R^t\left(n_1l_1n_2l_2, n_3l_3n_1l_1\right)\Bigg]. \tag{26}$$

– Interaction $l^N l_3 l_2^2 \leftrightarrow l^N l_1 l_2 l_3$:

$$\left\langle (nl^N\alpha_0S_0L_0, n_3l_3)S_1L_1, n_2l_2^2\alpha_2S_2L_2; SL\left|\mathbf{G^k}\right|(nl^N\alpha_0'S_0'L_0', n_1l_1)S_1'L_1', (n_2l_2n_3l_3)S_2'L_2'; SL\right\rangle =$$
$$\sqrt{2}N\sum_{\bar\alpha\bar S\bar L}\left(nl^N\alpha_0S_0L_0\{|nl^{N-1}\bar\alpha\bar S\bar L\right)\left(nl^N\alpha_0'S_0'L_0'\{|nl^{N-1}\bar\alpha\bar S\bar L\right)(-1)^{1+l+l_3+S_2'+L_2'}$$
$$\times [L_0,L_0',S_1,L_1,S_2,L_2,S_1',L_1',S_2',L_2']^{1/2}\begin{Bmatrix} S_0 & 1/2 & S_1 \\ 1/2 & 1/2 & S_2 \\ S_1' & S_2' & S \end{Bmatrix}\begin{Bmatrix} L_0 & l_3 & L_1 \\ l_2 & l_2 & L_2 \\ L_1' & L_2' & L \end{Bmatrix}$$
$$\times \Bigg[\delta(S_0,S_0')\;(-1)^{L_0+\bar L+L_0'+L_1'}\begin{Bmatrix} L_0 & \bar L & l \\ l & t & L_0' \end{Bmatrix}\begin{Bmatrix} L_0' & L_0 & t \\ l_2 & l_1 & L_1' \end{Bmatrix}\left(l \left\|\mathbf{C}^t\right\| l\right)\left(l_1 \left\|\mathbf{C}^t\right\| l_2\right) R^t\left(nln_1l_1, nln_2l_2\right)$$
$$+[S_0,S_0']^{1/2}\;(-1)^{3S_0+3S_0'}\;\begin{Bmatrix} \bar S & 1/2 & S_0' \\ S_1' & 1/2 & S_0 \end{Bmatrix}\begin{Bmatrix} \bar L & l & L_0 \\ l & t & l_2 \\ L_0' & l_1 & L_1' \end{Bmatrix}\left(l \left\|\mathbf{C}^t\right\| l_2\right)\left(l_1 \left\|\mathbf{C}^t\right\| l\right) R^t\left(nln_1l_1, n_2l_2nl\right)\Bigg]$$
$$+\sqrt{2}\;\delta(\alpha_0S_0L_0,\alpha_0'S_0'L_0')\;[L_1,L_2,L_1',L_2']^{1/2}$$
$$\times \Bigg[\delta(S_1,S_1')\delta(S_2,S_2')\;(-1)^{l_3+L_0'+L+L_1+L_1'}\;\begin{Bmatrix} L_2 & L_2' & t \\ l_3 & l_2 & l_2 \end{Bmatrix}\begin{Bmatrix} L_2 & L_2' & t \\ L_1' & L_1 & L \end{Bmatrix}\begin{Bmatrix} l_3 & l_1 & t \\ L_1' & L_1 & L_0' \end{Bmatrix}$$
$$\times \left(l_1 \left\|\mathbf{C}^t\right\| l_3\right)\left(l_3 \left\|\mathbf{C}^t\right\| l_2\right) R^t\left(n_1l_1n_3l_3, n_3l_3n_2l_2\right)$$
$$+(-1)^{1+l_1+l_3+3S_2+S_2'+L_2+L_2'}\;[S_1,S_2,S_1',S_2']^{1/2}\;\begin{Bmatrix} S_0' & 1/2 & S_1' \\ 1/2 & 1/2 & S_2' \\ S_1 & S_2 & S \end{Bmatrix}$$
$$\times \begin{Bmatrix} L_0' & l_1 & L_1' & L_2' \\ l_3 & t & l_3 & L_2 \\ L_1 & l_2 & L & l_2 \end{Bmatrix}\left(l_1 \left\|\mathbf{C}^t\right\| l_2\right)\left(l_3 \left\|\mathbf{C}^t\right\| l_3\right) R^t\left(n_1l_1n_3l_3, n_2l_2n_3l_3\right)\Bigg]. \tag{27}$$

– Interaction $l^N l_4 l_1^2 \leftrightarrow l^N l_1 l_2 l_3$:

$$\left\langle (nl^N\alpha_0S_0L_0, n_4l_4)S_1L_1, n_1l_1^2\alpha_2S_2L_2; SL\left|\mathbf{G^k}\right|(nl^N\alpha_0'S_0'L_0', n_1l_1)S_1'L_1', (n_2l_2n_3l_3)S_2'L_2'; SL\right\rangle =$$
$$\sqrt{2}\;\delta(\alpha_0S_0L_0,\alpha_0'S_0'L_0')\;(-1)^{l_1+l_2+1}\;[S_1,L_1,S_1',L_1',S_2,L_2,S_2',L_2']^{1/2}\begin{Bmatrix} S_0 & 1/2 & S_1 \\ 1/2 & 1/2 & S_2 \\ S_1' & S_2' & S \end{Bmatrix}\begin{Bmatrix} L_0 & l_4 & L_1 \\ l_1 & l_1 & L_2 \\ L_1' & L_2' & L \end{Bmatrix}$$
$$\times \Bigg[(-1)^{L_2'}\begin{Bmatrix} l_4 & l_2 & t \\ l_3 & l_1 & L_2' \end{Bmatrix}\left(l_4 \left\|\mathbf{C}^t\right\| l_2\right)\left(l_1 \left\|\mathbf{C}^t\right\| l_3\right) R^t\left(n_4l_4n_1l_1, n_2l_2n_3l_3\right)$$
$$+(-1)^{S_2'}\begin{Bmatrix} l_4 & l_3 & t \\ l_2 & l_1 & L_2' \end{Bmatrix}\left(l_4 \left\|\mathbf{C}^t\right\| l_3\right)\left(l_1 \left\|\mathbf{C}^t\right\| l_2\right) R^t\left(n_4l_4n_1l_1, n_3l_3n_2l_2\right)\Bigg]. \tag{28}$$

– Interaction $l^N l_1 l_4^2 \leftrightarrow l^N l_1 l_2 l_3$:

$$\begin{aligned}&\left\langle (nl^N\alpha_0 S_0 L_0, n_1 l_1) S_1 L_1, n_4 l_4^2 \alpha_2 S_2 L_2; SL \left| \mathbf{G^k} \right| (nl^N \alpha_0' S_0' L_0', n_1 l_1) S_1' L_1', (n_2 l_2 n_3 l_3) S_2' L_2'; SL \right\rangle = \\ &\sqrt{2}\ \delta(\alpha_0 S_0 L_0, \alpha_0' S_0' L_0')\ \delta(S_1 L_1, S_1' L_1')\ \delta(S_2 L_2, S_2' L_2')\ (-1)^{l_2+l_4+L_2} \\ &\times \begin{Bmatrix} l_4 & l_2 & t \\ l_3 & l_4 & L_2 \end{Bmatrix} \left(l_4 \left\| \mathbf{C}^t \right\| l_2\right) \left(l_4 \left\| \mathbf{C}^t \right\| l_3\right) R^t\left(n_4 l_4 n_4 l_4, n_2 l_2 n_3 l_3\right). \end{aligned} \tag{29}$$

– Interaction $l^N l_2^2 l_3 \leftrightarrow l^N l_1 l_2 l_3$:

$$\begin{aligned}&\left\langle nl^N\alpha_0 S_0 L_0, (n_2 l_2^2 \alpha_2 S_2 L_2, n_3 l_3) S_4 L_4; SL \left| \mathbf{G^k} \right| (nl^N \alpha_0' S_0' L_0', n_1 l_1) S_1' L_1', (n_2 l_2 n_3 l_3) S_2' L_2'; SL \right\rangle = \\ &\sqrt{2}\ (-1)^{S_0+L_0+S_2'+L_2'+S_4+L_4+S+L+l_3}\ [S_2, L_2, S_2', L_2', S_4, L_4, S_1', L_1']^{1/2} \begin{Bmatrix} 1/2 & 1/2 & S_2 \\ 1/2 & S_4 & S_2' \end{Bmatrix} \begin{Bmatrix} S_2' & S & S_1' \\ S_0 & 1/2 & S_4 \end{Bmatrix} \\ &\times \Bigg[\delta(S_0, S_0')\ (-1)^{L_0'+L_1'} \begin{Bmatrix} L_2' & L & L_1' \\ L_0 & l_2 & L_4 \end{Bmatrix} \begin{Bmatrix} l_2 & l_2 & L_2 \\ l_3 & L_4 & L_2' \end{Bmatrix} \begin{Bmatrix} l_1 & t & l_2 \\ L_0 & L_1' & L_0' \end{Bmatrix} \\ &\times \left\langle nl^N \alpha_0 S_0 L_0 \left\| \mathbf{U}^t \right\| nl^N \alpha_0' S_0' L_0' \right\rangle \left(l \left\| \mathbf{C}^t \right\| l\right) \left(l_2 \left\| \mathbf{C}^t \right\| l_1\right) R^t\left(nln_2l_2, nln_1l_1\right) \\ &+ N \sum_{\bar\alpha \bar S \bar L} \left(nl^N \alpha_0 S_0 L_0 \{| nl^{N-1} \bar\alpha \bar S \bar L\right) \left(nl^N \alpha_0' S_0' L_0' \{| nl^{N-1} \bar\alpha \bar S \bar L\right) \begin{Bmatrix} 1/2 & \bar S & S_0' \\ 1/2 & S_1' & S_0 \end{Bmatrix} \begin{Bmatrix} \bar L & l & L_0' \\ l & t & l_1 \\ L_0 & l_2 & L_1' \end{Bmatrix} \\ &\times \begin{Bmatrix} L_2' & L & L_1' \\ L_0 & l_2 & L_4 \end{Bmatrix} \begin{Bmatrix} l_2 & l_2 & L_2 \\ l_3 & L_4 & L_2' \end{Bmatrix} [S_0, L_0, S_0', L_0']^{1/2}\ (-1)^{l+S_0+S_0'} \left(l \left\| \mathbf{C}^t \right\| l_1\right) \left(l_2 \left\| \mathbf{C}^t \right\| l\right) R^t\left(nln_2l_2, n_1l_1nl\right) \\ &+ \delta(\alpha_0 S_0 L_0, \alpha_0' S_0' L_0') (-1)^{l_1+L_2} \begin{Bmatrix} L_2' & L & L_1' \\ L_0 & l_1 & L_4 \end{Bmatrix} \begin{Bmatrix} l_1 & l_2 & L_2 \\ l_3 & L_4 & L_2' \end{Bmatrix} \begin{Bmatrix} l_2 & l_1 & t \\ l_2 & l_2 & L_2 \end{Bmatrix} \\ &\times \left(l_2 \left\| \mathbf{C}^t \right\| l_1\right) \left(l_2 \left\| \mathbf{C}^t \right\| l_2\right) R^t\left(n_2 l_2 n_2 l_2, n_1 l_1 n_2 l_2\right) \Bigg] \\ &+ \sqrt{2}\ \delta(\alpha_0 S_0 L_0, \alpha_0' S_0' L_0')\ [S_4, L_4, S_1', L_1', L_2, L_2']^{1/2}\ (-1)^{S_0+L_0+S+L+S_4} \begin{Bmatrix} S_2' & S & S_1' \\ S_0 & 1/2 & S_4 \end{Bmatrix} \begin{Bmatrix} L_2' & L & L_1' \\ L_0 & l_1 & L_4 \end{Bmatrix} \\ &\times \Bigg[\delta(S_2, S_2')\ (-1)^{l_3+L_2+L_2'} \begin{Bmatrix} L_2 & L_2' & t \\ l_3 & l_2 & l_2 \end{Bmatrix} \begin{Bmatrix} L_2 & L_2' & t \\ l_1 & l_3 & L_4 \end{Bmatrix} \left(l_2 \left\| \mathbf{C}^t \right\| l_3\right) \left(l_3 \left\| \mathbf{C}^t \right\| l_1\right) R^t\left(n_2 l_2 n_3 l_3, n_3 l_3 n_1 l_1\right) \\ &+ (-1)^{l_1+l_2+S_2+S_2'+L_4}\ [S_2, S_2']^{1/2} \begin{Bmatrix} 1/2 & 1/2 & S_2 \\ S_4 & 1/2 & S_2' \end{Bmatrix} \begin{Bmatrix} l_2 & l_3 & L_2' \\ l_2 & t & l_1 \\ L_2 & l_3 & L_4 \end{Bmatrix} \left(l_2 \left\| \mathbf{C}^t \right\| l_1\right) \left(l_3 \left\| \mathbf{C}^t \right\| l_3\right) R^t\left(n_2 l_2 n_3 l_3, n_1 l_1 n_3 l_3\right) \Bigg]. \end{aligned} \tag{30}$$

– Interaction $l^N l_1^2 l_2 \leftrightarrow l^N l_3 l_1 l_4$:

$$\begin{aligned}&\left\langle nl^N\alpha_0 S_0 L_0, (n_1 l_1^2 \alpha_1 S_1 L_1, n_2 l_2) S_2 L_2; SL \left| \mathbf{G^k} \right| (nl^N \alpha_0' S_0' L_0', n_3 l_3) S_1' L_1', (n_1 l_1 n_4 l_4) S_2' L_2'; SL \right\rangle = \\ &\sqrt{2}\ \delta(\alpha_0 S_0 L_0, \alpha_0' S_0' L_0')\ [L_1, S_2, L_2, S_1', L_1', L_2']^{1/2}\ (-1)^{S_0+L_0+S+L+3S_2+l_1+l_3+1} \begin{Bmatrix} S_1' & S_2' & S \\ S_2 & S_0 & 1/2 \end{Bmatrix} \begin{Bmatrix} L_1' & L_2' & L \\ L_2 & L_0 & l_3 \end{Bmatrix} \\ &\times \Bigg[[S_1, S_2']^{1/2}\ (-1)^{3S_1+3S_2'+L_2} \begin{Bmatrix} 1/2 & 1/2 & S_1 \\ S_2 & 1/2 & S_2' \end{Bmatrix} \begin{Bmatrix} l_1 & l_3 & t \\ l_1 & L_2' & l_4 \\ L_1 & L_2 & l_2 \end{Bmatrix} \left(l_1 \left\| \mathbf{C}^t \right\| l_3\right) \left(l_2 \left\| \mathbf{C}^t \right\| l_4\right) R^t\left(n_1 l_1 n_2 l_2, n_3 l_3 n_4 l_4\right) \\ &+ \delta(S_1, S_2')\ (-1)^{l_4+L_1+L_2'} \begin{Bmatrix} t & l_1 & l_4 \\ l_1 & L_2' & L_1 \end{Bmatrix} \begin{Bmatrix} L_1 & t & L_2' \\ l_3 & L_2 & l_2 \end{Bmatrix} \left(l_1 \left\| \mathbf{C}^t \right\| l_4\right) \left(l_2 \left\| \mathbf{C}^t \right\| l_3\right) R^t\left(n_1 l_1 n_2 l_2, n_4 l_4 n_3 l_3\right) \Bigg]. \end{aligned} \tag{31}$$

– Interaction $l^N l_1^2 l_2 \leftrightarrow l^N l_2 l_1 l_3$:

$$\begin{aligned}
&\left\langle nl^N\alpha_0 S_0 L_0,(n_1l_1^2\alpha_1 S_1L_1,n_2l_2)S_2L_2;SL\left|\mathbf{G^k}\right|(nl^N\alpha_0'S_0'L_0',n_2l_2)S_1'L_1',(n_1l_1n_3l_3)S_2'L_2';SL\right\rangle =\\
&\sqrt{2}\;\delta(\alpha_0S_0L_0,\alpha_0'S_0'L_0')\;[S_2,L_2,S_1',L_1']^{1/2}\;(-1)^{1+S_0+L_0+S+L+3S_2}\\
&\times\Bigg[\delta(S_1L_1,S_2'L_2')\;(-1)^{L_1+L_2}\;\begin{Bmatrix}S_1 & S & S_1'\\ S_0 & 1/2 & S_2\end{Bmatrix}\begin{Bmatrix}L_1 & L & L_1'\\ L_0 & l_2 & L_2\end{Bmatrix}\begin{Bmatrix}l_1 & l_1 & t\\ l_3 & l_1 & L_1\end{Bmatrix}\\
&\times\left(l_1\left\|\mathbf{C}^t\right\|l_1\right)\left(l_1\left\|\mathbf{C}^t\right\|l_3\right)R^t\left(n_1l_1n_1l_1,n_1l_1n_3l_3\right)\\
&+[S_1,L_1,S_2',L_2']^{1/2}\;(-1)^{l_1+l_2+L_2+3S_1+3S_2'}\;\begin{Bmatrix}1/2 & 1/2 & S_1\\ S_2 & 1/2 & S_2'\end{Bmatrix}\begin{Bmatrix}S_1' & S_2' & S\\ S_2 & S_0 & 1/2\end{Bmatrix}\begin{Bmatrix}L_1' & L_2' & L\\ L_2 & L_0 & l_2\end{Bmatrix}\\
&\times\begin{Bmatrix}l_1 & l_1 & L_1\\ l_3 & t & l_2\\ L_2' & l_2 & L_2\end{Bmatrix}\left(l_1\left\|\mathbf{C}^t\right\|l_2\right)\left(l_2\left\|\mathbf{C}^t\right\|l_3\right)R^t\left(n_1l_1n_2l_2,n_2l_2n_3l_3\right)\\
&+\delta(S_1,S_2')[L_1,L_2']^{1/2}\;(-1)^{l_2+L_1+L_2'}\;\begin{Bmatrix}S_1' & S_2' & S\\ S_2 & S_0 & 1/2\end{Bmatrix}\begin{Bmatrix}L_1' & L_2' & L\\ L_2 & L_0 & l_2\end{Bmatrix}\\
&\times\begin{Bmatrix}t & l_1 & l_3\\ l_1 & L_2' & L_1\end{Bmatrix}\begin{Bmatrix}L_1 & t & L_2'\\ l_2 & L_2 & l_2\end{Bmatrix}\left(l_1\left\|\mathbf{C}^t\right\|l_3\right)\left(l_2\left\|\mathbf{C}^t\right\|l_2\right)R^t\left(n_1l_1n_2l_2,n_3l_3n_2l_2\right)\Bigg].
\end{aligned}\qquad(32)$$

– Interaction $l^N l_1^2 l_2 \leftrightarrow l^N l_1 l_2 l_3$ or $l^N l_1^2 l_2 \leftrightarrow l^N l_1 l_3 l_2$:

$$\begin{aligned}
&\left\langle nl^N\alpha_0 S_0 L_0,(n_1l_1^2\alpha_1 S_1L_1,n_2l_2)S_2L_2;SL\left|\mathbf{G^k}\right|(nl^N\alpha_0'S_0'L_0',n_1l_1)S_1'L_1',(n_2l_2n_3l_3)S_2'L_2';SL\right\rangle =\\
&\sqrt{2}\;N\sum_{\bar\alpha\bar S\bar L}\left(nl^N\alpha_0S_0L_0\{|nl^{N-1}\bar\alpha\bar S\bar L\right)\left(nl^N\alpha_0'S_0'L_0'\{|nl^{N-1}\bar\alpha\bar S\bar L\right)\\
&\times(-1)^{S_0+L_0+3S_1+S_2+S+2\bar S+l+l_3+1}\;[S_1,L_1,S_2,L_2,S_1',L_1',S_2',L_2',L_0,L_0']^{1/2}\\
&\times\Bigg[\delta(S_0,S_0')\;(-1)^{l_2+2S_2+2S+S_2'+\bar L+L_1+L_2'}\begin{Bmatrix}1/2 & 1/2 & S_1\\ S_2 & 1/2 & S_2'\end{Bmatrix}\begin{Bmatrix}S_1' & S_2' & S\\ S_2 & S_0 & 1/2\end{Bmatrix}\\
&\times\begin{Bmatrix}t & l & l\\ \bar L & L_0' & L_0\end{Bmatrix}\begin{Bmatrix}l_1 & L_1 & l_1 & L_1'\\ l_3 & l_2 & L_2' & L_0\\ t & L_2 & L_0' & L\end{Bmatrix}\left(l\left\|\mathbf{C}^t\right\|l\right)\left(l_1\left\|\mathbf{C}^t\right\|l_3\right)R^t\left(nln_1l_1,nln_3l_3\right)\\
&+(-1)^{l_1+L_2+2S_0'+L_0'+L_1'+L}\;[S_0,S_0']^{1/2}\;\begin{Bmatrix}S_1' & \bar S & S_1\\ 1/2 & 1/2 & S_0'\end{Bmatrix}\begin{Bmatrix}S_1' & \bar S & S_1\\ S & S_0 & S_2\\ S_2' & 1/2 & 1/2\end{Bmatrix}\\
&\times\sum_{\lambda=|l-l_<|,\ldots,l+l_<}[\lambda]\begin{Bmatrix}\bar L & l & L_0'\\ l & t & l_3\\ L_0 & l_1 & \lambda\end{Bmatrix}\begin{Bmatrix}L_0 & & \lambda & & l_3 & & l_2 & \\ & l_1 & & L_0' & & L_2' & & L_2\\ L_1 & & l_1 & & L_1' & & L &\end{Bmatrix}\left(l\left\|\mathbf{C}^t\right\|l_3\right)\left(l_1\left\|\mathbf{C}^t\right\|l\right)R^t\left(nln_1l_1,n_3l_3nl\right)\Bigg]\\
&+\sqrt{2}\;\delta(\alpha_0S_0L_0,\alpha_0'S_0'L_0')\;[S_1,L_1,S_2,L_2,S_1',L_1',S_2',L_2']^{1/2}\;(-1)^{S_0+L_0+S_2+L_2+S+L}\\
&\times\begin{Bmatrix}1/2 & 1/2 & S_1\\ 1/2 & S_2 & S_2'\end{Bmatrix}\begin{Bmatrix}S_1' & S_2' & S\\ S_2 & S_0 & 1/2\end{Bmatrix}\begin{Bmatrix}L_1' & L_2' & L\\ L_2 & L_0 & l_1\end{Bmatrix}\\
&\times\Bigg[(-1)^{L_1}\;\begin{Bmatrix}l_1 & l_3 & L_1\\ l_2 & L_2 & L_2'\end{Bmatrix}\begin{Bmatrix}l_1 & l_1 & t\\ l_3 & l_1 & L_1\end{Bmatrix}\left(l_1\left\|\mathbf{C}^t\right\|l_1\right)\left(l_1\left\|\mathbf{C}^t\right\|l_3\right)R^t\left(n_1l_1n_1l_1,n_1l_1n_3l_3\right)\\
&+(-1)^{S_2'+l_1+l_2}\;\begin{Bmatrix}l_1 & l_1 & L_1\\ l_2 & L_2 & L_2'\end{Bmatrix}\begin{Bmatrix}l_1 & l_2 & t\\ l_3 & l_2 & L_2'\end{Bmatrix}\left(l_1\left\|\mathbf{C}^t\right\|l_2\right)\left(l_2\left\|\mathbf{C}^t\right\|l_3\right)R^t\left(n_1l_1n_2l_2,n_2l_2n_3l_3\right)\\
&+(-1)^{2S_2'+L_2'+l_1+l_2}\;\begin{Bmatrix}l_1 & l_1 & L_1\\ l_2 & L_2 & L_2'\end{Bmatrix}\begin{Bmatrix}l_1 & l_3 & t\\ l_2 & l_2 & L_2'\end{Bmatrix}\left(l_1\left\|\mathbf{C}^t\right\|l_3\right)\left(l_2\left\|\mathbf{C}^t\right\|l_2\right)R^t\left(n_1l_1n_2l_2,n_3l_3n_2l_2\right)\Bigg],
\end{aligned}\qquad(33)$$

where $l_<$ is the smaller value of l_1 and l_3,

$$\begin{aligned}
&\left\langle nl^N\alpha_0 S_0 L_0,(n_1l_1^2\alpha_1 S_1L_1,n_2l_2)S_2L_2;SL\left|\mathbf{G^k}\right|(nl^N\alpha_0'S_0'L_0',n_1l_1)S_1'L_1',(n_3l_3n_2l_2)S_2'L_2';SL\right\rangle =\\
&(-1)^{l_2+l_3-S_2'-L_2'}\\
&\times\left\langle nl^N\alpha_0 S_0 L_0,(n_1l_1^2\alpha_1 S_1L_1,n_2l_2)S_2L_2;SL\left|\mathbf{G^k}\right|(nl^N\alpha_0'S_0'L_0',n_1l_1)S_1'L_1',(n_2l_2n_3l_3)S_2'L_2';SL\right\rangle.
\end{aligned}\qquad(34)$$

– Interaction $l^N l_1^2 l_4 \leftrightarrow l^N l_1 l_2 l_3$:

$$\begin{aligned}
&\left\langle nl^N \alpha_0 S_0 L_0, (n_1 l_1^2 \alpha_1 S_1 L_1, n_4 l_4) S_2 L_2; SL \left| \mathbf{G^k} \right| (nl^N \alpha_0' S_0' L_0', n_1 l_1) S_1' L_1', (n_2 l_2 n_3 l_3) S_2' L_2'; SL \right\rangle = \\
&\sqrt{2}\ \delta(\alpha_0 S_0 L_0, \alpha_0' S_0' L_0')\ (-1)^{l_1+l_2+S+L+S_0+L_0+S_2+L_2}\ [S_1, L_1, S_1', L_1', S_2, L_2, S_2', L_2']^{1/2} \\
&\times \begin{Bmatrix} 1/2 & 1/2 & S_1 \\ 1/2 & S_2 & S_2' \end{Bmatrix} \begin{Bmatrix} l_1 & l_1 & L_1 \\ l_4 & L_2 & L_2' \end{Bmatrix} \begin{Bmatrix} S_2' & S & S_1' \\ S_0 & 1/2 & S_2 \end{Bmatrix} \begin{Bmatrix} L_2' & L & L_1' \\ L_0 & l_1 & L_2 \end{Bmatrix} \\
&\times \Bigg[(-1)^{S_2'} \begin{Bmatrix} l_1 & l_2 & t \\ l_3 & l_4 & L_2' \end{Bmatrix} \left(l_1 \left\| \mathbf{C}^t \right\| l_2\right) \left(l_4 \left\| \mathbf{C}^t \right\| l_3\right) R^t\left(n_1 l_1 n_4 l_4, n_2 l_2 n_3 l_3\right) \\
&+ (-1)^{L_2'} \begin{Bmatrix} l_1 & l_3 & t \\ l_2 & l_4 & L_2' \end{Bmatrix} \left(l_1 \left\| \mathbf{C}^t \right\| l_3\right) \left(l_4 \left\| \mathbf{C}^t \right\| l_2\right) R^t\left(n_1 l_1 n_4 l_4, n_3 l_3 n_2 l_2\right) \Bigg].
\end{aligned} \tag{35}$$

– Interaction $l^N l_4^2 l_2 \leftrightarrow l^N l_1 l_2 l_3$:

$$\begin{aligned}
&\left\langle nl^N \alpha_0 S_0 L_0, (n_4 l_4^2 \alpha_1 S_1 L_1, n_2 l_2) S_2 L_2; SL \left| \mathbf{G^k} \right| (nl^N \alpha_0' S_0' L_0', n_1 l_1) S_1' L_1', (n_2 l_2 n_3 l_3) S_2' L_2'; SL \right\rangle = \\
&\sqrt{2}\ \delta(\alpha_0 S_0 L_0, \alpha_0' S_0' L_0')\ (-1)^{l_1+l_4+S+L+S_0+L_0+L_1+S_2+L_2}\ [S_1, L_1, S_1', L_1', S_2, L_2, S_2', L_2']^{1/2} \\
&\times \begin{Bmatrix} 1/2 & 1/2 & S_1 \\ 1/2 & S_2 & S_2' \end{Bmatrix} \begin{Bmatrix} l_1 & l_3 & L_1 \\ l_2 & L_2 & L_2' \end{Bmatrix} \begin{Bmatrix} S_2' & S & S_1' \\ S_0 & 1/2 & S_2 \end{Bmatrix} \begin{Bmatrix} L_2' & L & L_1' \\ L_0 & l_1 & L_2 \end{Bmatrix} \\
&\times \begin{Bmatrix} l_4 & l_1 & t \\ l_3 & l_4 & L_1 \end{Bmatrix} \left(l_4 \left\| \mathbf{C}^t \right\| l_1\right) \left(l_4 \left\| \mathbf{C}^t \right\| l_3\right) R^t\left(n_4 l_4 n_4 l_4, n_1 l_1 n_3 l_3\right).
\end{aligned} \tag{36}$$

– Interaction $l^N l_1 l_2 l_3 \leftrightarrow l^N l_1 l_2 l_4$:

$$\begin{aligned}
&\left\langle (nl^N \alpha_0 S_0 L_0, n_1 l_1) S_1 L_1, (n_2 l_2 n_3 l_3) S_2 L_2; SL \left| \mathbf{G^k} \right| (nl^N \alpha_0' S_0' L_0', n_1 l_1) S_1' L_1', (n_2 l_2 n_4 l_4) S_2' L_2'; SL \right\rangle = \\
&\delta(\alpha_0 S_0 L_0, \alpha_0' S_0' L_0')\ \delta(S_1 L_1, S_1' L_1')\ \delta(S_2 L_2, S_2' L_2')\ (-1)^{l_2+l_3} \\
&\times \Bigg[(-1)^{L_2} \begin{Bmatrix} l_2 & l_2 & t \\ l_4 & l_3 & L_2 \end{Bmatrix} \left(l_2 \left\| \mathbf{C}^t \right\| l_2\right) \left(l_3 \left\| \mathbf{C}^t \right\| l_4\right) R^t\left(n_2 l_2 n_3 l_3, n_2 l_2 n_4 l_4\right) \\
&+ (-1)^{S_2} \begin{Bmatrix} l_2 & l_4 & t \\ l_2 & l_3 & L_2 \end{Bmatrix} \left(l_2 \left\| \mathbf{C}^t \right\| l_4\right) \left(l_3 \left\| \mathbf{C}^t \right\| l_2\right) R^t\left(n_2 l_2 n_3 l_3, n_4 l_4 n_2 l_2\right) \Bigg] \\
&+ \delta(\alpha_0 S_0 L_0, \alpha_0' S_0' L_0')\ [L_1, L_2, L_1', L_2']^{1/2}\ (-1)^{l_1} \\
&\times \Bigg[\delta(S_1, S_1')\ \delta(S_2, S_2')\ (-1)^{L_0+L+L_1+L_1'+l_2+l_4} \begin{Bmatrix} L_2' & L_2 & t \\ l_3 & l_4 & l_2 \end{Bmatrix} \begin{Bmatrix} L_2' & L_2 & t \\ L_1 & L_1' & L \end{Bmatrix} \begin{Bmatrix} l_1 & l_1 & t \\ L_1 & L_1' & L_0 \end{Bmatrix} \\
&\times \left(l_1 \left\| \mathbf{C}^t \right\| l_1\right) \left(l_3 \left\| \mathbf{C}^t \right\| l_4\right) R^t\left(n_1 l_1 n_3 l_3, n_1 l_1 n_4 l_4\right) \\
&+ [S_1, S_2, S_1', S_2']^{1/2}\ (-1)^{S_2+L_2+S_2'+L_2'+l_3+1} \begin{Bmatrix} S_0 & 1/2 & S_1 \\ 1/2 & 1/2 & S_2 \\ S_1' & S_2' & S \end{Bmatrix} \\
&\times \begin{Bmatrix} L_0 & l_1 & L_1 & L_2 \\ l_1 & t & l_3 & L_2' \\ L_1' & l_4 & L & l_2 \end{Bmatrix} \left(l_1 \left\| \mathbf{C}^t \right\| l_4\right) \left(l_3 \left\| \mathbf{C}^t \right\| l_1\right) R^t\left(n_1 l_1 n_3 l_3, n_4 l_4 n_1 l_1\right) \Bigg] \\
&+ [L_1, L_2, L_1', L_2']^{1/2}\ (-1)^{S_2+S_2'+L_0} \\
&\times \Bigg[\delta(S_0, S_0')\ \delta(S_1, S_1')\ \delta(S_2, S_2')\ (-1)^{L+l_1+l_2+l_4} \begin{Bmatrix} L_1' & L_1 & t \\ L_0 & L_0' & l_1 \end{Bmatrix} \begin{Bmatrix} L_1' & L_1 & t \\ L_2 & L_2' & L \end{Bmatrix} \begin{Bmatrix} l_4 & l_3 & t \\ L_2 & L_2' & l_2 \end{Bmatrix} \\
&\times \left\langle nl^N \alpha_0 S_0 L_0 \left\| \mathbf{U}^t \right\| nl^N \alpha_0' S_0' L_0' \right\rangle \left(l \left\| \mathbf{C}^t \right\| l\right) \left(l_3 \left\| \mathbf{C}^t \right\| l_4\right) R^t\left(nln_3 l_3, nln_4 l_4\right) \\
&+ [S_0, L_0, S_0', L_0', S_1, S_2, S_1', S_2']^{1/2}\ (-1)^{3S_0+S_0'+3S_1+S_1'+L_0'+L_1+L_1'+L_2+L_2'+1}
\end{aligned}$$

$$\times N \sum_{\bar\alpha\bar S\bar L} \left(nl^N\alpha_0 S_0 L_0\{|nl^{N-1}\bar\alpha\bar S\bar L\right) \left(nl^N\alpha_0' S_0' L_0'\{|nl^{N-1}\bar\alpha\bar S\bar L\right) (-1)^{\bar L} \begin{Bmatrix} 1/2 & 1/2 & S_2 & S_1 \\ 1/2 & \bar S & S_0 & S_1' \\ S_2' & S_0' & S & 1/2 \end{Bmatrix}$$
$$\left. \times \begin{Bmatrix} l_4 & & L_2' & & L_1' & & L_0' & & l & \\ & l_2 & & L & & l_1 & & \bar L & & t \\ l_3 & & L_2 & & L_1 & & L_0 & & l & \end{Bmatrix} \left(l \left\|\mathbf{C}^t\right\| l_4\right) \left(l_3 \left\|\mathbf{C}^t\right\| l\right) R^t\left(nln_3l_3, n_4l_4nl\right) \right]. \tag{37}$$

– Interaction $l^N l_1 l_2 l_3 \leftrightarrow l^N l_2 l_3 l_4$ or $l^N l_1 l_2 l_3 \leftrightarrow l^N l_2 l_4 l_3$:

$$\left\langle (nl^N\alpha_0 S_0 L_0, n_1l_1)S_1L_1, (n_2l_2n_3l_3)S_2L_2; SL \left|\mathbf{G^k}\right| (nl^N\alpha_0' S_0' L_0', n_2l_2)S_1'L_1', (n_3l_3n_4l_4)S_2'L_2'; SL\right\rangle =$$
$$\delta(\alpha_0S_0L_0, \alpha_0'S_0'L_0')\ [S_1, L_1, S_2, L_2, S_1', L_1', S_2', L_2']^{1/2} \begin{Bmatrix} S_0 & 1/2 & S_1 \\ 1/2 & 1/2 & S_2 \\ S_1' & S_2' & S \end{Bmatrix} \begin{Bmatrix} L_0 & l_1 & L_1 \\ l_2 & l_3 & L_2 \\ L_1' & L_2' & L \end{Bmatrix}$$
$$\times \left[(-1)^{L_2'+1} \begin{Bmatrix} l_1 & l_3 & t \\ l_4 & l_3 & L_2' \end{Bmatrix} \left(l_1 \left\|\mathbf{C}^t\right\| l_3\right) \left(l_3 \left\|\mathbf{C}^t\right\| l_4\right) R^t\left(n_1l_1n_3l_3, n_3l_3n_4l_4\right) \right.$$
$$\left. + (-1)^{S_2'+1} \begin{Bmatrix} l_1 & l_4 & t \\ l_3 & l_3 & L_2' \end{Bmatrix} \left(l_1 \left\|\mathbf{C}^t\right\| l_4\right) \left(l_3 \left\|\mathbf{C}^t\right\| l_3\right) R^t\left(n_1l_1n_3l_3, n_4l_4n_3l_3\right) \right]$$
$$+ \delta(\alpha_0S_0L_0, \alpha_0'S_0'L_0')\ [L_1, L_2, L_1', L_2']^{1/2}$$
$$\times \left[\delta(S_1, S_1')\ \delta(S_2, S_2')\ (-1)^{S_2+L_2+L+L_0+L_1+L_1'+l_4} \begin{Bmatrix} L_2' & L_2 & t \\ l_2 & l_4 & l_3 \end{Bmatrix} \begin{Bmatrix} L_2' & L_2 & t \\ L_1 & L_1' & L \end{Bmatrix} \begin{Bmatrix} l_2 & l_1 & t \\ L_1 & L_1' & L_0 \end{Bmatrix} \right.$$
$$\times \left(l_1 \left\|\mathbf{C}^t\right\| l_2\right) \left(l_2 \left\|\mathbf{C}^t\right\| l_4\right) R^t\left(n_1l_1n_2l_2, n_2l_2n_4l_4\right)$$
$$+ [S_1, S_2, S_1', S_2']^{1/2}\ (-1)^{S_2'+L_2'+l_1+l_3+1} \begin{Bmatrix} S_0 & 1/2 & S_1 \\ 1/2 & 1/2 & S_2 \\ S_1' & S_2' & S \end{Bmatrix}$$
$$\left. \times \begin{Bmatrix} L_0 & l_1 & L_1 & L_2 \\ l_2 & t & l_2 & L_2' \\ L_1' & l_4 & L & l_3 \end{Bmatrix} \left(l_1 \left\|\mathbf{C}^t\right\| l_4\right) \left(l_2 \left\|\mathbf{C}^t\right\| l_2\right) R^t\left(n_1l_1n_2l_2, n_4l_4n_2l_2\right) \right]$$
$$+ (-1)^{S_2'+L_2'+l_1+l_3+l_4+1}\ [S_1, L_1, S_2, L_2, S_1', L_1', S_2', L_2']^{1/2} \begin{Bmatrix} S_0' & 1/2 & S_1' \\ 1/2 & 1/2 & S_2' \\ S_1 & S_2 & S \end{Bmatrix} \begin{Bmatrix} L_0' & l_2 & L_1' \\ l_4 & l_3 & L_2' \\ L_1 & L_2 & L \end{Bmatrix}$$
$$\times \left[\delta(S_0, S_0')\ (-1)^{L_1+L_0'} \begin{Bmatrix} l_4 & l_1 & t \\ L_0 & L_0' & L_1 \end{Bmatrix} \left\langle nl^N\alpha_0S_0L_0 \left\|\mathbf{U}^t\right\| nl^N\alpha_0'S_0'L_0'\right\rangle \right.$$
$$\times \left(l \left\|\mathbf{C}^t\right\| l\right) \left(l_1 \left\|\mathbf{C}^t\right\| l_4\right) R^t\left(nln_1l_1, nln_4l_4\right)$$
$$+ [S_0, L_0, S_0', L_0']^{1/2}\ (-1)^{S_0+S_0'+l}\ N \sum_{\bar\alpha\bar S\bar L} \left(nl^N\alpha_0 S_0 L_0\{|nl^{N-1}\bar\alpha\bar S\bar L\right) \left(nl^N\alpha_0' S_0' L_0'\{|nl^{N-1}\bar\alpha\bar S\bar L\right)$$
$$\left. \times \begin{Bmatrix} 1/2 & \bar S & S_0' \\ 1/2 & S_1 & S_0' \end{Bmatrix} \begin{Bmatrix} \bar L & l & L_0' \\ l & t & l_4 \\ L_0 & l_1 & L_1 \end{Bmatrix} \left(l \left\|\mathbf{C}^t\right\| l_4\right) \left(l_1 \left\|\mathbf{C}^t\right\| l\right) R^t\left(nln_1l_1, n_4l_4nl\right) \right], \tag{38}$$

$$\left\langle (nl^N\alpha_0 S_0 L_0, n_1l_1)S_1L_1, (n_2l_2n_3l_3)S_2L_2; SL \left|\mathbf{G^k}\right| (nl^N\alpha_0' S_0' L_0', n_2l_2)S_1'L_1', (n_4l_4n_3l_3)S_2'L_2'; SL\right\rangle =$$
$$(-1)^{l_3+l_4-S_2'-L_2'}$$
$$\times \left\langle (nl^N\alpha_0 S_0 L_0, n_1l_1)S_1L_1, (n_2l_2n_3l_3)S_2L_2; SL \left|\mathbf{G^k}\right| (nl^N\alpha_0' S_0' L_0', n_2l_2)S_1'L_1', (n_3l_3n_4l_4)S_2'L_2'; SL\right\rangle. \tag{39}$$

Table 1. Comparison of the experimental and calculated energy values [cm^{-1}] and *hfs* A and B constants [MHz] for Ta even configuration system.

$E_{\rm exp}$	$E_{\rm calc}$	%	Main comp.	%	Sec. comp.	$g_{J_{\rm calc}}$	$g_{J_{\rm exp}}$	$A_{\rm exp}$		$A_{\rm calc}$	$B_{\rm exp}$		$B_{\rm calc}$
J=9/2													
5621.12	5651	69.97	$5d^3(^4F)6s^2\ ^4F$	14.79	$5d^3(^2G)6s^2\ ^2G$	1.288	1.272	256.617	(0.002)	198	−650.388	(0.044)	−654
10690.41	10653	37.93	$5d^3(^2H)6s^2\ ^2H$	34.57	$5d^3(^2G)6s^2\ ^2G$	1.062	1.063	327.0	(1.0)	293	2138	(20)	2276
13351.55	13365	90.08	$5d^4(^5D)6s\ ^6D$	3.06	$5d^4(^3F)6s\ ^4F$	1.535	1.533	1083.628	(0.0031)	1105	−2701.537	(0.008)	−2702
15391.02	15324	46.20	$5d^3(^2H)6s^2\ ^2H$	34.21	$5d^3(^2G)6s^2\ ^2G$	1.012	1.014	298.4	(1.0)	265	1993	(30)	1996
21153.40	21169	53.26	$5d^4(^3H)6s\ ^4H$	22.89	$5d^4(^3G)6s\ ^4G$	1.071	1.089	731.5	(1.0)	752	1275	(10)	2013
23912.93	23918	33.44	$5d^4(^3H)6s\ ^4H$	29.51	$5d^4(^3F)6s\ ^4F$	1.181	1.185	811.4	(0.5)	807	−610	(12)	−602
25376.47	25416	50.39	$5d^4(^3G)6s\ ^4G$	14.33	$5d^4(^3F)6s\ ^4F$	1.184		983.0	(0.1)	974	−960	(22)	−1026
29116.26	29044	39.88	$5d^4(^3H)6s\ ^2H$	15.17	$5d^4(^3F)6s\ ^4F$	1.052		690.0	(2.0)	733	546	(40)	387
32192.70	32149	28.50	$5d^4(^1G)6s\ ^2G$	27.77	$5d^4(^3H)6s\ ^2H$	1.056		1015.3	(0.8)	1031	323	(30)	422
33978.88	33989	52.75	$5d^4(^3G)6s\ ^2G$	10.81	$5d^4(^3F)6s\ ^4F$	1.127		45	(10)	96	100	(100)	−257
	36004	62.22	$5d^4(^3F)6s\ ^4F$	11.89	$5d^4(^3G)6s\ ^2G$	1.279				1067			1729
	41076	54.11	$5d^4(^1G)6s\ ^2G$	23.09	$5d^4(^1G)6s\ ^2G$	1.118				1347			1072
	43850	83.34	$5d^5\ ^4G$	3.56	$5d^5\ ^4F$	1.171				136			782
45636.87	45612	90.27	$5d^3(^4F)6s7s\ ^6F$	2.98	$5d^3(^4F)6s7s\ ^4F$	1.421	1.411	1411	(3.0)	1276	−939	(50)	−751
48269.58	48276	26.03	$5d^3(^4F)6s7s\ ^4F$	24.76	$5d^3(^4F)6s7s\ ^4F$	1.301		982.5	(2.0)	1130	−1065	(20)	−432
49200.69	49190	27.09	$5d^3(^4F)6s6d\ ^6G$	26.14	$5d^2(^3F)6s6p^2\ ^6G$	1.209		774	(11)	740	−413	(17)	−1161
49907.10	49910	28.11	$5d^5\ ^4F$	15.28	$5d^5\ ^2H$	1.195		389	(6)	436	−450	(120)	−631
50322.75	50340	28.83	$5d^3(^4F)6s6d\ ^6H$	24.01	$5d^2(^3F)6s6p^2\ ^6G$	1.146		673.8	(3.0)	630	−1025	(50)	−1086

– Interaction $l^N l_1 l_2 l_3 \leftrightarrow l^N l_2 l_4 l_5$:

$$\left\langle (nl^N\alpha_0 S_0 L_0, n_1 l_1) S_1 L_1, (n_2 l_2 n_3 l_3) S_2 L_2; SL \left| \mathbf{G^k} \right| (nl^N \alpha'_0 S'_0 L'_0, n_2 l_2) S'_1 L'_1, (n_4 l_4 n_5 l_5) S'_2 L'_2; SL \right\rangle =$$

$$\delta(\alpha_0 S_0 L_0, \alpha'_0 S'_0 L'_0)\ (-1)^{l_3+l_4+1} \begin{Bmatrix} S_0 & 1/2 & S_1 \\ 1/2 & 1/2 & S_2 \\ S'_1 & S'_2 & S \end{Bmatrix} \begin{Bmatrix} L_0 & l_1 & L_1 \\ l_2 & l_3 & L_2 \\ L'_1 & L'_2 & L \end{Bmatrix}$$

$$\times \Bigg[(-1)^{L'_2} \begin{Bmatrix} l_1 & l_4 & t \\ l_5 & l_3 & L'_2 \end{Bmatrix} \left(l_1 \left\| \mathbf{C}^t \right\| l_4\right) \left(l_3 \left\| \mathbf{C}^t \right\| l_5\right) R^t\left(n_1 l_1 n_3 l_3, n_4 l_4 n_5 l_5\right)$$

$$+ (-1)^{S'_2} \begin{Bmatrix} l_1 & l_5 & t \\ l_4 & l_3 & L'_2 \end{Bmatrix} \left(l_1 \left\| \mathbf{C}^t \right\| l_5\right) \left(l_3 \left\| \mathbf{C}^t \right\| l_4\right) R^t\left(n_1 l_1 n_3 l_3, n_5 l_5 n_4 l_4\right) \Bigg]. \quad (40)$$

4 Results

The examples of the results of the semi-empirical fine and hyperfine structure analysis for tantalum atom are shown in table 1. Details of the calculations and a more extensive comparison to the experiment are contained in a previously published paper [9].

5 Conclusions

The fine structure analysis should be carried out in for the broadest possible configuration basis. The derived and programmed formulae allowed us to analyse the spectra of elements with complex configurations systems. As a result of those analyses, it should be possible to predict the positions of new energy levels and determine the intermediate coupling wave functions, which is necessary to understand the strength of the transitions or the observed hyperfine structure splittings.

The results of the *fs*, *hfs* and oscillator strengths analyses in the multi-configurations systems show that a close connection between experimental work and semi-empirical calculations can be very fruitful in investigations of the complicated structure and spectra of complex atoms.

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