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ω symbol was erroneously capitalized in several equations and in the text of the article. The Publisher's erratum provides the correct article.

The Publisher apologizes for the inconvenience.

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E. Cartan's attempt at bridge-building between Einstein and the Cosserats – or how translational curvature became to be known as *torsion*[★]

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Abstract. Élie Cartan's "généralisation de la notion de courbure" (1922) arose from a creative evaluation of the geometrical structures underlying both, Einstein's theory of gravity and the Cosserat brothers generalized theory of elasticity. In both theories groups operating in the infinitesimal played a crucial role. To judge from his publications in 1922–24, Cartan developed his concept of generalized spaces with the dual context of general relativity and non-standard elasticity in mind. In this context it seemed natural to express the translational curvature of his new spaces by a rotational quantity (via a kind of Grassmann dualization). So Cartan called his translational curvature "torsion" and coupled it to a hypothetical rotational momentum of matter several years before spin was encountered in quantum mechanics.

1 Introduction

In a series of notes in the *Comptes Rendus* of the Paris Academy of Sciences submitted between February and April 1922 Élie Cartan sketched the basic ideas of a new type of geometry which was centrally based on the method of differential forms (Cartan, 1922*d,f,g,c,b,a*). The notes were an outgrowth of his investigations of Einstein's gravity theory under the perspective of his own differential geometry. He completed this first round of publications by applying his methods to the problem of space as it had recently been re-formulated by Hermann Weyl in the light of relativity. A detailed presentation of the ideas followed during the next years.¹

[★] A French version of this article appears in E. Haffner, D. Rabouin, eds. 2019. *L'épistémologie du dedans. Mélange en l'honneur d'Hourya Benis-Sinaceur*. Paris: Garnier.

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¹See Chorlay (2010) and Nabonnand (2016).

One aspect of Cartan's peculiar approach to differential geometry consisted in formulating the curvature concept of Riemannian differential geometry in terms of differential forms with values in the inhomogeneous Euclidean group operating in the infinitesimal neighbourhoods of any point. But the core of his new geometry lay elsewhere; it generalized the concept of curvature in two respects. The first generalization consisted in adding a translational component to the connection and correspondingly to the curvature. For the latter he chose the somewhat surprising name "torsion". The earliest public presentation of his idea was given in his note *Sur une généralisation de la notion de courbure de Riemann et les espaces à torsion* of February, 22nd (Cartan, 1922g).

The second, perhaps even more consequential, generalization lay in his proposal for allowing different types of groups operating in the infinitesimal neighbourhoods, rather than just concentrating on the group of Euclidean motions (respectively their Lorentzian counterpart, the Poincaré group). This made it possible to study various types of geometries arising from the conformal, the affine, the projective groups, or even more general Lie groups, with their respective pairing of inhomogeneous/homogeneous constituents. By this move Cartan reshaped the Kleinian program of structuring different types of geometry according to their automorphism groups in the context of differential geometry. In his note of March 13th, *Sur les espaces généralisés et la théorie de la Relativité* (Cartan, 1922c) the idea was first stated in some generality. The double aspect of infinitesimalizing the Kleinian view of geometry and of taking into account a translational component of connection and curvature was crucial for Cartan's *espaces généralisés* which later came to be known as *Cartan spaces*.²

The following paper concentrates on the first aspect of Cartan's generalization of differential geometry and the peculiar contexts which lay at the base of the, *prima facie* paradoxical, terminology of *torsion* for the *translational* component of the curvature. In the paper in which Cartan announced this new concept he described it in quite intuitive terms. He expressed the difference of his approach to classical (Euclidean) geometry similarly to what had been done by Levi-Civita and Weyl. That is, he considered the change a vector would undergo, if it is transported along an infinitesimal closed path according to the rules established by the generalized connection:

En définitive, à tout contour fermé infiniment petit de l'espace donné sont associées une translation et une rotation infiniment petites (...) qui manifestent la divergence entre cet espace et l'espace Euclidien (Cartan, 1922g, p. 594).³

The mentioned infinitesimal translation and rotation expresses the curvature properties of the space. Cartan immediately identified the well known case of Riemannian geometry with its Levi-Civita connection as the situation in which the translational component of the curvature vanishes.

A little later in the note he came back to the difference to the more classical geometries again and introduced a new terminology for the translational curvature mentioned above:

Dans le cas général où il y a une translation associée à tout contour fermé infiniment petit, on peut dire que l'espace donné se différencie de l'espace euclidien de deux manières: 1° par une *courbure* au sens de Riemann,

²For a modern presentation see Sharpe (1997).

³Definitely, to any infinitesimally closed curve of the space an infinitesimal translation and a rotation are associated (...); they express the divergence between this space and Euclidean space. – Translations in emphasized letters by ES; other translations in quotes with source indicated.

qui se traduit par la rotation; 2° par une *torsion*, qui se traduit par la translation (Cartan, 1922g, 594f.).⁴

But why did he call the translational curvature “torsion”?

A first clue follows immediately; but at first glance it enhances the riddle and introduces an even wider *quid pro quo*:

La rotation peut être représentée par un vecteur d’origine A et la translation par un couple (ibid.).⁵

Now everything has been turned upside down, rotations were expressed by vectors, translations by couples.

The last word of the sentence indicates that a mechanical context stood behind this move. In fact, Cartan indicated that one can study the equilibrium of an elastic medium in terms of his connection and curvature. This led him to formulate a geometrical picture of the constellation of forces:

On a ainsi une image géométrique d’un milieu matériel continu en équilibre, mais dans le cas où ces forces se manifesteraient sur chaque élément de surface, non seulement par une force unique (tension ou pression), mais par un couple (torsion) (Cartan, 1922g, 594).⁶

By *couple* Cartan referred to the traditional (18th and 19th century) expression for a rotational momentum (torque) by a pair of forces of the same norm, acting along different parallel lines in opposing orientations. This might superficially explain the rephrasing of translational curvature as “torsion”. But for the unprepared reader it still remains a riddle why Cartan identified the infinitesimal *rotations* with forces (vectors) and infinitesimal *translations* with rotational momentums (couples). From a purely geometrical point of view this identification would not appear particularly plausible. But at the end of the note Cartan gave a hint for the motivation of such an interchange. He indicated that

...les considérations précédentes (...) du point de vue mécanique, s’apparentent aux beaux travaux de MM. E. et F. Cosserat sur, l’action Euclidienne ...”. (ibid.).⁷

In addition he mentioned another link, namely to Weyl’s studies of the problem of space; but this does not lead us further for our question.⁸

If we want to understand the background of Cartan’s choice of terminology for the translational curvature we have to reconstruct the historical context of the unconventional theory of elastic media of the brothers Eugène and François Cosserat, which Cartan referred to. On the other hand, the geometrical picture of the elastic medium Cartan had in mind arose from his way of reading Einstein’s gravity theory in a mathematical analogy to elasticity. In order to understand Cartan’s intentions expressed in the note (Cartan, 1922g) we have to follow the traces of a “threefold knot” tied by

⁴To any closed infinitesimal loop there is generally an associated translation; in this case one can say that the given space differs from Euclidean space in two respects: 1. by a *curvature* in the sense of Riemann, which is expressed by the rotation; 2. by a *torsion* which is expressed by the translation (emphasis in the original).

⁵The rotation can be represented by a vector of origin A and the translation by a couple.

⁶One thus has a geometrical picture of a continuous material medium in equilibrium, but in the case where the forces express themselves not only by a single force (tension or pressure) but also by a couple (torsion).

⁷...from the mechanical point of view (...) the preceding considerations are similar to the beautiful works of the Messieurs E. and F. Cosserat on the Euclidean action ...

⁸For Weyl’s space problem see, among others (Bernard, 2018; Scholz, 2016); for the elasticity of the Cosserats (Brocato, 2009; Hehl, 2007; Pommaret, 1997).

Cartan between the *mathematical methods* developed for his new type of geometry, *Einstein's theory* of gravity, and the *generalized theory of elasticity* of the Cosserats.

We therefore start this paper with a short description of Cartan's mathematical arsenal used for constituting his generalized geometries (Sect. 2), continue with a résumé of Cosserat elasticity and its historical context (Sect. 3) before we shed a glance at Cartan's reading of Einstein gravity (Sect. 4). This allows us to reconstruct how Cartan linked these three components in an intriguing interplay between his geometrical picture of a Cosserat type elasticity theory and a (speculative) generalization of Einstein gravity by torsion (Sect. 5). We then look back at his practice of organizing the three-sided interplay between mathematics/geometry, elasticity theory, and gravity (Sect. 6), and give some indications of its repercussions on the work of physicists in the second half of the 20th century (Sect. 7).

2 A short outline of basic ideas of Cartan geometry

The usual differential geometric description of a *metric* ds^2 (Euclidean, Minkowski or (pseudo-)Riemannian) uses the differentials dx_i of the coordinates of a point $x = (x_1, \dots, x_n)$

$$ds^2 = \sum_{i=1}^n g_{ij} dx_i^2 dx_j^2$$

This corresponds to a choice of a coordinate basis in the tangent spaces (infinitesimal neighbourhood) of any point

and an expression of the metric with regard to this basis.

Cartan, in contrast, preferred to describe the metric in terms of differential forms $\omega_1, \dots, \omega_n$ which diagonalize the metric:

$$ds^2 = \sum_{i=1}^n \epsilon_i \omega_i^2, \quad \epsilon_i = \pm 1 \quad (1)$$

The ϵ_i (used by Cartan himself) account for different signatures of the metric, most importantly Euclidean/Riemannian and Minkowski/Lorentzian.⁹

This form can be arrived at by linear algebraic considerations in each infinitesimal neighbourhood. In his papers of 1922ff. Cartan emphasized that geometrically the diagonalization indicates a choice of point-dependent (“mobile”) *orthonormal reference systems*.¹⁰ Cartan called them Euclidean reference systems, “système de référence euclidien” (ibid., p. 151) or “trièdre trirectangle” (Cartan, 1922g) etc.

⁹(Cartan, 1922d, p. 150, Eq. (9))

¹⁰The paper (Cartan, 1922d) was written in 1921 and published only in the following year. Cartan remarked that the “germs” of his new geometry can be found at the beginning and the end of this paper. Before it was published, Cartan announced the basic ideas of his new geometry in several *Comptes Rendus* notes (Cartan, 1922f,g,c,b,a,e). Technical details followed in his long mémoire *Les*

and denoted them by $e_1, e_2, \dots, e_n$ (orthonormal basis ONB, or frame).

If we introduce the analogous symbols for the dual basis (at every point),

$$\{e^1, \dots, e^n\} \quad \text{dual basis of 1-forms to ONB } \{e_1, \dots, e_n\}, \quad (2)$$

Cartan's ω_i turn out to be nothing but these, $\omega_i = e^i$. In a coherent use of lower and upper indexes one therefore better writes Cartan's component forms as ω^i . In fact Cartan often, although not always, used upper and lower indices like in the tensor calculus,¹¹ e.g., ω^i_k in place of ω_{ik} . He also applied the Einstein summation convention abbreviating, e.g., $\sum_k \omega_{ik} \omega_{kj}$ by $\omega^i_k \omega^k_j$, etc.

If one moves between infinitesimally close points x, x' the reference systems undergo an infinitesimal rotation given by a system of coefficients (ω_{ij}) depending on the start point x and $\delta x = x' - x$:

Cartan realized that the coefficients of ω_{ij} can be understood as a system of differential forms (antisymmetric in the indices i, j). They encode the *rotational connection* of the space.

By analogy Cartan interpreted the ω_i as assigning to any δx a translational shift of the reference system identical to δx :

This was a new idea which paved the way for the first of Cartan's two innovations mentioned above. In addition to the role of the ω_i for representing the metric in diagonal form (and for specifying a "trièdre trirectangle") he used them for assigning

variétés à connexion affine et la théorie de la relativité généralisée (Cartan, 1923/1924b, 1925) and subsequent publications.

¹¹Upper indices ones for vector like, and lower ones for differential form like transformation behaviour under change of coordinates or reference systems.

a translation with components $\omega_i(\delta x)$ to any infinitesimal shift δx from the point x to an infinitesimally close one x' (Cartan, 1922d, p. 152). This was a first step towards turning the ω_i into a *translational connection* which complements the rotational one of the reference systems:

An important feature of Cartan's approach was that both parts of connection, the rotational and the translational one, were given component-wise by (real valued) differential forms. Present day readers may prefer to read them collectively as two differential forms, one with values in the Lie algebra of the rotational group $\bar{\omega} = (\omega^i_j)$ and one with values in the translations $\omega = (\omega^i)$. Moreover, a present reader might like to see an explicit expression for the covariant derivative ∇ of Cartan's connection $\tilde{\omega} = (\omega, \bar{\omega})$, which would generalize the Levi-Civita connection of the metric (1).¹² Cartan emphasized calculations which could be expressed in the calculus of differential forms, rather than rewriting the bulk of Ricci's and Levi-Civita's covariant tensor calculus in his symbolism.

For calculating the analogue of exterior differentials of the connection forms Cartan had to take the rotational coefficients into account. If both generalized exterior differentials are zero,

$$\begin{aligned} d\omega_i + \epsilon_i \sum_k \omega_{ik} \omega_k &= 0, \\ d\omega_{ij} + \sum_k \epsilon_k \omega_{ik} \omega_{kj} &= 0, \end{aligned} \quad (3)$$

so Cartan noticed, the space is *Euclidean* (flat).¹³

But in general this needs not be the case, and one encounters an *espace généralisé* (generalized space). If one then lets a point M traverse an infinitesimal loop starting and ending in A

... on ne retrouvera pas dans l'espace euclidien le trièdre initial mais il faudra, pour l'obtenir, effectuer un déplacement complémentaire dont les composantes sont bien définies par rapport au trièdre initial (Cartan, 1922g, p. 594, emphasis in the original).¹⁴

Cartan noted explicitly that this *déplacement complémentaire* is *independent* of the choice of reference systems. Today it is called the *Cartan curvature* of the space.

¹²For a lucid modern presentation see Sternberg (2012); for more technicalities (Gasparini, 2017, appendix).

¹³(Cartan, 1922d, pp. 145, 148)

¹⁴... one does not find the initial three-frame in the *Euclidean space* (meant is the tangent space in modern terms, ES), in order to arrive at it one rather has to apply a complementary displacement the components of which are well defined with regard to the original three-frame.

The translational and rotational *déplacement complémentaire* Ω^i and Ω^i_j , i.e. the deviations from zero of the above given expressions are the 2-forms:

$$\Omega^i = d\omega^i + [\omega^i_k \omega^k] = A^i_{jk} [\omega^j \omega^k] \quad (\text{“torsion”}) \quad (4)$$

$$\Omega^i_j = d\omega^i_j + [\omega^i_k \omega^k_j] = A^i_{jkl} [\omega^k \omega^l] \quad (\text{“courbure”}), \quad (5)$$

where the square brackets denote alternating products and the Einstein sum conventions is applied. They were adapted by Cartan from Hermann Grassmann.¹⁵ Cartan called the equations (3) and (4) “les équations de structure” (structural equations) of the generalized space (Cartan, 1923/1924b, p. 368). For vanishing torsion the A^i_{jkl} characterize the Riemannian curvature in Cartan’s symbolism (Cartan, 1922d, p. 154). But it may also happen that the rotational curvature vanishes, while the torsion is non-trivial.

A curve with a tangent vector field which is parallel in the sense of the Cartan connection is called an *autoparallel*, while a curve of extremal length is called a *geodesic*. In Riemannian geometry both concepts agree, but in Cartan geometry (modeled on the Euclidean or pseudo-Euclidean group) they usually fall apart. But this need not be so. Cartan gave a simple example of a structure in dimension $n = 3$ with vanishing rotational curvature and non-trivial torsion, in which geodesics and autoparallels coincide (Cartan, 1922g, p. 595).¹⁶

Cartan generalized this approach to allow for more general groups than the orthogonal ones, operating in the infinitesimal neighbourhoods. At the moment we need not follow this generalization in more details; but in general, the metric lost its central place and the “trièdre trirectangulaires” had to be replaced by more general frames. Cartan called the arising spaces *espaces non-holonomes* (non-holonomous spaces).¹⁷ During the 1920s he studied such spaces of increasingly complex type with the following groups:

- The Poincaré group in papers on the geometrical foundation of general relativity (Cartan, 1922c, 1923), (Cartan, 1923/1924b). For torsion $\Omega^i = 0$ such a Cartan space reduces to a Lorentzian manifold. Cartan could use this reduction for treating Einstein’s theory in his own geometric terms.
- The inhomogeneous similarity group. For torsion = 0, this case reduces to Weylian manifolds (Cartan, 1923).
- The conformal group (Cartan, 1922b).
- The projective group (Cartan, 1924c).

In this way, Cartan developed a wide conceptual frame for studying different types of differential geometries, Riemannian, Lorentzian, Weylian, affine, conformal, projective. All were enriched by the possibility to allow for the new phenomenon of torsion, and all arose from Cartan’s unified method of adapting the Kleinian viewpoint to infinitesimal geometry. But if we want to understand his first papers of the year 1922 and the immediately following ones, we have to know a bit the “beaux travaux de MM. E. and F. Cosserat” (Cartan, 1922g).

¹⁵Later authors, more precisely Ernst Kähler, introduced the now common symbolism $[ab] = a \wedge b$.

¹⁶In the context of the studies of (generalized) Cosserat media this structure attracts attention until today as an example with intriguing geometrical properties (Hehl, 2007; Lazar, 2010).

¹⁷For the historical background of this terminology see Nabonnand (2009).

3 Generalized elasticity theory

In the early 19th century a group of mainly French authors developed the foundations of the linear elasticity theory of solid bodies. Augustin J. Fresnel (1821) and Claude-Louis Navier (1827) derived their theories on the basis of a molecular theory of matter with central forces acting between the discrete units of matter. When Augustin-Louis Cauchy jumped in between 1823 and 1828, he first approached the question from the point of view of a continuum theory of matter and derived his influential representation of the linear relationship between the *strain* matrix characterizing the deformation of the material and the *stress* matrix (both later understood as tensors) from a phenomenological *Ansatz*.¹⁸

But the molecular theory of matter behind these different approaches remained dominant. In 1827 also Cauchy presented a derivation of Navier's equations on the basis of a molecular approach. A year later Simon D. Poisson developed the linear elasticity theory of molecular matter a step further and brought it home to the Laplacian program of physics.¹⁹

Poisson's theory was built upon the hypothesis of central forces acting between point-like centers inside the radius of a “molecular sphere” outside of which the forces are no longer to be felt. The phenomenological forces in the material on a surface element were derived by summing up all the forces in the range of the “molecular spheres” of points intersecting or touching the surface element. For isotropic solid matter the calculations resulted in a linear relation between strain (deformation) and stress (surface forces), which depended on a single material constant. The basic structure of the theory seemed empirically convincing; but the 1-parameter assumption turned out to be untenable even for isotropic matter; a second elastic constant had to be assumed to fit the data.²⁰

An alternative derivation of the equations of elasticity was proposed by George Green in 1838. He avoided any hypothesis about the basically unknown molecular structure of matter and based his analysis on a potential function ϕ from which the forces in the elastic medium could be derived by very general formal considerations (Darrigol, 2012, p. 234ff.). Although this approach led to quite acceptable results, including the empirically necessary two elastic constants in the case of an isotropic medium, Green's theory did not manage to replace the research program following the molecular hypothesis in France (Timoshenko, 1953, pp. 217ff.) and was only partially accepted in Germany (Darrigol, 2012, p. 226). But it became an important input for the generalized theory of elasticity of the brothers Cosserat to whom Cartan referred in his note of 1922.

In the late 1880s Woldemar Voigt (1850–1919) gave a detailed analysis of the actual status of the molecular theory of elasticity in a report to the *Göttingen Gesellschaft der Wissenschaften* (Voigt, 1887). He carefully reviewed the molecular elasticity theory of the French tradition and proposed a refinement of it, which would take into account that the molecules are extended bodies of different shapes. In general, the form of the molecules breaks the rotational symmetry of the old pointlike force centers; thus not only the coordinates of the centers of the molecules, but also their directional properties had to be considered.

As a result, the molecular interactions could no longer be represented by forces alone but had to be complemented by the consideration of rotational momenta,

¹⁸For Cauchy's contributions to elasticity see (Dahan-Dalmedico, 1992) and (Belhoste, 1991).

¹⁹A little later the Laplacian program started to be undermined from different sides: Fourier's theory of heat, electricity, magnetism, optics (Fox, 1974).

²⁰For more details on this development see (Capecchi, 2010; Fox, 1974; Timoshenko, 1953), (Darrigol, 2005, Chap. 3) and (Grattan-Guinness, 1990, Chap. 7).

torque, which depend on the relative “polarity” of the molecules.²¹ For a full representation of the position and the orientation of the molecules the coordinates of their barycenters and the directions of a system of axis, tied to the molecule and changing from one to the other had to be taken into account.²² From a mathematical point of view, Voigt’s description resembled point dependent *répères mobiles* linked to the different orientations (“polarisations”) of the molecules in a material structure. But neither he nor mathematicians at the time took up this analogy.

For studying equilibrium conditions on the macro-level, Voigt considered forces and rotational momenta on surface or volume elements, given with regard to an axis system by the components (Y, Y, Z) and (L, M, N) respectively. They came about from the summation of the corresponding actions on the micro-level and had to be studied in the rest state and, if subject to external forces, in a deformed state (ibid. p. 10). A clear and quite detailed study of Voigt’s further derivation is given in (Capecci, 2010). We need not go into the details here, because in the course of his calculations Voigt introduced the assumption that for all practical purposes *the point dependence of the rotational momenta induced in the material even by deformations could be neglected*.²³

So the bulk of Voigt’s enriched structure theory on the micro-level (the point dependence of the axis systems linked to the molecules and their deformations) remained without visible consequences, once one turned to the phenomenological level. One effect remained however. Voigt’s calculations led to introducing a second parameter resulting from a global rotational momentum which was not present in the older molecular theory. It filled the gap which had arisen between the older molecular theory of elasticity and the experimental findings. In the end, this was the main achievement of Voigt’s approach.

Other authors explored alternatives in the framework of continuum mechanics. Particularly important in our context was the joint research of *François Cosserat* (1853–1914) and his younger brother *Éugène Cosserat* (1866–1931) during the two decades between 1896 and 1914. François was a civil engineer working for the French railroad system. He studied at the École Polytechnique and graduated at the École des Ponts et Chaussées. Éugène studied mathematics at the École Normale Supérieure under the guidance of Paul É. Appell, Jean-Gaston Darboux, Gabriel Koenigs and

²¹ “Wir denken uns das homogene krystallinische Medium bestehend aus einem System von Molekülen, welche durch ihre Wechselwirkungen einander im Gleichgewicht halten. Diese Wechselwirkungen sind Kräfte und Drehmomente, deren Componenten in unbekannter Weise mit der relativen Lage der Moleküle variieren.” (Voigt, 1887, p. 5)

We conceive the homogeneous crystalline medium as consisting of a system of molecules which stand in equilibrium by their mutual interactions. These interactions are forces and rotational momenta, the components of which vary with the relative position of the molecules in an unknown way.

²² “Da die Moleküle nach unserer Annahme eine Polarität besitzen, so muss man sie wie endliche Körper behandeln und ihre Lage ausser durch die Coordinaten ihres Schwerpunktes noch durch die Richtung eines fest mit ihnen verbundenen Axensystemes bestimmen.” (ibid., p. 6)

According to our assumption the molecules possess a polarity, one therefore has to treat them like finite bodies and has to specify their position in addition to the coordinates of their barycenter by the direction of an axis system rigidly tied to them.

²³ “...sind die in den Ausdrücken für die Drehungsmomente vorkommenden Coefficienten als unendlich klein gegen die in den Componenten X_x ... auftretenden anzusehen. Dies hat den Effect ihre Differentialquotienten neben den übrigen Gliedern zu vernachlässigen sind, – in Übereinstimmung mit der Umstände, dass bei allen bekannten Problemen an der Oberfläche der elastischen Körper $\bar{L}_n, \bar{M}_n, \bar{N}_n$ (the rotational momentum represented as a vector normal to the surface, ES) gleich Null zu setzen ist ...” (Voigt, 1887, p. 23).

... the expressions for the coefficients in the rotational momenta have to be considered as infinitely small with respect to those appearing in the components X_x ... This leads to the effect that their differential quotients can be neglected in comparison with the other terms – in accordance with the fact that in all known problems at the surface of elastic bodies $\bar{L}_n, \bar{M}_n, \bar{N}_n$ (the rotational momentum represented as a vector normal to the surface, ES) may be equated to zero ...

Émile Picard. After his graduation in 1886 and a few months of teaching at a Lycée in Rennes he became an assistant astronomer at the Observatory in Toulouse. Parallel to observational work on binary stars he wrote a dissertation in mathematics with a topic in differential geometry. In 1889 he finished his PhD in Paris under the supervision of Appell, Darboux and Koenigs. In 1896 he succeeded Thomas Stieltjes as a professor in mathematics at Toulouse University. Roughly a decade later (1908) he became the director of Toulouse Observatory and professor of astronomy. He was elected corresponding member of the Paris Académie des Sciences in 1911 and became a full member in 1919 (Levy, 1971). After the early death of his older brother he discontinued work on elasticity theory. The last joint publication of the two appeared after François' death. It was an extended French version of Aurel Voss essay on rational mechanics for the *Encyclopédie des sciences mathématiques pure et appliquées*. It provided the occasion for explaining the wider perspective of their research in rational mechanics (Brocato, 2009, Sect. 4).

The two brothers studied elasticity theory in a strictly deductive Lagrangian approach to continuum mechanics, while acknowledging that its aim was a rational understanding of inductively generalized empirical knowledge. Their first paper (“premier mémoire”) appeared in the year Éugène became a professor of mathematics in Toulouse (Cosserat, 1896). A series of papers followed; but their main result did not become mature before 1909. In this year they managed to derive, on the basis of two principles, a set of generalized equations for the equilibrium of an elastic medium carrying forces and torques. They presented their new theory in several variants to the scientific public (Cosserat, 1909a,b,c).²⁴ The main presentation is contained in their book (Cosserat, 1909c); for a short historical evaluation of its content and reception see (Mauguin, 2014, Chap. 8).

Their first principle was the *invariance* of the action under transformations of the *inhomogeneous Euclidean group* for elastic continua of dimension $n = 1, 2, 3$ (elastic rod, plate, body). In their terminology they worked with an “action euclidienne”. As a second principle they characterized the elements of the elastic continuum by *point dependent “trièdres”* (orthonormal frames) rather than studying elastic deformations of a simple point continuum. In this form they proposed to consider “polarized” (directionally oriented) molecules. They incorporated them into the continuum mechanics framework and gave them a form which nicely corresponded to Darboux style differential geometry.²⁵

In their last paper they described a limit idea underlying this approach. In the older approach to elasticity ordinary geometric space was considered as an adequate mathematical representation of the physical medium (“milieu”). But, according to the Cosserats, the studies of elasticity, crystallography, electricity and of light made it necessary to consider a more complex notion of the continuum (“une notion plus complexe du milieu continu”).

... This notion is derived in all generality from a discontinuous collection of *point systems with an arbitrary number of degrees of freedom* by passing to the limit. (Cosserat, 1915, p. 72, emphasis ES)²⁶

²⁴For list of all common papers of Éugène and François Cosserat and a discussion see Brocato (2009).

²⁵In his PhD dissertation Éugène had already investigated infinitesimal circles as space elements, combining ideas of Plücker's generalized space elements with Darboux' differential geometry.

²⁶“Cette notion se déduit dans toute sa généralité, par un passage à la limite, de celle d'un ensemble discontinu de systèmes de points à un nombre quelconque de degrés de liberté” citation from (Brocato, 2009, p. xx). With regard to crystallography one may easily recognize the discrete “point systems” as an abstract representation of the lattice structure of polyhedral molecules studied by Bravais or the more refined structures of the 1890s according to the research tradition of Fedorow or Schoenflies.

This may be read as a late reflection on the motivations which had brought them to study the influence of the directionally oriented elements on the equilibrium conditions in all generality (not only under the restrictive assumptions used by Voigt).²⁷

The Cosserats characterized an element of the undeformed continuum (“état primitif” or “état naturel”) by the coordinates $x = (x_1, x_2, x_3)$ of a point p with regard to a fixed Euclidean frame $\mathfrak{D}$ and orthonormal frame (“trièdre trirectangle”) $\{e_1, e_2, e_3\}$ attached to the point and specified by a point dependent rotation $o(x)$ with regard to the reference system $\mathfrak{D}$. For the sake of brevity we denote such an oriented continuum element here by $(x, o(x))$.²⁸ The coordinates x could be changed by a smooth coordinate transformation. A deformed state $(x', o'(x'))$ of the medium, on the other hand, was described by transformations $x' = f(x)$ and $o'(x) = g(o(x))$ with smooth functions f, g . The range of possible infinitesimal deformations was then characterized by the 9 partial derivatives of the three components of f , which we denote collectively by ∂f and 9 partial derivatives of the angle transformations of type ∂g .²⁹

To begin with, they assumed a time-independent action density W for the deformation of the continuous medium and analyzed it step by step for the dimensions $n = 1, 2, 3$. For dimension $n = 3$ the (non-kinetic part of the) action was of the general form

$$\mathfrak{S} = \int_{A_o} W(x, \partial f, \partial g) dx, \quad (6)$$

with A_o the space region occupied by the elastic body in the natural state. Thus their action depended on $21 = 3 + 9 + 9$ continuous parameters (Cosserat, 1909a, p. 559). Its form was constrained only by the *postulate of invariance under (infinitesimal) Euclidean motions*. Analyzing the variation of a not necessarily “natural”, i.e. force free, state they were able to derive formal expressions for the (external) *forces* (X_i) and *rotational moments* (L_i) acting on the volume elements.³⁰ The

$$\begin{array}{lll} \text{surface densities of} & \text{force } (F_1, F_2, F_3) & \text{and torque } (J_1, J_2, J_3) \\ \text{and volume densities of} & \text{force } (X_1, X_2, X_3) & \text{and torque } (L_1, L_2, L_3) \end{array}$$

arose from the variation of (6) with regard to infinitesimal changes of the point coordinates δx_i and to infinitesimal rotations δj_i of the *trièdres*. This accorded to the venerated principle of virtual velocities:

$$\begin{aligned} \delta \int_{A_o} W dx &= \int_{S_o} \sum_i (F_i \delta x_i + J_i \delta j_i) d\sigma \\ &+ \int_{A_o} \sum_i (X_i \delta x_i + L_i \delta j_i) dx, \end{aligned}$$

where $d\sigma$ denotes the surface element on S_o (Cosserat, 1909a, p. 597). By dissecting the medium along a surface S inside A analog expressions could be discerned for

²⁷The Cosserats saw and commented the relationship between Voigt’s and their work. They were clearly aware of their own achievements; see fn 34.

²⁸The original notation of the Cosserats for the trièdre $M_o x_o, M_o y_o, M_o z$ was given by angle cosinus to the fixed reference system (Cosserat, 1909a, p. 559).

²⁹Expressed in coordinates of the fixed reference system $\mathfrak{D}$, the Cosserats gave ∂f as (ξ_i, η_i, ζ_i) and ∂g by (p_i, g_i, r_i) , where the index $i = 1, 2, 3$ indicates the partial derivative with regard to x_i (Cosserat, 1909a, pp. 559, 596), similar in Cosserat (1909c).

³⁰Volume forces may result from fields permeating the continuum; but the authors did not discuss the origin of them.

the surface densities of *internal force* (F_i) and *torque* (J_i) (“effort et moment de déformation”).³¹

The evaluation of the invariance under Euclidean motions became a complicated task. After diverse transformations the authors derived two sets of equations involving auxiliary quantities p_{ij} and q_{ij} (“dix-huit nouvelles auxiliaires”) indicating the possibility of asymmetric stress and a corresponding torque.³² Later readers would read these quantities as tensors of the surface density of *force stress* and *torque stress* respectively (sometimes also called “proper stress” and “couple stress” or “spin momentum stress” respectively). The Cosserats usually talked about *effort of deformation* (“effort de déformation” and *moment of deformation* (“moment de déformation”).

In slightly streamlined notation the equations of the Cosserats read as

$$X_j = \sum_i \frac{\partial p_{ij}}{\partial x_i} \quad (7)$$

$$L_j = \sum_i \frac{\partial q_{ij}}{\partial x_i} + p_{kl} - p_{lk} \quad (8)$$

where the (j, k, l) in the last line form cyclical permutations of $(1, 2, 3)$. Moreover they found that the internal forces and torques can be expressed by the auxiliary quantities:

$$F_j = \sum_i p_{ij} n_i \quad J_j = \sum_i q_{ij} n_i; \quad (9)$$

here the n_1, n_2, n_3 are the components of an interior directed unit normal of a surface element (of unit area) at any point (Cosserat, 1909a, p. 601, Eqs. (29), (30)).³³

The equations (7–9) are now known as the *fundamental equations of elastostatics*. This theory includes the older linear theory of elasticity as a special case: If the densities of (external) volume forces and torque and the (internal) torque stresses vanish, $X_j = 0, L_j = 0, q_{ij} = 0$, equation (8) implies a symmetric stress tensor, $p_{ij} = p_{ji}$, which satisfies $\sum_j \frac{\partial p_{ij}}{\partial x_i} = 0$. This may be considered as a “conservation condition”; but here, in the statical context, it indicates the equilibrium of the integrated forces acting on a closed surface inside the medium.

After having derived the equilibrium conditions for the surface torque momentum (see below), the Cosserats stated:

Les auxiliaires que nous venons d’introduire et les équations qui les lient ne paraissent pas avoir été jusqu’ici envisagées sous une forme aussi générale; à notre connaissance, elles n’ont été considérées que dans les cas particuliers où les neuf quantités $q_{xx}, \dots, q_{zz}$ (Cosserat’s expression for the components of the surface density of torque, here denoted by q_{ij} , ES) sont nulles, et le premier travail qui traite alors de la question semble être celui de M. Voigt (Cosserat, 1909c, p. 137).³⁴

³¹The original notation was (F, G, H) for our (F_i) , I, J, K for our (J_i) , and (X, Y, Z) for (X_i) , respectively (L, M, N) for the (L_i) .

³²Non-symmetric stress coefficients and torque had already been investigated by MacCullagh in his non-conventional ether theory of the 1840s (Darrigol, 2012, p. 243ff.). This parallel was indicated to me by an anonymous referee.

³³Again our notation is slightly streamlined; the original notation was $p_{xx}, p_{xy}, \dots, p_{zz}$ for the stress densities $q_{xx}, q_{xy}, \dots, q_{zz}$ for torque (Cosserat, 1909a, p. 601). In the book (Cosserat, 1909c) the equations appear on p. 137 in exactly the same form.

³⁴“The auxiliary functions that we just introduced and the equations that relate them do not appear to have been envisioned in a form that was that general up till now; to our

In a footnote added by the Cosserats they gave precise references to [Voigt \(1887\)](#) and [Voigt \(1900\)](#).

Given the generality of the assumptions, this was a great achievement. But the great complexity of the calculations made the results extremely difficult to absorb.³⁵ Only in retrospect could the Cosserats' theory be put into the context of wider mathematical theories and their derivations be justified on the basis of general theorems, which involved less, or at least different, calculations: The equations (7) and (8) were identified as the Noether equations with regard to translational, respectively rotational invariance of the action ([Hehl, 2007](#)). [Pommaret \(1997\)](#) sees them as a special case of non-linear Spencer transformations in the theory of partial differential equations. Elasticity theorists had to develop their own viewpoint which gave reasons to address the study of general elastic media. In any case, Cosserat theory did not enter the broader theoretical or even experimental research for at least half a century. It was revived only in the 1950/60s. Even today it cannot be considered mainstream, although it now seems to form an interesting sidestream of its own ([Brocato, 2009](#); [Mauguin, 2010](#), pp. xi ff.).

During the course of their work, the Cosserats developed a perspective of a grand unifying scheme for theoretical mechanics, covering hydrodynamics, heat conduction, electrodynamics, and elasticity ([Cosserat, 1915](#)).³⁶ This turned out to be an untimely enterprise: the relativity theories, special and general, and the rising quantum mechanics were just changing the role of rational mechanics in mathematical physics. Although classical mechanics was not invalidated in its core, it lost its central and foundational role for natural philosophy of the 20th century. In consequence the overarching perspective of Cosserats' research program lost much of its power of persuasion. This may have contributed to the relative neglect of their generalized elasticity theory, in addition to its intrinsic technical difficulties.

On the other hand, the theory of elasticity proposed by E. and F. Cosserat was highly valued by a small group of mathematical scientists, mainly in France but also internationally.³⁷ Our protagonist, Élie Cartan, was one of the admirers. His re-reading of the Cosserats' elasticity theory took place in the wider context of his investigations of Einstein's gravity theory, which we consider next.

4 Cartan's re-reading of Einstein gravity

As already remarked, Cartan's new geometric ideas were spelled out at the occasion of his studies of Einstein's general theory of relativity. He started with analyzing the form of the Einstein equation from a mathematician's point of view. In Riemannian geometry, with metric $g = \sum_{ij} g_{ij} dx_i dx_j$, it may be written summarily as

$$G = \kappa T, \quad (10)$$

knowledge, they have been considered only in the particular case in which the nine quantities $q_{xx}, \dots, q_{zz}$ are null, and the first work to treat that question seems to be that of [Voigt \(1909c\)](#), p. 132, Delphenich's English translation). Moreover they recommend to compare with the papers ([Combebiac, 1902](#); [Larmor, 1891](#); [Love, 1892/1906](#)). In part of the literature P. Duhem is mentioned as a possible source for the Cosserats' turning towards oriented elements of the continuum. This seems implausible, however, because they did not mention him at this place, but only later with regard to the use of reversible transformations ([Cosserat, 1915](#), p. 73f). The other way round, Duhem quotes the Cosserats positively in his ([Duhem, 1906](#), p. 3); see ([Brocato, 2009](#), pp. xxv, xxxv).

³⁵Readers interested in technical details may like to consult ([Badur, 1989](#)).

³⁶For a detailed discussion of this point see ([Brocato, 2009](#), Sect. 4, pp. xxxvi ff.)

³⁷([Brocato, 2009](#), Sect. 5, pp. xxxivff.)

where T denotes the energy-momentum-stress tensor of matter, κ the gravitational constant ($\kappa = 1$ for Cartan).³⁸

- (o) $G = \sum_{ij} G_{ij} dx_i dx_j$, abbreviated G_{ij} , is a symmetric covariant 2-tensor, which contains the first and second partial derivatives of the components g_{ij} only. Equivalently it can be perceived as a vector valued 1-form G^i_j .

In Einstein gravity the left hand side is the *Einstein tensor*,

$$G = Ric - \frac{R}{2}g, \quad (11)$$

where Ric and R stand for the Ricci, respectively scalar curvature of the Levi-Civita connection associated to g . In Cartan's view, the study of gravitational equations (plural!) boils down to the question which covariants may serve on the left hand side of equation (10) as the (non-linear) partial differential operator on g . In any case, one should take into account two constraints which Einstein had emphasized as basic principles:

- (i) G is linear in the second partial derivatives $\partial\partial g$,
- (ii) G satisfies the *conservation law* ("loi de conservation").

(i) is necessary for avoiding too complicated differential equations. (ii) is a consequence of demanding a vanishing covariant divergence of the energy momentum tensor. By (10) this translates to the left hand side as $\nabla_i G^i_j = 0$ in Ricci calculus (with ∇ the covariant derivative associated to g). Cartan preferred to express conservation as the vanishing of exterior covariant differential of G , which we denote here as

$$d_\omega G = 0, \quad (12)$$

because it is defined with regard to a Cartan connection $\tilde{\omega} = (\omega^i, \omega^j_k)$.³⁹ In his *Comptes rendus* notes of 1922 Cartan apparently considered it as a criterion for an elastic medium with no external forces standing in equilibrium.

In his first paper on Einstein gravity Cartan introduced his method of differential forms (outlined in Sect. 2) for Riemannian geometry only. Using the Cartanized coefficients of the Riemann curvature (Eq. (5)) he showed that a G satisfying conditions (o) and (i) is a linear combination of Ric , Rg and g , all three expressed in terms of the basic differential forms $\omega^1, \dots, \omega^n$ (Cartan, 1922d, p. 196). If also the constraint (ii) is taken into account only the Einstein tensor form (11) plus a linear term g remains, in the symbols introduced above:

$$G = \alpha \left(Ric - \frac{R}{2}g \right) + \beta g, \quad (13)$$

with two arbitrary constants α, β (Cartan, 1922d, p. 203).

From a mathematician's point of view, that was a highly pleasing result. Cartan was cautious, however, whether something similar had not perhaps been already derived (in terms of the Ricci calculus) and published elsewhere in the international

³⁸Following Einstein, Cartan used a different sign $G = -T$. This is a question of conventions, expressing the choice of a different sign for the Ricci contraction of the Riemann tensor and the signature dependence of energy momentum.

³⁹For the torsion-free case see (Cartan, 1922d, 199); for the general case (Cartan, 1923/1924b); modern presentations in, e.g., (Hehl, 1986), (Gasperini, 2017, pp. 269ff.) etc.

literature which, due to the effects of the great war, may have remained unknown in Paris.⁴⁰ In fact, more or less at the same time at which Cartan wrote his manuscript of [Cartan \(1922d\)](#) Hermann Weyl proved that in Riemannian geometry the scalar curvature R is the only invariant containing not more than the first and second derivatives in g , and the second ones only linearly. The proof was published about the time of Cartan's submissions of his notes to the *Comptes Rendus* in an appendix to the fourth edition of *Raum - Zeit - Materie* ([Weyl, 1921](#), p. 287f., Anhang II).⁴¹ In the framework of a Lagrangian approach Weyl's theorem implied the same restriction for the Einstein tensor, which Cartan had derived.⁴² Weyl's proof had the advantage of being much shorter, but Cartan's analysis went deeper to the basic principles and was more general, independent of a Lagrangian approach to the Einstein equation.

In our representation of the general form of (13) we have assimilated Cartan's result to the more common notation of tensor calculus. But we have to keep in mind that Cartan used a different mathematical representation. That influenced also his interpretation of the Einstein tensor:

Nour regarderons ses composantes comme des coefficients entrant dans l'expression de la projection sur une direction fixe d'une tension appliquée à un élément à trois dimension de l'univers à quatre dimensions ([Cartan, 1922d](#), p. 199).⁴³

In other words, he conceptualized the “tenseur gravitationnel” G as a *vector valued* (alternating) *3-form* $\tilde{G}$, which, by analogy to classical elasticity, expresses the respective stress force (“tension”) exercised on a 3-dimensional volume element in the 4-manifold (“l'univers”). From a later point of view $\tilde{G}$ may be understood in the Riemann geometric view as the Hodge dual of the vector-valued 1-form G .

Following Cartan we shall use the terminology *gravitational tensor* (“tenseur gravitationnel”) or *Einstein form* and the notation $\tilde{G}$ if we conceptualize it as a $(n-1)$ -form (in dimension n), while the *Einstein tensor* (notation G) will generally be understood as a the symmetric covariant tensor with coefficients G_{ij} .

To understand better what Cartan meant, we have to go into more detail. Cartan decomposed $\tilde{G}$ into its (real-valued) 3-form components Π_i , such that it may be written as

$$\tilde{G} = \sum_i e_i \Pi^i \quad (14)$$

(e_i the basis vectors of the Cartan orthonormal frame).⁴⁴

Close to the end of analyzing the general form of $\tilde{G}$ he defolded an intriguing argument involving his representation of the Riemannian curvature in terms of rotation coefficients A^i_{jkl} (Eq. (5)) and found that the components Π^i can be written

⁴⁰ “Étant donné la difficulté qu'on rencontre à avoir connaissance des Mémoires parus à l'étranger pendant la guerre et depuis la guerre, je ne suis pas absolument sûr qu'aucune démonstration de ce théorème n'ait été donnée” ([Cartan, 1922d](#), p. 142).

Taking into account the difficulty for gaining knowledge of foreign publications during or after the war, I am not absolutely sure that no demonstration of this theorem has perhaps already been given.

⁴¹ In the French translation ([Weyl, 1922](#), p. 279f.).

⁴² Condition (ii) is here a result of the contracted Bianchi identities.

⁴³ We consider its components as the coefficients appearing in the expression of the projection along a fixed direction, applied to tension exercised on an element of three dimension in the four-dimensional universe.

A similarly intuitive description is given in [Cartan \(1922f\)](#).

⁴⁴ ([Cartan, 1922d](#), p. 203), ([Cartan, 1923/1924b](#), (vol 41) p. 13)

as⁴⁵:

$$\Pi^i = \sum \epsilon_k \epsilon_l \operatorname{sgn}(i, j, k, l) [\omega_j \Omega_{kl}] \quad (15)$$

where the system of indices (i, j, k, l) is any cyclic permutation of $(0, 1, 2, 3)$, $\operatorname{sgn}(i, j, k, l)$ its sign, and the summation runs over all such cyclic permutations (Cartan, 1922a, p. 203). This form of the gravitational tensor reappears in (Cartan, 1923/1924b, (vol. 41) p.13f.), where he called it the *kinetic quantity of the mass* “quantité de mouvement masse”.⁴⁶ Adopting the signature convention $\operatorname{sig} g = (+ - - -)$ for the metric, Cartan gave the “kinetic quantity of mass” (the gravitational tensor) in a form which displays its character as a vector valued 3-form openly (ibid. Eq. (7')):

$$\tilde{G} = \sum_{(ijkl)} \operatorname{sgn}(ijkl) e_i [\omega_j \Omega_{kl} + \omega_k \Omega_{lj} + \omega_l \Omega_{jk}] \quad (16)$$

(with summation over all cyclic permutations $(ijkl)$ of $(0 \dots 3)$).⁴⁷

During the course of his study of the gravitational tensor Cartan started to think geometrically about it and, as a consequence of the Einstein equation, also about the matter tensor. In the note of February 13, 1922, he stated:

On sait que, dans la théorie de la relativité généralisée d'Einstein, le tenseur qui caractérise complètement l'état de la matière au voisinage d'un point d'Univers est *identifié* à un tenseur faisant intervenir uniquement les propriétés *géométriques* de l'Univers au voisinage de ce point (Cartan, 1922f, p. 437, 1-st emphasis ES, 2-nd emphasis in the original).⁴⁸

This differed from how physicists usually understand the Einstein equation, although Eddington thought similarly. For them equation (10) expresses a kind of communication between two aspects of reality, spacetime and matter, not a reduction of one to the other. Einstein fought strongly against the claim that his theory of general relativity had geometrized gravity.⁴⁹ But for Cartan the idea that the Einstein equation justifies an identification of its left hand (geometrical) side and the right hand (matter) side became a guiding motif for his further investigations. In his notes of 1922 he used the notions “tenseur de matière” and “tenseur d'énergie d'Einstein” etc. synonymously and understood them to be *defined* by geometrical curvature properties.⁵⁰ This was a clue for his way of generalizing Einstein gravity in Cartan (1923/1924b), Cartan (1925) and, to my knowledge, remained so in the years to come.

In order to explain what he meant Cartan used a 3-dimensional analogue of the Einstein equation. Then the right hand side reduces to the classical (symmetric) stress tensor of matter, and the left hand side analogue may be described by curvature properties of a space with the correct properties of infinitesimal trièdres (3-frames) which, if one wants so, define their own metric different from the classical Euclidean metric of the ordinary embedding space. In Cartan's conceptualization of curvature

⁴⁵Up to the factor α in equation (13) and an equivalent to the “cosmological” term βg which we suppress here.

⁴⁶Cf. A. Trautman's commentary in (Cartan, 1986, p. 17).

⁴⁷In this formula Cartan wrote $[me_i]$ in the place of e_i , apparently to make the point dependence of the basis vectors e_i immediately visible in his notation.

⁴⁸One knows that in Einstein's generalized theory of relativity the tensor which completely characterizes the state of matter in a neighbourhood of the Universe is identified with a tensor which is exclusively made up by the *geometric properties of the Universe at this point*. (1-st emphasis ES, 2-nd emphasis in the original)

⁴⁹(Lehmkuhl, 2014)

⁵⁰This identification is announced already in the title of (Cartan, 1922f) and referred to in the next notes, e.g. (Cartan, 1922g, 593).

the latter expresses itself in a “rotation complémentaire” (complementary rotation) which is to be applied after parallel transporting a trièdre around an infinitesimal loop. Presupposing the quid pro quo mentioned in our introduction he continued:

Cette rotation peut se représenter par un vecteur. L'état de divergence entre l'espace donné et l'espace euclidien peut donc être traduit par un vecteur attaché à chaque élément de surface orienté de l'espace. (Cartan, 1922*f*, p. 438)⁵¹

In the same note he declared that the assignment of vectors to surface elements results in a tensor from which one can show symmetry and “conservation law” just like for the original Einstein equation. In 3-dimensional (Euclidean embedding) space the expression of an infinitesimal rotation by a vector was a standard procedure. Cartan concluded:

Il résulte de ce qui précède qu'on peut expliquer l'état d'un milieu élastique en équilibre en admettant que l'espace qui le contient est déformé et que l'état de tension du milieu traduit physiquement cette déformation géométrique. (ibid.)⁵²

This is an interesting phrase. For the moment we leave it open whether we ought to understand “expliquer” in the sense of making something explicit in a mathematical sense, or even stronger as an explanation in the physical sense.

In order to understand what the mathematics behind this sentence is, one has to see the context. Immediately after this discussion of 3-dimensional classical elasticity, Cartan explained a geometrical interpretation of the Einstein equation (10) in the light of (15) derived in Cartan (1922*d*).⁵³ Although he did not discuss elasticity in the latter, his notes show that in early 1922 he thought about *classical elasticity as a 3-dimensional analogue of the Einstein equation* and vice versa. In this case the right hand side reduces to the tension tensor which Cartan would understand as a vector valued 2-form with real valued 2-forms $\tilde{T}^i$ as components ($i = 1, 2, 3$). It expresses the stress force $\tilde{T}^i(\sigma)$ exercised on any infinitesimal surface element σ . In the following we use the simplified notation T^i etc. also for the 2-form.

In a 3-dimensional version of (14), (15) the gravitational 3-form reduces to a 2-form and the signature coefficients are all $\epsilon_j = 1$.⁵⁴ With Cartan's choice $\kappa = 1$ a 3-dimensional analogue of the Einstein equation would be

$$\Pi^i = \sum_{k,l} \text{sgn}(i, k, l) \Omega_{kl} = T^i \quad (17)$$

with Ω_{kl} the components of the curvature 2-form of a 3-dimensional Cartanized Riemannian geometry. The alternating signs in the summation of (17) associate a vector to the rotational coefficients just like in the vector product representation of infinitesimal rotations.⁵⁵ This would underpin what Cartan intuitively circumscribed

⁵¹ This rotation may be represented by a vector. The state of divergence between the given space and Euclidean space can thus be expressed by a vector attached to any oriented surface element of the space.

⁵² From the preceding it follows that one can explain/express the state of an elastic medium in equilibrium by assuming that the space in which it is contained is deformed and that the state of tension of the medium reflects this deformation physically.

⁵³ Remember that the Cartan (1922*a*) was already written at the time of submission of the *Comptes Rendus* notes.

⁵⁴ If we consider the Einstein tensor in dimension $n = 3$ and allow us (anachronistically) to apply Hodge duality, we arrive at $\tilde{G}$ as a vector valued 2-form.

⁵⁵ For $\Omega = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$ equation (17) gives $T = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

in his note (Cartan, 1922f) quoted above and explain half of the *quid pro quo* cited in the Section 1, the expression of a rotation (curvature) by a vector.

Two years later, in the second lot of Cartan (1923/1924b), Cartan gave a more technical explanation in terms of the 4-dimensional Einstein equation. Here he concentrated on a 3-dimensional spacelike hypersurface S (in the infinitesimal represented by a hyperplane) corresponding to $x_o = 0$ and “projected” the 4-dimensional rotations onto S . If a vector $\xi = (\xi^i)$ is rotated by Ω_j^i , the hyperplane projection of the change is $\Delta\xi^i = \Omega_j^i \xi^j$. Cartan expressed this rotation in two ways:

Elle peut, dans cet hyperplan, être représentée par le bivecteur

$$(*) \quad [e_2 e_3] \Omega^{23} + [e_3 e_1] \Omega^{31} + [e_1 e_2] \Omega^{12}$$

ou encore par le vecteur polaire de même mesure

$$(**) \quad \frac{1}{\sqrt{g_{11}g_{22}g_{33}}} (e_1 \Omega_{23} + e_2 \Omega_{31} + e_3 \Omega_{12})$$

(Cartan, 1923/1924b, (vol. 41) p. 16, (marks (*), (**) added, ES)).⁵⁶

(*) was a Grassmann type characterization of (infinitesimal) rotations and (**) its equivalent in terms of a vector product representation. In this way the rotation of the hyperplane may be given given by a vector with components $(\Omega_{23}, \Omega_{31}, \Omega_{12})$.

This transition from “bivectors” to “polar vectors” was a special case of what Grassmann had introduced as a more general duality (called *Ergänzung* by him).⁵⁷ In dimension 3 this type of dualization was particularly well known, even without any reference to Grassmann. But Cartan was well aware of the general nature of Grassmann dualization and used it in his discussion of the invariants of his geometry (Cartan, 1923/1924b, vol. (40) p. 400ff.).

From this point of view the first half of Cartan’s *quid pro quo* resulted from a *Grassmann type duality transformation* (of infinitesimal rotations in dimension $n = 3$ to polar vectors), which Cartan introduced for gaining a *geometrical understanding* of the spacelike part of the *Einstein equation*. The result of the transformation led to a new geometrical picture of *classical elasticity*: stress could be expressed in terms of the curvature of a space with a connection and metric adapted to the mechanical properties of the material medium under investigation.

⁵⁶ In this hyperplane it (the rotation, ES) can be represented by the bivector

$$(*) \quad [e_2 e_3] \Omega^{23} + [e_3 e_1] \Omega^{31} + [e_1 e_2] \Omega^{12}$$

or also by the polar vector of the same measure (norm)

$$(**) \quad \frac{1}{\sqrt{g_{11}g_{22}g_{33}}} (e_1 \Omega_{23} + e_2 \Omega_{31} + e_3 \Omega_{12})$$

(Cartan, 1923/1924b, (vol. 41) p. 16, (marks (*), (**) added, ES)).

⁵⁷ In modernized notation, Grassmann established an equivalence (isomorphism) between $\Lambda^k V$ and $\Lambda^{(n-k)} V$ for any n dimensional *Ausdehnungsgebiet* ($0 \leq k \leq n$) with a volume form, respectively a basis $e_1, \dots, e_n$ with the property $e_1 \wedge \dots \wedge e_n = 1$ (using modern notation for alternating products). He assigned to basis elements $e_{i_1} \wedge \dots \wedge e_{i_k}$ ($i_1 < \dots < i_k$) in $\Lambda^k V$ the basis elements $e_{j_1} \wedge \dots \wedge e_{j_{n-k}}$ ($j_1 < \dots < j_{n-k}$) of $\Lambda^{(n-k)} V$ for which $e_{i_1} \wedge \dots \wedge e_{i_k} \wedge e_{j_1} \wedge \dots \wedge e_{j_{n-k}} = e_1 \wedge \dots \wedge e_n = 1$ and used linear continuation (Graßmann, 1862, §89, p. 57f.), cf. (Scholz, 1984). As Grassman introduced an inner product in V , which made $e_1, \dots, e_n$ an orthonormal basis, this can be considered as a linear algebraic isomorphism serving as the basis for the later *Hodge duality*.

5 Einstein gravity in analogy to geometrized Cosserat elasticity

The second part of the *quid pro quo* resulted from Cartan's generalization of Einstein's theory of gravity and involved an adaptation of Cosserat elasticity to his research program of 1921/22. At the time of submitting his *Comptes Rendus* notes, in February and March 1922, Cartan had all this in mind, but it took some time to work out the mathematical details. They are contained in the two-part paper (Cartan, 1923/1924b, 1925) the first part of which came in two lots (vol. 40, 41 of the *Annales ENS*).⁵⁸

In this paper, he showed that the vacuum Maxwell equations are compatible with any (Cartan-) connection of the Poincaré group, also those including torsion; but taking Lorentz forces into account may run into difficulties. For a kinetic quantity of energy like in (16) the Lorentz force exercised on an electric current density came out correctly, i.e., in agreement with special relativity, only if the “universe” has vanishing torsion (Cartan, 1923/1924b, p.20f.). That was disappointing; but Cartan indicated a way out:

La conclusion précédente (vanishing torsion, ES) ne serait pas logiquement nécessaire si l'on admettait une conception de la Mécanique des milieux continus plus large que la conception habituelle, la “quantité de mouvement-masse” élémentaire étant représentée par une système de vecteurs et de bivecteurs

$$G = [me_i] \Pi^i + [e_i e_j] \Pi^{ij}$$

(Cartan, 1923/1924b, p. 21).⁵⁹

If such a modification of the Einstein form is accepted, the laws of electromagnetism, including the Lorentz forces, were compatible with a non-vanishing torsion. In this context, it was natural to assume that the “quantité de mouvement-masse élémentaire”, i.e. $\tilde{G}$, should remain a geometric integral invariant, like in Einstein's theory. This, so Cartan declared, was easy to achieve. One had only to replace the rotation associated to any surface element by the total displacement of the full Cartan curvature (“déplacement total (*rotation et translation*)”) assigned to the surface element.

The rotations had been transmuted into vectors by Grassmann duality in 3-dimensional spacelike hyperplanes and this transmutation was taken over to $n = 4$ (see above). In an analogous manner Cartan transmuted translations into bivectors (Grassmann duality in the 3-dimensional spacelike projection, but here transferred to $n = 4$). In this way $\tilde{G}$ became a fully Cartanized variant of the Einstein form (Cartan, 1923/1924b, p. 22, Eq. (11)):

$$\tilde{G} = \sum_{(ijkl)} \text{sgn}(ijkl) (e_i [\omega_j \Omega_{kl} + \omega_k \Omega_{lj} + \omega_l \Omega_{jk}] - [e_i e_j] [\omega_k \Omega_l - \omega_l \Omega_k]) \quad (18)$$

Expressed in more recent terminology Cartan proposed a 3-form with values in the Grassmann algebra of the tangent bundle as a generalization of the gravitational

⁵⁸Reprint in (Cartan, 1955), English translation with a commentary (foreword) by A. Trautman in (Cartan, 1986).

⁵⁹“The above conclusion is not logically forced upon us if we accept a broader framework for mechanics of continuous media and represent the energy-momentum density by a system of vectors and bivectors: $G = [me_i] \Pi^i + [e_i e_j] \Pi^{ij}$ ” (Cartan, 1986, p. 123).

Compare fn. 47.

tensor. It consists of two terms, the first one with values in TM ⁶⁰ contracts rotational curvature and transmutes it into a vector. The second one with values in the bivector bundle $\Lambda^2(TM)$ *transmutes translational curvature into a bivector*. But what would be the right hand side of a correspondingly generalized Einstein equation? Because of his geometrical reduction of the “quantité de mouvement masse” this seemed no particularly burning problem for him.⁶¹

When he wrote his *Comptes rendus* notes of 1922 he envisaged a geometrical expression for force and torque stresses of Cosserat elasticity in dimension $n = 3$. At that time he apparently had in mind a 3-dimensional analogue of (18) as a generalization of his geometrization of the energy-momentum tensor:

$$\tilde{G} = e_i \Pi^i + [e_i e_j] \Pi^{ij} \quad \text{with} \quad \Pi^i = \epsilon^{ijl} \Omega_{jl} \quad \text{and} \quad \Pi^{ij} = \epsilon^{ijl} \Omega_k \quad (21)$$

(using the Levi-Civita symbol ϵ^{ikl} in place of Cartan’s own notation for the sign of permutations).

Cartan’s understanding of the Einstein equation suggested reading the Π^i as the components of the stress force $f = \Pi^i(\sigma) e_i$ exercised on a surface element σ . They have been gained via Grassmann dualization, from the rotational component of Cartan curvature.

In the same vein Cartan interpreted the Π^{ij} as components of the torque stress τ expressed as a bivector, $\tau(\sigma) = \Pi^{ij}(\sigma) e_i \wedge e_j$. They are derived from the translational curvature Ω_k by Grassmann duality. He even found that, due to the specific form of the torsion in his case, the translation associated to any surface element is normal to the latter (Cartan, 1923/1924b, p. 21, fn).⁶² In this specific constellation the torque stress exercised on a unit surface element with unit normal n can be written, in terms of the vector calculus in $\mathbb{R}^3$ (which Cartan did not use), by taking the vector product with $u := \Omega^i e_i$, $\tau(\sigma) = u \times n$. In this respect Cartan’s terminology choice *torsion* for the translational curvature was in good agreement with the Cosserats. The latter had introduced the term “moments de torsion” during their discussion of a 2-dimensional medium for the components of $\tau(\sigma)$ in the tangential direction of σ , while the orthogonal component was called “moment de flexion” (Cosserat, 1909c, §35), (Cosserat, 1909a, p. 589).⁶³

So far the mathematics behind Cartan’s verbal description of Cosserat elasticity in the *Comptes rendus* note (Cartan, 1922g) can be reconstructed from the 1923/24 paper (compare the Cartan quotes in Sect. 6). It solves the riddle of the *quid pro quo* mentioned in our introduction and sheds light on the choice of terminology for the

⁶⁰In fact, Cartan put square brackets about his symbols of the vector basis $[m e_i]$, apparently in order to emphasize the Grassmann character of the term.

⁶¹In the later *Einstein-Cartan theory* (see Sect. 7) the two terms are separated and, with an appropriate matter Lagrangian term added, give rise to two equations. In his commentary to the English translation of Cartan’s 1923-25 paper Trautman writes them in the form (Cartan, 1986, p. 17)

$$\frac{1}{2} \epsilon_{ijkl} \omega^j \wedge \Omega^{kl} = -8\Pi t_i \quad (19)$$

$$\epsilon_{ijkl} \omega^k \wedge \Omega^l = 8\Pi s_{ij} \quad (20)$$

with t_i and s_{ij} (vector-respectively bivector-valued) 3-forms of energy-momentum and spin density (ϵ_{ijkl} the anti-symmetric Levi-Civita symbol).

⁶²In a different context he showed that this property implies the identity of autoparallels of the Cartan connection with the geodesics of the related Riemannian structure (Cartan, 1923/1924b, §66, p. 407).

⁶³For 1-dimensional media, i.e. elastic curves, they also used the term *torsion*, but here in the sense of curve theory (second curvature). I have not found any usage of the term in their discussion of 3-dimensional media.

translational curvature. But one point is still missing: the equilibrium condition of forces and torques announced by Cartan in 1922.

In 1922 he apparently expected a vanishing covariant differential of the respective expressions not only for the Einstein form (16) and (17) but also for his generalization. By 1923/24 he knew that this is not the case. In his understanding that violated the “conservation law”. He therefore proposed to impose an appropriate constraint for the Einstein equation in dimension $n = 4$. As the easiest he proposed (ibid., p. 22)

$$\sum \text{sgn}(ijkl) (\Omega_j \Omega_{kl} + \Omega_k \Omega_{lj} + \Omega_l \Omega_{jk}) = 0 \quad \text{for all } i = 0, \dots, 3. \quad (22)$$

Later authors replaced the condition of vanishing covariant divergence, by the more general one of a contracted Bianchi identity for Riemann-Cartan geometry. Only for vanishing torsion or Cartan’s algebraic constraint (22) it boils down to the “conservation law” (Trautman, 1973, p. 152f.). Unhappily the relation (22) gives an *algebraic* constraint for torsion which thus cannot play the role of a dynamical field.

All this may have been a reason for Cartan to hesitate claiming immediate physical relevance for his theory.⁶⁴ The only point he emphasized was:

On a ainsi une généralisation, au moins mathématique, de la théorie d’Einstein, généralisation compatible avec toutes les lois de l’Électromagnétisme (Cartan, 1923/1924b, p. 22).⁶⁵

He also indicated that this generalization of Einstein gravity can be formulated in a Lagrangian field approach. He proposed a generalization of the Hilbert action expressed in terms of his geometrical quantities (Cartan, 1923/1924b, p. 23),

$$\mathfrak{L}_{grav} = \sum_{(ijkl)} [\omega_i \omega_j \Omega_{kl}], \quad (23)$$

but did not consider a separate matter Lagrangian (in agreement with his “unphysical” identification of the left hand and the right hand side of the Einstein equation).

At this point, our author stopped his physics related studies without even shedding a first glance at the dynamical effects of his theory:

... je me contente de cette indication, sans entrer dans plus de détails à ce sujet (Cartan, 1923/1924b, p. 23).⁶⁶

He rather continued with studies of Weyl’s dilational gauge metric and in part II, according to the title of the paper, with studies of Cartan spaces of the affine group or subgroups of it (Cartan, 1925).

In other papers, e.g. (Cartan, 1924c), he turned towards generalizations of the group. Where he stuck to the Euclidean group he tended to investigate more classical questions. In particular he showed that from his point of view Clifford parallelism in elliptic space can be understood as due to a connection with torsion but vanishing rotational curvature, while the autoparallels coincide with elliptic straight lines (Cartan, 1924a).⁶⁷ All these topics turned his and his readers’ attention away from the Cosserat inspired view of torsion. It rather hinted in the direction of what a little

⁶⁴Cf. the remark by A. Trautman in (Cartan, 1986, p. 17).

⁶⁵“This yields – at least on a mathematical level – a generalization of Einstein’s theory which is compatible with all the laws of electromagnetism” (Cartan, 1986, p. 124).

⁶⁶... I make to do with this indication without going into more details of this subject.

Translation in (Cartan, 1986, p. 124): “However, we shall not discuss this issue in more detail.”

⁶⁷Cf. (Cogliati, 2018, p. 46f.).

later became the study of distant parallelism or *absolute parallelism*, as Cartan would call it.

6 Cartan's practice of mathematical analogies

It is not completely clear, why Cartan stopped short of pursuing the physical considerations further. One obvious reason might have been that he was more interested in studying the mathematical side of his approach than going deeper into the physics. But he seems also to have developed doubts with regard to the interpretation and the physical relevance of his findings. The algebraic constraint for torsion which he considered more or less necessary for the “conservation law” as he understood it must have been a stumbling block for him.⁶⁸

We also find a clear hint in [Cartan \(1923/1924b\)](#) that during the continuation of his work he developed doubts regarding the feasibility of his geometric interpretation of Cosserat elasticity in the *Compte Rendus* notes. In §60 of his paper he discussed the differential forms of a (Cartan) space with the Euclidean group, invariant under changes of the Cartan reference system (change of Cartan gauge in modern terminology). One of the invariants combines a “système de vecteur et de couples” of a form similar to (18):

$$[me_1]\Omega_{23} + [me_2]\Omega_{31} + [me_3]\Omega_{12} + [e_1e_3]\Omega_1 + [e_3e_1]\Omega_2 + [e_1e_2]\Omega_3 \quad (24)$$

This was Cartan's mode of writing our (21), the 3-dimensional analogue of the generalized Einstein form.

Here the relation between translations and bivectors is simpler and more direct than in the 4-dimensional case. As already pointed out, in this dimension it is nothing but a Grassmann duality between vectors and bivectors. Cartan commented:

Le système de vecteurs et de couples

$$[me_1]\Omega_{23} + [me_2]\Omega_{31} + [me_3]\Omega_{12} + [e_1e_3]\Omega_1 + [e_3e_1]\Omega_2 + [e_1e_2]\Omega_3$$

représente de même le déplacement associé à un élément de surface. Sa dérivée extérieure est nulle. Si l'on regardait ce système comme représentant les tensions qui s'exercent sur un milieu matériel (tensions comportant des couples), ce milieu serait en équilibre ([Cartan, 1923/1924b](#), vol. 40, p. 401, fn).⁶⁹

In other words, an invariant of this type looked as if it could be interpreted as a geometrical expression for the system of tensions/momenta inside a Cosserat medium with vanishing exterior forces and moments. But, different from his announcement in [Cartan \(1922g\)](#) of February 1922, in which he had referred so positively to the “beaux travaux de MM. E. et F. Cosserat”, Cartan now continued with a methodological reflection which stepped back from an interpretation in this sense and came even close to a disassociation:

⁶⁸Cf. fn 64.

⁶⁹“The system $[me_1]\Omega_{23} + [me_2]\Omega_{31} + [me_3]\Omega_{12} + [e_1e_3]\Omega_1 + [e_3e_1]\Omega_2 + [e_1e_2]\Omega_3$ of vectors and torques represents the same displacement associated with a surface element. Its exterior derivative is zero. If this system is viewed as describing tensions (with torque) in a material medium, the medium would be in equilibrium.” ([Cartan, 1986](#), p. 96, fn. (9))

Il y a là une des nombreuse analogies, plus ou moins trompeuses, qui existent entre la Géométrie et la Mécanique. En fait, ce n'est qu'une analogie. (ibid.)⁷⁰

He neither stated that the analogy *was* “trompeuse” (deceptive), nor did he repeat a positive claim of relevance for it. But it seems that he was having second thoughts about this point between February 1922 and the final preparation of the paper (Cartan, 1923/1924b). In the meantime it had become clear to him that the analogy between Cosserat elasticity and the generalized Einstein equation was less perfect than he had initially hoped, probably because of the non-vanishing covariant exterior derivative in dimension $n = 4$. Perhaps this disillusionment was also the reason why he avoided any explicit reference to the Cosserats' work. He even did not mention their name any longer in the new paper.⁷¹

That could not change the crucial heuristic role which, according to Cartan's own testimony in the notes of early 1922, Cosserat elasticity played during his early work on the Einstein equation and its generalization in the light of his geometrical ideas. The interpretation of Cosserat torque stresses for the transmutation from translations to bivectors (rotational momenta) by a Grassmann type duality in 3-dimensions (24) would strongly underpin his approach, *if it could be considered as physical*. In 1921/22 Cartan was apparently strongly impressed by the analogy between geometry and mechanics. The parallel helped Cartan to structure his argumentation in which he tried to build a bridge between Einstein gravity and geometry. Once that bridge was built in the form of our equation (18), the reference to Cosserat media could be downgraded to the status of a mere analogy without a further epistemic claim.

In spite of this the *terminological residuum* of Cartan's early heuristics remained unaffected: the translational curvature baptized under the impression of the 3-dimensional analogy with Cosserat media when it was still fresh and strong, continued to be called *torsion* and remains so until today.

7 The aftermath

This is not the end of the story. We should not finish our's without having a short glance at the reception and further developments connected to Cartan's early papers on Einstein gravity and Cosserat theory. The early idea of a potentially intimate connection between Cosserat elasticity and Einstein gravity did not play a role in the reception for many decades to come, while Cartan geometry attracted the attention of mathematicians by other reasons. Jan Arnoldus Schouten got interested in Cartan's proposals of torsion in his general studies of connections. He contacted Cartan in 1924. The two mathematicians communicated on linear connections in Lie groups and found that left and right translations in any Lie group lead to distant parallelism structures, i.e. connections with torsion but vanishing curvature, with autoparallels which agree with geodesics of the Riemannian metric on the group manifold induced by the Cartan-Killing form. They even were able to show that, with the exception of the 7-sphere, the Lie groups are the only Riemannian manifolds with this property (Cogliati, 2018).

⁷⁰ We have here one of the numerous, more or less misleading, analogies which exist between geometry and mechanics. In fact this is not more than an analogy.

The translation in (Cartan, 1986, p. 96, fn. (9)) (“We have here one of the numerous somewhat misleading analogies between geometry and mechanics.”) omits the last phrase of Cartan's remark.

⁷¹ Later research on this question, starting in the 1960s, showed that a viable usage of Cartan geometry in Cosserat type theory of elastic media needed, in fact, a more sophisticated approach than was available in 1922 (see below).

Distant parallelism became a “hot topic” at the end of the the 1920s, when Albert Einstein started to study it as the framework for one of his attempts to unify gravity with electromagnetism. After Cartan reminded Einstein that this approach could be well framed in his geometrical method and had been mentioned by him during their conversations in 1922 (Cartan, 1979, p. 4), Einstein hurried to give credit to Cartan and accepted that his study of gravity in terms of *distant parallelism* (also called *teleparallel* gravity) used a specific type of Cartan geometry. In this setting the deviation of flat space was encoded in the torsion part of curvature only, while the rotational curvature was set to zero.⁷² Also Roland Weitzenböck had studied flat linear connections (vanishing Riemannian curvature) with torsion in the course of his study of differential invariants, i.e. in a pure mathematics context (Weitzenböck, 1923, pp. 317ff.). He did not relate this to Einstein gravity at the time but was keen to get acknowledgement from Einstein and published a note on the topic in 1929 (Sauer, 2006). Neither Cartan nor Cartan geometry was ever mentioned by him.

Although Cartan himself had given an example of a teleparallel Cartan space in his note (Cartan, 1922g), the general outlook of this example was a far cry from his early idea of interpreting torsion by rotational momenta as an additional feature of the gravitational field. In a way it even was opposite to his proposal for the physical interpretation of translational curvature. But even so, the studies of distant parallelism in the gravity context demonstrated the openness of general mathematical structures for different physical interpretations. Even those which were designed with definite physical interpretations in mind, like Cartan geometry of the Euclidean and Poincaré group, did not carry the mark of their original interpretation with them as some sort of inbuilt, although perhaps hidden finality.

We also have to be aware that, by a constellation of historical contingency, the late 1920s was also the time in which quantum physicists started to realize that the new complex (wave) fields could carry an internal rotational momentum, called *spin* (Dirac, 1928; Pauli, 1927). But at this time no author had the idea that this new internal torque-like momentum might give new support to Cartan’s idea of torsion.

This changed only much later, in the 1960s. An important contribution for renewing the interest in Cartan torsion among physicists arose in the wake of the work of Dennis Sciama and Thomas Kibble in gravity theory. Without knowing it, the two authors independently reinvented much of the Cartan geometric field structures by considering what physicists call the “localization” of the Poincaré group (Kibble, 1961; Sciama, 1962). They found that the spin of elementary particle fields might play a role for a generalized theory of Einstein gravity which was close to what Cartan had anticipated in his early papers. The close relationship of their theory to Cartan geometry was not clear to Sciama and Kibble; but it was soon made explicit by other authors, at first by Friedrich Hehl in his PhD dissertation (Hehl, 1966) and independently by Andrzej Trautman (Trautman, 1973).

A group of authors joined and extended this research program.⁷³ They realized that Cartan geometry offered a tailor-made geometric framework for infinitesimalizing (“localizing” in the language of physicists) energy-momentum and spin currents known from Minkowski space and special relativity. The Cartan geometry of this approach was modeled on the Poincaré group and has both, rotational curvature and torsion, like in Cartan’s work of the early 1920s. If the gravitational Lagrangian was chosen as closely as possible to the Hilbert action of Einstein gravity, it turned out to be the one Cartan mentioned in his side remark quoted above. Cartan’s constraint (22) had to be relaxed and the spin coupling to torsion had to be incorporated in the matter Lagrangian. Trautman called the resulting theory with dynamical

⁷²(Goenner, 2004), cf. (Cartan, 1929).

⁷³Much information on this development is collected in the reader Blagojević (2013) which contains very helpful commentaries. For systematic surveys see Hehl (2017); Trautman (2006).

equations (19), (20) *Einstein-Cartan gravity*. It is considered as a “viable” alternative to Einstein gravity – although one which can be distinguished from the latter only under the conditions of extremely high energy densities.⁷⁴

Motivated by the successes of non-abelian gauge field theory in the rise of the *standard model* of elementary particle physics, gauge field dynamics related to the Poincaré group was studied in the 1970/80s. It turned out that Einstein-Cartan gravity can be reconstructed in a Cartan space modeled on the Poincaré group, where the dynamical equation for translational curvature – paradoxically still called “torsion” – couples to the energy-momentum current (of matter and the gravitational field), and the rotational curvature is coupled to the spin-current (Hehl, 1980, p. 337ff.). This was an intriguing result. Conceptually it set the couplings right while avoiding the surprising crossover of translational and rotational aspects for geometry and dynamics of Einstein gravity as seen by Cartan in analogy to Cosserat elasticity. In this way it generalized the “teleparallel” representation of Einstein gravity by adding spin. But the authors presented the respective Lagrangian as a special case, even a degenerate one, and directed their attention towards more general quadratic Lagrangians in Poincaré gauge field theory.⁷⁵

This was not the only path which led back to Cartan’s ideas of the early 1920s. Also Cartan’s sparse remarks on Cosserat elasticity found successors a generation later, although only after a specific turn taken by authors interested in the mathematical study of the yielding of plastic materials. Kazuo Kondo in Japan proposed to model dislocation in crystal matter by the torsion of a linear connection (Kondo, 1952). Ekkehart Kröner, an expert in classical solid state physics, and Hehl’s PhD advisor at the *Bergakademie/TH Clausthal-Zellerfeld*, was attracted by the ideas of the Cosserats on generalized elasticity and its link to Cartan’s generalized geometry. He was one of those authors who brought in the idea that such a material structure could be studied in a Cartan geometry modeled on the Euclidean group, called *Riemann-Cartan space* by these authors.⁷⁶ This led to the attempt of studying dislocations and proper tensions in metals in terms of Cartan-type geometrical methods and was also the background from which Hehl entered gravity.

A complicated story started; we have to cut it short and hope for more detailed historical investigations to come. The different components of the Cartan connection had to be linked to physical quantities expressing the deformation of the material, different types of dislocation inside the material, and the hypothetical force and torque stresses.⁷⁷ A new generation of scientists studied non-symmetric “micro-deformations” accompanied by “micro-rotations” conjointly related to proper stress and torque inside continuous media, now often called *Cosserat media*.⁷⁸ For the protagonists of a geometrization perspective a Cartan geometry with translational curvature but no rotational curvature, thus with “distant parallelism” in the language of gravity theorists, became an option. The intuitive idea behind this was the choice of a Cartan reference system adapted to the geometry of the lattice structure of the material. The rotational curvature was thus set to zero, while the torsion could be expressed as a *closing defect* arising from parallel transporting a vector u along an infinitesimal shift vector $\delta x = v$ and v along $\delta' x = u$.⁷⁹ Such closing defects can

⁷⁴(Hehl, 2017; Trautman, 2006).

⁷⁵See also the surveys in (Hehl, 2017, p. 164f.) and (Blagojević, 2013).

⁷⁶(Kröner, 1963a,b)

⁷⁷A recent survey of the resulting theory can be found in Hehl (2007), a short historical glance in (Mauguin, 2014, Chap. 8).

⁷⁸See Mauguin (2010); for a short historical outlook see the end of chapter 8 in Mauguin (2014). A contemporaneous survey talk cited there is (Schaefer, 1967).

⁷⁹The “would-be” infinitesimal parallelogram arising from this procedure does not close (thus it is *no* parallelogram). Mathematically, the defect is expressed by the asymmetry of the corresponding linear connection Γ_{jk}^i in the lower indexes, $\Gamma_{jk}^i \neq \Gamma_{kj}^i$.

be used to model dislocations in the material, once they turn up sufficiently often (densely). This results in a mathematical description of materials with densely distributed dislocations by a “teleparallel” Cartan structure. Its *torsion*, here better to be interpreted in its *translational connotation*, is related to the dislocation field.⁸⁰

Finally the tables have been turned another time. At the origin of Cartan’s theory the context of continuum mechanics was Cartan’s motivation for introducing his slightly paradoxical terminology of “torsion”. Now even in part of the field of continuum mechanics we find an epistemic constellation for which the geometrical naming of *translational curvature* would be closer to the matter than the terminology chosen by Cartan. But in the meantime the latter has been widely established.

In the end, Cartan’s papers of the early 1920s have found new readers also among present day theorists of continuum mechanics. In this recent development it became clear that a reliable connection between the physics of matter and geometry needs much more sophistication than could be imagined by Cartan (or even the Cosserats). It turned out that, in the long run, Cartan’s analogy between generalized elasticity and gravity, which became apparent in the framework of his geometry, was not as misleading (“trompeuse”) as he may have feared in 1924. The mathematical analogy established by Cartan became a stimulating input for these studies, even though the structural analogy had to be disentangled, before it could bare fruits.

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⁸⁰Cf. [\(Hehl, 2007, p. 167ff.\)](#). In this context, not any Riemann-Cartan geometry with distant parallelism is feasible. Additional constraints have to be observed, in order to lead to acceptable deformation quantities related to the Cartan structure [\(Hehl, 2007, p. 163\)](#).

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