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Subtractions and the effective Salpeter term for the Lamb shift in muonic atoms with the nuclear spin $I \neq 1/2$

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Abstract. While taking into account the nuclear-structure contributions to the Lamb shift, one has to make various subtractions for the two-photon exchange contributions. Such subtractions should be consistent with the structureless part of theory. We study here the subtractions for a two-body atomic systems which consist of a pointlike lepton (an electron or a muon) and a nucleus with spin 0, 1/2, and 1, and find the recoil contribution in order $(Z\alpha)^5$ due to the subtractions for $I = 0, 1$. (The related contribution to the energy levels for $I = 1/2$ of order $(Z\alpha)^5 m$ is called the Salpeter term.)

1 Introduction

Quantum electrodynamics (QED) for free particles and bound-state QED deal with Feynman diagrams, but in a different way. Free-particle QED addresses the diagrams as they are. In the meantime, a consideration within bound-state QED suggests that at a certain stage the bound-state problem is solved. The latter means a summation of a certain group of diagrams and a rearrangement of the rest. The rearrangement assumes a subtraction of the contributions which are related to lower-order diagrams (say, a subtraction of some specific one-photon contributions from the two-photon diagrams). Such a subtraction appears in any approach, be it the effective Dirac equation [1–4] or the non-relativistic QED [5].

The need of subtraction is also clearly seen from a pragmatic point of view. There are some irreducible two-photon diagrams. They should contribute into the final results. They are not gauge invariant. We should somehow combine them with the related reducible contributions. The latter are infra-red divergent, and, to obtain a meaningful result, those divergences should be subtracted. Indeed, it is important to use a method which would do such subtractions correctly, however, a need for subtractions is undeniable.

Technically, some subtractions have been introduced in evaluation of two-photon nuclear-structure-dependent effects for pragmatic reasons, leaving aside the discussion whether the pointlike part of the theory is treated in a way, consistent with those subtractions (see, e.g., a discussion in [6]). Here we question this consistency for the case of the Lamb shift in light muonic atoms with bosonic nuclei.

Study of the Lamb shift in light muonic atom is a powerful tool to explore the nuclear structure. The experimental results, which have already been published on muonic hydrogen and deuterium [7,8] and are expected on muonic helium-3 and helium-4 ions [9], can deliver us an accurate value of the rms nuclear charge radius only if the higher-order nuclear-structure effects are under control. The most important part of such higher-order effects is due to the two-photon exchange contribution (see Fig. 1). The related contribution to the Lamb shift is defined as an average over the hyperfine interaction.

It is also necessary indeed to develop a theory for a muonic atom with a structureless nucleus. That is a quantum-electrodynamics theory. It is important to define the structureless and structure-dependent parts of the theoretical expression for the Lamb shift in such a way that nothing is missed and nothing is counted twice, which is not quite straightforward.

The complete theoretical result includes soft and hard physics. The soft one with atomic momentum transfer is marginally sensitive to the nuclear structure. However, it would be wrong to equalize those soft near pointlike part to a certain fundamental theory for pointlike particle. A very good illustrative example is with the hyperfine-splitting theory of the hydrogen atom. The fundamental pointlike theory should deal with a pointlike Dirac's proton without the anomalous magnetic moment. However, all the contributions within the external field approximation may be found ignoring the electric and magnetic form factors (i.e., ignoring the space distribution of the electric charge and magnetic moment), but keeping

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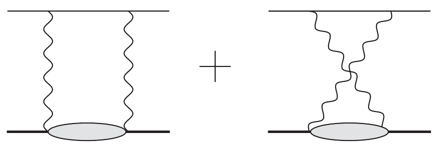

Fig. 1. The total two-photon exchange contribution includes the pointlike, elastic and inelastic terms. The figure is for the nuclear spin $I = 1/2$. For the bosonic nuclei, one has also deal with a seagull diagram.

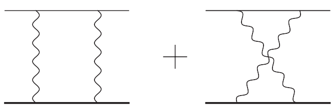

Fig. 2. The Salpeter contribution: the two-photon part ($I = 1/2$).

the actual value of the proton magnetic moment. A consideration of the recoil effects on the same ground is not possible. The two-photon recoil 'pointlike' contribution is divergent [10,11]. In an atom with an extended nucleus the nuclear size plays a role of the cut-off. The 'pointlike' part of a theory of an actual atom is always a certain convention, on which contributions are included into the pointlike part and which are a part of the nuclear-structure term.

Such an effective QED theory is still a pointlike theory in a sense that it does not involve any form factors or charge distributions etc. However, it is not defined by itself. It is defined when the separation of the complete expression into its pointlike part and its nuclear-structure part is considered. Usually, the separation is performed by the introduction of certain 'pointlike' subtractions in the nuclear-structure part. Here we consider those pointlike subtractions, introduced by other authors while calculating the nuclear-structure contribution to the Lamb shift in muonic atoms (see below), and check what effective pointlike theory is related to them.

A somewhat similar situation is with the recoil two-photon effects in the Lamb shift in muonic atoms, which are involved once one are to calculate higher-order nuclear-structure effects to the Lamb shift. The structure of the two-photon diagrams includes the pointlike part depicted in Figure 2. Such a pointlike part is nuclear-spin-dependent. The result for the nuclear spin 1/2 is well known [1,12–15] (for more details see [16,17] and references therein) as the Salpeter term. The exact results for pointlike nuclei with other spins are unknown and in some cases they are not well defined and may be ultraviolet (UV) divergent. We consider them in this paper.

A straightforward study of the two-photon exchange (with neglecting bound effects in the internal propagators in Fig. 2), which is the only correct starting point for any rigorous consideration of the hard part of the two-photon contribution, also leads to a divergence, which is an infrared (IR) one. That is because at low momentum transfer the dominant physics is with free particles, i.e., it is reduced to the Salpeter contribution, which has a soft part (see Fig. 3). The soft part involves the atomic momentum transfers and therefore the additional exchange by a Coulomb photon is not a small effect. Additionally to the two-photon exchange, one has to include the exchange by an arbitrary number of soft Coulomb photons.

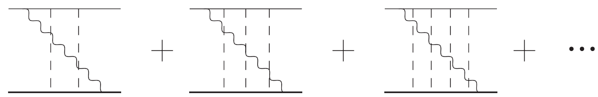

Fig. 3. The Salpeter contribution (the many-photon part). The Coulomb exchanges are presented with the dashed line.

To make the total two-photon exchange IR finite, one has to make a subtraction of the IR divergent part of the soft contribution. It is not that important whether such a subtraction is related to a standard ab initio consideration of a free pointlike particle. It is important that the contribution of the subtracted terms, whatever it is, is taken into account additionally to the nuclear-structure two-photon exchange.

The point-like two-photon contribution (with an extension to many-Coulomb effects for its soft part) was first studied for the nuclear spin 1/2 in [12–14] (see also [1]). The obtained result for that relativistic recoil correction was found then only in the leading order in m/M. The result is nuclear-spin independent. Later on the higher-order corrections in m/M were found (see, e.g., [16,17]. Those results were valid only for $I = 1/2$. we refer to the results of the point like physics as to a Salpeter-type contribution.

Let's start with the consideration of the subtraction for the standard case ($I = 1/2$), which is exactly related to the Salpeter contribution. We extend our consideration to the nuclear spins $I = 0, 1$. While the 'true' pointlike physics can be derived ab initio, the subtraction for a calculation with a structured nuclei are chosen [for some reasons]. To complete the calculation of the subtracted nuclear-structure contributions we have to add an effective pointlike term, to which we refer to as to the effective Salpeter contribution or the subtraction contribution. As we just mentioned for $I = 1/2$ those two contributions coincide. Below we consider the effective Salpeter contribution for $I = 0, 1$.

The soft physics deals with nonrelativistic nuclei and the result does not depend on the value of the nuclear spin. The soft correction is a recoil one [12–14]. The hard contribution depends on the nuclear spin and it differs for different I. For $I = 1/2$ it is a recoil contribution, but it has been not studied for other values of I. Summarizing, we repeat that the nuclear-spin-dependent terms have the same soft-physics effects, but different hard contributions. Because of that, while studying the subtraction for $I = 0$ and $I = 1$, it is advantageous to study a difference of two-photon subtraction terms for nuclei with those spins and with a nucleus with the same mass and spin 1/2. The soft contributions vanishes in the difference, while the hard ones survives and becomes IR finite. Therefore we have to evaluate only a hard contribution, i.e. the one with the momentum transfer $\sim m$ or $\gg m$, which involves only the two-photon exchange.

The two-photon structure contribution is of the order $(Z\alpha)^5 m$. To make the two-photon expression IR finite, one has to subtract the divergences related to the pointlike $(Z\alpha)^2 m$ term (the Bohr energy), pointlike $(Z\alpha)^4 m$ term (the Breit correction) and leading nuclear-finite-size term of order $(Z\alpha)^4 m^3 r_N^2$. However, different subtractions may have a different finite part and we need to work with such finite terms in a consistent way.

2 Nuclear spin 1/2: the Salpeter term and the subtraction in the two-photon nuclear-structure contribution

The standard Salpeter term ($I = 1/2$) is well known (see, e.g., [16,17])

$$\Delta E(ns) = \left\{ \frac{2}{3}\ln\frac{1}{Z\alpha} - \frac{8}{3}\ln k_0(ns) - \frac{1}{9} - 2\ln\frac{m}{m_r} + \frac{14}{3}\left(\ln\frac{2}{n} + \psi(n+1) - \psi(1) + \frac{2n-1}{2n}\right) + \frac{2m^2}{M^2-m^2}\ln\frac{M}{m}\right\}\frac{(Z\alpha)^5}{\pi n^3}\frac{m^2}{M}\left(\frac{m_r}{m}\right)^3, \quad (1)$$

and

$$\Delta E(np) = -\left\{\frac{7}{18} + \frac{8}{3}\ln k_0(np)\right\}\frac{(Z\alpha)^5}{\pi n^3}\frac{m^2}{M}\left(\frac{m_r}{m}\right)^3. \quad (2)$$

Here, ψ function stands for the logarithmic derivative of the Γ function, and $\ln k_0(nl)$ is the standard Bethe logarithm, which is tabulated in many text books.

We are interested in the two-photon contribution due to the nuclear-structure. For the nuclear spin 1/2 it is defined (for the hydrogen and muonic hydrogen) in [18,19]. It includes a subtraction of the following pointlike terms:

$$\Delta E^{1/2,1/2}(ns) = \frac{m(Z\alpha)^2}{M(M^2-m^2)}\left(\phi_{n0}(0)\right)^2\int_0^\infty\frac{dQ^2}{Q^2} \times\left\{-\left(\frac{\gamma_2(\tau_N)}{\sqrt{\tau_N}} - \frac{\gamma_2(\tau_l)}{\sqrt{\tau_l}}\right)\frac{1}{\tau_N} + \left(\frac{\gamma_1(\tau_N)}{\sqrt{\tau_N}} - \frac{\gamma_1(\tau_l)}{\sqrt{\tau_l}}\right) - \frac{M^2-m^2}{m^2}\frac{\gamma_1(\tau_l)}{\sqrt{\tau_l}}\right\}, \quad (3)$$

where Q stands for the Euclidian momentum $Q^2 = -q^2$, $\phi_{nl}(\mathbf{r})$ is the non-relativistic wave function of the hydrogen-like atom in the coordinate space, and, following [18], we introduce

$$\tau_N = \frac{Q^2}{4M^2},$$
$$\tau_l = \frac{Q^2}{4m^2},$$
$$\gamma_1(\tau) = (1-2\tau)\left(\sqrt{1+\tau} - \sqrt{\tau}\right) + \sqrt{\tau},$$
$$\gamma_2(\tau) = (1+\tau)^{3/2} - \tau^{3/2} - \frac{3}{2}\sqrt{\tau},$$

the subscript l stands for the (orbiting) lepton which may be an electron or a muon and N is for the nucleus.

That is related to the pointlike Salpeter-type contribution, which means that its two-photon part in a hard-photon approximation is equal to the subtraction term. The latter is IR divergent. The IR divergent part of the subtraction term is

$$\Delta E_{\rm IR}^{1/2,1/2}(ns) = -\frac{16(Z\alpha)^5\, m_r^4}{\pi n^3}\int\frac{dQ}{Q^4}\left[1 - \frac{Q^2}{8mM}\right]. \quad (4)$$

Let's remind the main steps of the calculation of the Salpeter term. Following [1,12–14,17], it is useful to use the Coulomb gauge and calculate the sum of the ladder and cross diagrams. There are three types of the contributions depending on the type of the photons in the two-photon exchange: CC (both photons are Coulomb photons), CT (one photon is a Coulomb photon, while the other is a transverse photon), and TT (both photons are the transverse photons). If the soft momentum area dominates, then some additional Coulomb photons may be involved (cf. Fig. 3).

The results for the individual contributions are [17]

$$\Delta E_{\rm CC}^{1/2,1/2}(ns) = -\frac{4}{3}\frac{(Z\alpha)^5 m}{\pi n^3}\frac{m}{M}\left(\frac{m_r}{m}\right)^3,$$
$$\Delta E_{\rm CT}^{1/2,1/2}(ns) = \frac{(Z\alpha)^5 m}{\pi n^3}\frac{m}{M}\left(\frac{m_r}{m}\right)^3 \times\frac{8}{3}\left\{\ln\frac{1}{Z\alpha} - \ln k_0(ns) + \ln\frac{4}{n} + (\psi(n+1) - \psi(1)) - \frac{1}{2n} + \frac{5}{6}\right\},$$
$$\Delta E_{\rm TT}^{1/2,1/2}(ns) = \frac{(Z\alpha)^5 m}{\pi n^3}\frac{m}{M}\left(\frac{m_r}{m}\right)^3 \times\left[2\ln Z\alpha + 2\ln\frac{2}{n} - 2\ln\left(1 + \frac{m}{M}\right) + 2\left(\psi(n+1) - \psi(1)\right) + \frac{n-1}{n} + \frac{8(1-\ln 2)}{3} + \frac{m^2}{M^2-m^2}\ln\frac{M^2}{m^2}\right]. \quad (5)$$

The contribution to the energy of the level with $l \neq 0$ comes from the CT and TT terms (see, e.g. [17]). They are the soft-photon contributions and are not our concern in this paper.

3 Nuclear spin 0: the pointlike two-photon exchange and the subtraction term

The two-photon exchange for a scalar nucleus due to the nuclear-structure effects was considered in the case of muonic helium in [20]. The pointlike subtraction term for the Lamb shift in an ordinary or muonic atom with a scalar nucleus is [20]

$$\Delta E^{1/2,0}(ns) = \frac{m(Z\alpha)^2}{M(M^2-m^2)}\,\phi^2_{n0}(0)\int_0^\infty \frac{dQ^2}{Q^2} \times\left\{-\left(\frac{\gamma_2(\tau_d)}{\sqrt{\tau_d}}-\frac{\gamma_2(\tau_l)}{\sqrt{\tau_l}}\right)\frac{1}{\tau_d} -\frac{M^2-m^2}{m^2}\,\frac{\gamma_1(\tau_l)}{\sqrt{\tau_l}}\right\}, \tag{6}$$

where we use the same notation as in (3).

The IR divergent part of the subtraction term

$$\Delta E^{1/2,0}_{\rm IR}(ns) = -\frac{16(Z\alpha)^5\,m_r^4}{\pi n^3}\int\frac{dQ}{Q^4}\left[1-\frac{Q^2}{8mM}+\frac{Q^2}{4M^2}\right] \tag{7}$$

is somewhat different from the result for the nucleus with spin 1/2 (cf. (4)). The difference is due to the fact that the Breit-type corrections for the nuclear spin 0 and nuclear spin 1/2 are not the same in order $(Z\alpha)^4m^3/M^2$ [23–25]. As we mentioned above, the IR divergent terms follow in part the Breit-type contributions[1] and should be subtracted. If those Breit-type terms are different, the subtractions are also somewhat different.

The effective Salpeter contribution is the contribution of the subtracted terms (see (6)), where one should subtract the IR divergent part (see (7)) and properly restore the atomic effects at the soft part of the contribution. Instead of restoring the atomic effects for the soft part one may consider the difference of the Salpeter contribution for the nuclear spin 1/2 and the effective contribution for the nuclear spin 0. The soft part cancels out in such a difference and, as the result, to find the difference one does not need to restore the details of atomic effects.

One may also consider an ab initio calculation of the ‘true’ Salpeter-type contribution for the nuclear spin 0. In this case the result for the integrand follows from the Feynman rules and the realization of the chosen strategy for the solution of the bound state problem. In the case of the nuclear spin 0, the effective Salpeter contribution, which follows from the subtraction for the two-photon nuclear-structure effects, is exactly related to the pointlike Salpeter-type contribution, i.e., to complete contributions of the Feynman diagrams with a pointlike scalar particle as the nucleus.

The results for the individual contributions with the scalar nucleus differ from those for the nucleus with $I=1/2$ (cf. (5))

$$\begin{aligned}
\Delta E^{1/2,0}_{\rm CC}(ns) &= \frac{(Z\alpha)^5 m}{\pi n^3}\,\frac{m}{M}\left(\frac{m_r}{m}\right)^3 \times\left[-\frac{4}{3}+\frac{m^2}{M^2-m^2}\ln\frac{M^2}{m^2}\right]\\
&= \Delta E^{1/2,1/2}_{\rm CC}(ns) + \frac{(Z\alpha)^5 m}{\pi n^3}\,\frac{m}{M}\left(\frac{m_r}{m}\right)^3 \times\frac{m^2}{M^2-m^2}\ln\frac{M^2}{m^2}\,,\\
\Delta E^{1/2,0}_{\rm CT}(ns) &= \Delta E^{1/2,1/2}_{\rm CT}(ns)\,,\\
\Delta E^{1/2,0}_{\rm TT}(ns) &= \frac{(Z\alpha)^5 m}{\pi n^3}\,\frac{m}{M}\left(\frac{m_r}{m}\right)^3 \times\Big[2\ln Z\alpha + 2\ln\frac{2}{n} - 2\ln\left(1+\frac{m}{M}\right)\\
&\quad +2\left(\psi(n+1)-\psi(1)\right) +\frac{n-1}{n}+\frac{8(1-\ln 2)}{3}\Big]\\
&= \Delta E^{1/2,1/2}_{\rm TT}(ns) - \frac{(Z\alpha)^5 m}{\pi n^3}\,\frac{m}{M}\left(\frac{m_r}{m}\right)^3 \times\frac{m^2}{M^2-m^2}\ln\frac{M^2}{m^2}\,.
\end{aligned} \tag{8}$$

Summing up those three terms in (8), we obtain a new original result for $\Delta E^{1/2,0}(ns)$. We note, that while the individual contributions for $I=1/2$ and $I=0$ differ the complete result

$$\Delta E^{1/2,0}(ns) = \Delta E^{1/2,1/2}(ns) \tag{9}$$

is the same.

In the case of the scalar nucleus there is an additional diagram, the seagull one. The terms above (for $I=0$) are related to the sum of the ladder, cross, and seagull diagrams, while the related terms for $I=1/2$ are for the sum of the ladder and cross diagrams. The seagull one contributes only in the TT term.

The contribution to the energy of the level with $l\geq 1$, which comes from the CT and TT terms [17], does not depend on the nuclear spin.

4 Nuclear spin 1: the subtraction term

As we mention here the contribution of the subtractions terms (to which we refer as to an ‘effective Salpeter contribution’) has a soft and hard part. The soft one dealt with a nonrelativistic nucleus and is nuclear-spin-independent. The hard one is determined by the subtraction introduced in calculations of the nuclear structure effects and in principle is chosen to a certain extend arbitrary. For example, there is no doubt that starting with an ultraviolet finite consideration of a not-pointlike vector particle, it is possible to separate the nuclear-structure effects and a soft physics with a pointlike nucleus in such a way that both terms are finite. In the meantime, a QED theory with a

[1] The Breit equation (see, e.g., [26]) is an equation for two particles with spin 1/2. Following the same logics one could extend the consideration for other spins; we refer to such considerations as ‘Breit-type’ ones.

pointlike vector nucleus is divergent, while a fundamental theory (such as a consideration of the W^+ gauge boson0 as a nucleus is finite, but involves such a contributions (e.g., the Goldstone modes) [28] which have nothing to do with a pointlike limit for a realistic system, such as muonic deuterium. That means that whatever we consider as a pointlike QED part of the complete theory, it is rather an effective theory, than a fundamental one. While building a fundamental theory one has no choices. On contrary, a construction of an effective theory gives some options. It is worth to mention different possibility for the form factors, nuclear radii (see, e.g., [29,30] for $I = 1$), a consideration a pointlike limit with an actual value of the nuclear magnetic moment, which is a standard procedure for the external-field approximation for the hyperfine structure (see, e.g., [16,17]) and which may be applied to two-photon recoil contributions for the Lamb shift in muonic atoms with $I = 1/2$ [31,32]. A pointlike part of theory for the Lamb shift of a muonic deuterium and atoms with other $I = 1$ nuclei should differ from a fundamental one, since we are to build former as an ultraviolet finite one, while the latter is divergent.

The two-photon nuclear-structure effects for an extended vector nucleus were considered for muonic deuterium in [21,22]. The subtraction applied there (see, e.g., [21]) is exactly the same as in the scalar case (see (6)). As we mention above the soft part of the two-photon contributions is nuclear-spin independent. The hard part of the effective Salpeter term is completely both defined and determined by the subtraction terms. If the subtraction terms are the same for the nuclear spin $I = 0$ and $I = 1$, the hard part of the effective Salpeter contribution is also the same. Therefore we arrive at a new result, describing the subtraction contribution for $I = 1$

$$\Delta E^{1/2,1} = \Delta E^{1/2,0} \,. \quad (10)$$

It might be interesting to compare the effective Salpeter terms and the 'true' one. The latter is to be derived from a fundamental ab initio theory. As well as in the case of the effective term its soft part, determined by the nonrelativistic nucleus, is nuclear-spin-independent and is the same as for the effective term. The difference is only in the definition of the hard part. In contrast to the case of the scalar nucleus, the subtraction with a vector nucleus is *not* related directly to the Salpeter-type contribution, in the sense that the subtraction relates to a certain 'pointlike' construction, but not to a *complete* contribution of a pointlike nuclear vector particle. The two-photon contribution to the Lamb shift for a hydrogen-like atom with a pointlike vector nucleus is UV divergent (see, e.g., [28]). Such divergences do not cause any practical problem, because the form factor of a real nucleus serves as a regularization function which efficiently cuts the divergences off once the momentum of the integration is above the inverse radius of the nucleus.

As we mentioned above, the infrared divergences for $I = 0$ and $I = 1/2$ are somewhat different (cf. Eqs. (4) and (7)). That was interpreted as a consequence of the differences in the Breit-type contributions. The situation with a vector nucleus is somewhat more tricky than in the case of $I = 0$. The results for the pointlike Breit-type contributions for $I = 1/2$ and $I = 1$ are different [27]. However, one cannot see it in an 'easy' way because the difference in the pointlike results is absorbed by a difference in the definition of the charge radius [29,30]. Still, we have to subtract some divergences which are related to all the contributions in order $(Z\alpha)^4 m$, which includes both the Breit-type contributions and the leading nuclear charge contribution. Independently of which definition for the nuclear charge is applied, we have to make the same subtraction for the pointlike terms.

5 Conclusion

Concluding, we have studied the effective Salpeter term, the effective pointlike contribution which arises from the introduction of the 'pointlike' subtractions while calculating the elastic part of the nuclear-structure two-photon-exchange diagrams, depicted in Figure 1. While the case of the nuclear spin $I = 1/2$ has been considered for a while, the results on $I = 0, 1$ are obtained in this paper for the first time.

Since the subtractions for $I = 0$ and $I = 1$ are the same, the effective Salpeter contribution is also the same. It is a structureless QED contribution in a sense that it involves a nucleus without any internal structure. In the case of $I = 0$ it exactly matches the contribution for a hydrogen-like atom with a pointlike scalar nucleus, while for $I = 1$ it does not relate to a pointlike $I = 1$ nucleus. But that is not a problem for the consistency of the consideration of the 'structured' and 'structureless' parts of the complete contribution.

The results for pointlike nuclei with $I = 0$ and $I = 1/2$ are the same. However, that is just a coincident. There is no 'deep' principle why it should be. In appendix we consider the case of the atom with a scalar orbiting particle and a scalar pointlike nucleus and the result is different from the consideration above.

We have also to comment on the composition of the nuclear-structure effects. As it was explained in [31,32], there are three kinds of them in the case of the nuclear spin 1/2. One is due to excitation of the nuclear degrees of freedom (the so-called nuclear polarizability contribution). The second is the nuclear-size correction. If we have an explicit parameter, which describes the nuclear size, say, the charge radius, the correction should vanish at the limit of zero radius. The last is the anomalous-magnetic-moment contribution. This contribution does not vanish in the limit of the zero radius. It is well-defined (and finite) for the two-photon contribution to the Lamb shift for $I = 1/2$. (It is, however, not finite in the case of the two-photon contribution for the hyperfine structure (see, e.g., [10,11])).

In the case of different values of the nuclear spin the situation changes. For $I = 0$, apparently there is no anomalous-magnetic-moment contribution. In the case of $I = 1$, to discuss the result in the limit of zero radius we have to return to the 'true' pointlike diagrams. The result is divergent and contains a logarithmic term

$$\frac{(Z\alpha)^5 m}{\pi n^3}\,\frac{m}{M}\left(\frac{m_r}{m}\right)^3 \times \frac{(g-1)^2}{2}\,\frac{m^2}{M^2}\,\ln\frac{\Lambda^2}{m^2}\,,$$

where Λ is the UV cut-off.

The divergence does not vanish at the limit of the zero nuclear radius. In ordinary atoms with $m \ll 1/r_N$ the inverse nuclear radius enters the related would-be divergent expression as the cut-off parameter and produces a logarithmically enhanced recoil contribution (cf. with HFS in hydrogen [10,11]). In muonic atoms with $m \sim 1/r_N$ there is no enhanced term. In the case of $g = 1$ the divergent term disappear, however, there is no actual nucleus with $g = 1$. Besides, there are finite additional contributions for $g = 1$ (see [28] for detail).

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Author contribution statement

The calculations have been performed by the authors together.

Appendix A: Two-photon exchange in the bound system of two scalar particles

As it is shown in Section 3 the results for $\Delta E^{1/2,0}$ and $\Delta E^{1/2,1/2}$ coincide (see (9)). We note that the result for $\Delta E^{1/2,0}$ is exact in m/M and therefore we may consider also the case of an atom with a light orbiting scalar and a heavy spin-1/2 nucleus

$$\Delta E^{0,1/2} = \Delta E^{1/2,0} = \Delta E^{1/2,1/2}. \tag{A.1}$$

To show that that is a coincident we compare those results with $\Delta E^{0,0}$. The hard part of this contribution has been considered in classical works [33–35] (see also [36,37]). The result is quite different from those discussed above for other spins. The result for $\Delta E^{0,0}$ is UV divergent (but renormalizable). Due to that the contribution of the two-photon exchange should be considered together with a 4-scalar contact term, which is present in the Lagrangian of the scalar theory and the QED renormalization of which comes from the two-photon diagram.

The situation with two scalar particles is different from the case of two-photon exchanges for an atom with at least one particle with spin-1/2 also in another aspect. The Salpeter-type contributions, which are discussed in the main part of the paper (such as 1/2-1/2, 1/2-0, and 0-1/2) are recoil contributions with the leading term of the order $(Z\alpha)^5 m^2/M$. Their differences are recoil effects of higher order and the related differential contributions are of the order $(Z\alpha)^5 m^4/M^3$. The above mentioned divergence in the 1/2-1 case is also of the order $(Z\alpha)^5 m^4/M^3$. In contrast to that, the [divergent] difference between the 1/2-1/2 contribution and the 0-0 one is of the order $(Z\alpha)^5 m^2/M$, i.e., it is of the same order that the leading term.

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