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Conformal symmetry in quantum gravity

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Abstract We study the problem of how to derive conformal symmetry in the framework of quantum gravity. We start with a generic gravitational theory which is invariant under both the general coordinate transformation (GCT) and Weyl transformation (or equivalently, local scale transformation), and then construct its BRST formalism by fixing the gauge symmetries by the extended de Donder gauge and scalar gauge conditions. These gauge-fixing conditions are invariant under global $GL(4)$ and global scale transformations. The gauge-fixed and BRST invariant quantum action possesses a huge Poincaré-like $IOSp(10|10)$ global symmetry, from which we can construct an extended conformal symmetry in a flat Minkowski background in the sense that the Lorentz symmetry is replaced with the $GL(4)$ symmetry. Moreover, we construct the conventional conformal symmetry out of this extended symmetry. With a flat Minkowski background $\langle g_{\mu\nu} \rangle = \eta_{\mu\nu}$ and a non-zero scalar field $\langle \phi \rangle \neq 0$, the $GL(4)$ and global scale symmetries are spontaneously broken to the Lorentz symmetry, thereby proving that the graviton and the dilaton are respectively the corresponding Nambu–Goldstone bosons, and therefore they must be exactly massless at nonperturbative level. One of remarkable aspects in our findings is that in quantum gravity, a derivation of conformal symmetry does not depend on a classical action, and its generators are built from only the gauge-fixing and the FP ghost actions. Finally, we address a generalized Zumino theorem in quantum gravity.

1 Introduction

There is no question that both local and global symmetries play an important role in elementary-particle physics and quantum gravity. For instance, in quantum chromodynamics (QCD) it has been found that a local symmetry is the non-abelian gauge symmetry based on the gauge group $SU(3)$, and that this local symmetry gives rise to physically significant effects, such as the asymptotic freedom and the confinement of quarks and gluons. In addition to the $SU(3)$ gauge symmetry, there is a $U(1)_V \times U(1)_A$ global symmetry of the quark action. The $U(1)_A$ symmetry is anomalous and the effect of a $U(1)_A$ transformation is to change the value of the theta angle, which requires us to consider an instanton to make the vacuum energy density to depend on the theta angle.

On the other hand, in quantum gravity, the meaning of symmetries is more subtle than that in elementary-particle physics. Although general relativity has been beautifully established in the Riemannian geometry on the basis of the general coordinate invariance and the equivalence principle, it seems that the feature of the non-renormalizability of general relativity would need a more huge local symmetry such as a local supersymmetry or a Weyl symmetry (or equivalently, a local scale symmetry). As for a global symmetry, in classical general relativity, it could be broken by the no-hair theorem of black holes [1]. Moreover, in string theory, which is a strong candidate of quantum gravity, we never get any additive conservation laws and at least in known string vacua, the additive global symmetries turn out to be either gauge symmetries or explicitly violated [2].

In such a situation, it might appear to be strange to seek for a new global symmetry in quantum gravity, but even in quantum gravity, we have important global symmetries such as the BRST symmetry and the conservation law of the ghost number. The BRST symmetry, which is a residual global symmetry emerging after the gauge-fixing procedure, plays a role in proving the unitarity of the theory and deriving the Ward identities among the Green functions [3]. On the other hand, the conservation of the ghost number is violated in a two-dimensional quantum theory, and leads to the ghost number anomaly, which is closely related to the Riemann-Roch theorem on the closed Riemann surfaces [4].

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From this viewpoint, it is valuable to derive global symmetries, which include the BRST symmetry and the symmetry of the FP ghost number, in quantum gravity. If such the symmetries are purely built from quantum fields such as the FP (anti)ghosts and the Nakanishi–Lautrup auxiliary fields, they could escape from the no-hair theorem of black holes, and might have some important applications for the study of their anomalies and the confinement of massive ghost in higher-derivative gravities.

In a flat Minkowski space-time, conformal symmetry occupies a special position among many global symmetries in that conformal symmetry is ubiquitous in physics ranging from elementary-particle physics to condensed matter physics [5,6]. Even if conformal symmetry plays an important role, it must be spontaneously broken in reality since our world has a built-in scale in it. Moreover, it is of interest that conformal symmetry is connected with a classical gravity formulated in a curved space-time. In fact, the Zumino theorem advocates that the theories which are invariant under the GCT and Weyl transformation have conformal invariance in the flat Minkowski background at the classical level [7]. Then, it is natural to ask ourselves whether the Zumino theorem can be generalized even in quantum gravity or not. In our past works [8–11], we have attacked this question. We have shown that in Weyl invariant scalar-tensor gravity [9], the Zumino theorem is valid whereas in conformal gravity [11] it is not so. In these works, since we have specified a classical theory, it is not clear if the answer is independent of the choice of the classical action or not. It is therefore desirable to start with not a specific action but a more general gravitational action, which is invariant under both the GCT and Weyl transformation, and investigate the validity of the Zumino theorem in quantum gravity. This is one of our motivations behind the present study. Actually, we will find that it is not necessary to fix such a classical action at all in order to understand the Zumino theorem in quantum gravity.

The outline of this article is as follows: in the next section we construct a BRST formalism of our theory. In Sect. 3, we carry out the canonical quantization and calculate the equal-time (anti)commutation relations (ETCRs). In Sect. 4, we prove the existence of a Poincaré-like $IOSp(10|10)$ global symmetry in a gauge-fixed and BRST invariant quantum Lagrangian and compute its algebra. In Sect. 5, we derive conformal symmetry in a flat Minkowski space-time from the Poincaré-like $IOSp(10|10)$ symmetry. In Sect. 6, we investigate spontaneous symmetry breakdown of $GL(4)$ symmetry and global scale symmetry to Lorentz symmetry. Through this mechanism of the symmetry breaking, we can precisely prove that the graviton and the dilaton are exactly massless owing to the Nambu–Goldstone theorem. The final section is devoted to a conclusion. In particular, two important problems, those are, conformal anomaly and non-unitarity owing to the presence of the massive ghost, are pointed out.

Four appendices are put for technical details. In Appendix A, we show that the extended de Donder gauge condition is invariant under the global $GL(4)$ transformation. In Appendix B, we derive the ETCR, $[\dot{g}_{\mu\nu}, b'_\rho]$. In Appendix C, we present a proof of the ETCR, $[\dot{b}_\mu, b'_\nu] = -i\tilde{f}\phi^{-2}(\partial_\mu b_\nu + \partial_\nu b_\mu)\delta^3$ without recourse to the Einstein's equation. In Appendix D, we give two different proofs for an algebra $[P_\mu, K^\nu] = -2i(G^\rho{}_\rho - D)\delta^\nu_\mu$.

2 BRST formalism

We wish to perform a manifestly covariant BRST quantization of a gravitational theory which is invariant under both general coordinate transformation (GCT) and Weyl transformation (or equivalently, local scale transformation).[1] To take a more general theory into consideration, without specifying a concrete expression of the gravitational Lagrangian, we will start with the following classical Lagrangian[2]

$$\mathcal{L}_c = \mathcal{L}_c(g_{\mu\nu}, \phi), \tag{1}$$

which includes the metric tensor field $g_{\mu\nu}$ and a scalar field ϕ as dynamical variables.[3] We assume that $\mathcal{L}_c$ does not involve more than first order derivatives of the metric and matter fields.

We are ready to fix the general coordinate symmetry and the Weyl symmetry by suitable gauge conditions. It is a familiar fact that after introducing the gauge conditions, instead of such the two local gauge symmetries, we are left with two kinds of global symmetries named as the BRST symmetries. The BRST transformation, which is denoted as δ_B, corresponding to the GCT is defined as

$$\begin{aligned}
\delta_B g_{\mu\nu} &= -(\nabla_\mu c_\nu + \nabla_\nu c_\mu) \\
&= -(c^\alpha \partial_\alpha g_{\mu\nu} + \partial_\mu c^\alpha g_{\alpha\nu} + \partial_\nu c^\alpha g_{\mu\alpha}), \\
\delta_B \tilde{g}^{\mu\nu} &= \sqrt{-g}(\nabla^\mu c^\nu + \nabla^\nu c^\mu - g^{\mu\nu}\nabla_\rho c^\rho), \\
\delta_B \phi &= -c^\lambda \partial_\lambda \phi, \quad \delta_B c^\rho = -c^\lambda \partial_\lambda c^\rho, \\
\delta_B \bar{c}_\rho &= i B_\rho, \quad \delta_B B_\rho = 0,
\end{aligned} \tag{2}$$

where c^ρ and $\bar{c}_\rho$ are respectively the Faddeev-Popov (FP) ghost and antighost, B_ρ is the Nakanishi–Lautrup (NL) field,

[1] There are a lot of papers dealing with gravitational theories which are invariant under the GCT and Weyl transformation. The partial list is given in [12–18].

[2] We follow the notation and conventions of MTW textbook [1]. Greek little letters $\mu, \nu, \ldots$ and Latin ones $i, j, \ldots$ are used for space-time and spatial indices, respectively; for instance, $\mu = 0, 1, 2, 3$ and $i = 1, 2, 3$. The Riemann curvature tensor and the Ricci tensor are respectively defined by $R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\sigma\nu} - \partial_\nu \Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\lambda\mu}\Gamma^\lambda_{\sigma\nu} - \Gamma^\rho_{\lambda\nu}\Gamma^\lambda_{\sigma\mu}$ and $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$. The Minkowski metric tensor is denoted by $\eta_{\mu\nu}$; $\eta_{00} = -\eta_{11} = -\eta_{22} = -\eta_{33} = -1$ and $\eta_{\mu\nu} = 0$ for $\mu \neq \nu$.

[3] It is straightforward to add the other fields such as gauge fields and spinor fields.

and we have defined $\tilde{g}^{\mu\nu} \equiv \sqrt{-g} g^{\mu\nu}$. For later convenience, in place of the NL field B_ρ we introduce a new NL field defined as

$$b_\rho = B_\rho - i c^\lambda \partial_\lambda \bar{c}_\rho, \tag{3}$$

and its BRST transformation reads

$$\delta_B b_\rho = -c^\lambda \partial_\lambda b_\rho. \tag{4}$$

The other BRST transformation, which is denoted as $\bar{\delta}_B$, corresponding to the Weyl transformation is defined as

$$\bar{\delta}_B g_{\mu\nu} = 2c g_{\mu\nu}, \quad \bar{\delta}_B \tilde{g}^{\mu\nu} = 2c \tilde{g}^{\mu\nu}, \\ \bar{\delta}_B \phi = -c\phi, \quad \bar{\delta}_B \bar{c} = iB, \quad \bar{\delta}_B c = \bar{\delta}_B B = 0, \tag{5}$$

where c and $\bar{c}$ are respectively the FP ghost and FP antighost, and B is the NL field. Note that the two BRST transformations are nilpotent, i.e.,

$$\delta_B^2 = \bar{\delta}_B^2 = 0. \tag{6}$$

To complete the two BRST transformations, we have to fix not only the GCT BRST transformation δ_B on c, $\bar{c}$ and B but also the Weyl BRST transformation $\bar{\delta}_B$ on c^ρ, $\bar{c}_\rho$ and b_ρ. It is easy to determine the former BRST transformation since the fields c, $\bar{c}$ and B are such scalar fields that to match the transformation law for scalar fields their BRST transformations should take the form:

$$\delta_B B = -c^\lambda \partial_\lambda B, \quad \delta_B c = -c^\lambda \partial_\lambda c, \quad \delta_B \bar{c} = -c^\lambda \partial_\lambda \bar{c}. \tag{7}$$

On the other hand, there is an ambiguity in fixing the latter BRST transformation, but we would like to propose a recipe for achieving this goal. The recipe [9] is to just assume that the two BRST transformations anti-commute with each other, that is,

$$\{\delta_B, \bar{\delta}_B\} \equiv \delta_B \bar{\delta}_B + \bar{\delta}_B \delta_B = 0, \tag{8}$$

which requires us to take

$$\bar{\delta}_B b_\rho = \bar{\delta}_B c^\rho = \bar{\delta}_B \bar{c}_\rho = 0. \tag{9}$$

As the gauge condition for the GCT, we will take "the extended de Donder gauge" [9]

$$\partial_\mu (\tilde{g}^{\mu\nu} \phi^2) = 0, \tag{10}$$

which is invariant under the Weyl transformation (5) as required from the condition (8). As for the Weyl transformation, we have to choose an appropriate gauge condition, which is invariant under the GCT, that is, a scalar quantity. As such a gauge condition, we will take the so-called "the scalar gauge condition" [9]

$$\partial_\mu (\tilde{g}^{\mu\nu} \phi \partial_\nu \phi) = 0, \tag{11}$$

which can be alternatively written as

$$\Box \phi^2 = 0. \tag{12}$$

The key observation here is that both the extended de Donder gauge condition (10) and the scalar gauge condition (11) are invariant under both a *global* $GL(4)$ transformation and a *global* scale transformation.[4] For instance, a proof of the $GL(4)$ invariance of the extended Donder gauge condition (10) is given in Appendix A.

After taking the extended de Donder gauge condition (10) for the GCT and the scalar gauge condition (11) for the Weyl transformation, the gauge-fixed and BRST invariant quantum Lagrangian is given by

$$\begin{aligned} \mathcal{L}_q &= \mathcal{L}_c + \mathcal{L}_{GF+FP} + \bar{\mathcal{L}}_{GF+FP} \\ &= \mathcal{L}_c + i\delta_B(\tilde{g}^{\mu\nu}\phi^2 \partial_\mu \bar{c}_\nu) + i\bar{\delta}_B \left[\bar{c}\partial_\mu(\tilde{g}^{\mu\nu}\phi\partial_\nu\phi)\right] \\ &= \mathcal{L}_c - \tilde{g}^{\mu\nu}\phi^2(\partial_\mu b_\nu + i\partial_\mu \bar{c}_\lambda \partial_\nu c^\lambda) \\ &\quad + \tilde{g}^{\mu\nu}\phi\partial_\mu B \partial_\nu \phi - i\tilde{g}^{\mu\nu}\phi^2 \partial_\mu \bar{c}\partial_\nu c, \end{aligned} \tag{13}$$

where surface terms are dropped. Let us note that this quantum Lagrangian is also invariant under the global $GL(4)$ transformation and the global scale transformation.

Let us rewrite this Lagrangian concisely as

$$\mathcal{L}_q = \mathcal{L}_c - \frac{1}{2}\tilde{g}^{\mu\nu} E_{\mu\nu}, \tag{14}$$

where we have defined

$$E_{\mu\nu} \equiv \phi^2(\partial_\mu b_\nu + i\partial_\mu \bar{c}_\lambda \partial_\nu c^\lambda) - \phi\partial_\mu B\partial_\nu\phi + i\phi^2\partial_\mu\bar{c}\partial_\nu c \\ + (\mu \leftrightarrow \nu). \tag{15}$$

Moreover, it is sometimes more convenient to introduce the dilaton $\sigma(x)$ by defining

$$\phi(x) \equiv e^{\sigma(x)}, \tag{16}$$

and rewrite (14) further into the form

$$\mathcal{L}_q = \mathcal{L}_c - \frac{1}{2}\phi^2 \tilde{g}^{\mu\nu}\hat{E}_{\mu\nu} = \mathcal{L}_c - \frac{1}{2}e^{2\sigma(x)}\tilde{g}^{\mu\nu}\hat{E}_{\mu\nu}, \tag{17}$$

where we have defined

$$\hat{E}_{\mu\nu} \equiv \partial_\mu b_\nu + i\partial_\mu\bar{c}_\lambda\partial_\nu c^\lambda - \partial_\mu B\partial_\nu\sigma + i\partial_\mu\bar{c}\partial_\nu c \\ + (\mu \leftrightarrow \nu). \tag{18}$$

Note that the relation between $E_{\mu\nu}$ and $\hat{E}_{\mu\nu}$ is given by

$$E_{\mu\nu} = \phi^2 \hat{E}_{\mu\nu} = e^{2\sigma}\hat{E}_{\mu\nu}. \tag{19}$$

The field equations obtained from variations of b_ν, B, c^ρ, $\bar{c}_\rho$, c and $\bar{c}$ in the Lagrangian $\mathcal{L}_q$, take the form:

$$\partial_\mu(\tilde{g}^{\mu\nu}\phi^2) = 0, \quad \partial_\mu(\tilde{g}^{\mu\nu}\phi\partial_\nu\phi) = 0, \\ \partial_\mu(\tilde{g}^{\mu\nu}\phi^2\partial_\nu\bar{c}_\rho) = 0, \quad \partial_\mu(\tilde{g}^{\mu\nu}\phi^2\partial_\nu c^\rho) = 0, \\ \partial_\mu(\tilde{g}^{\mu\nu}\phi^2\partial_\nu\bar{c}) = 0, \quad \partial_\mu(\tilde{g}^{\mu\nu}\phi^2\partial_\nu c) = 0. \tag{20}$$

[4] The unitary gauge condition $\phi =$ constant and the Lorenz condition $\nabla_\mu S^\mu = 0$ [10], where S_μ is a Weyl gauge field, are also invariant under these global transformations.

Then, the two gauge-fixing conditions in (20), or equivalently (10) and (11), lead to a very simple d'Alembert-like equation for the dilaton:

$$g^{\mu\nu}\partial_\mu\partial_\nu\sigma = 0. \tag{21}$$

It is worthwhile to notice that it is not the scalar field ϕ but the dilaton σ that satisfies this type of equation. Furthermore, the extended de Donder gauge condition, i.e., the first field equation in (20) gives us the similar simple equation for the (anti)ghosts:

$$g^{\mu\nu}\partial_\mu\partial_\nu\bar{c}_\rho = 0, \qquad g^{\mu\nu}\partial_\mu\partial_\nu c^\rho = 0,$$
$$g^{\mu\nu}\partial_\mu\partial_\nu\bar{c} = 0, \qquad g^{\mu\nu}\partial_\mu\partial_\nu c = 0. \tag{22}$$

In order to derive the field equation for B, it is useful to take the Weyl BRST transformation for the field equation for the antighost $\bar{c}$ and the result again provides a simple equation:

$$g^{\mu\nu}\partial_\mu\partial_\nu B = 0. \tag{23}$$

Finally, the field equation for b_ρ can be obtained by taking the GCT BRST transformation for the field equation for $\bar{c}_\rho$ as follows:[5] Taking the GCT BRST transformation for the field equation for $\bar{c}_\rho$, i.e., $g^{\mu\nu}\partial_\mu\partial_\nu\bar{c}_\rho = 0$ leads to

$$(-\partial_\lambda g^{\mu\nu}c^\lambda + 2g^{\mu\lambda}\partial_\lambda c^\nu)\partial_\mu\partial_\nu\bar{c}_\rho + ig^{\mu\nu}\partial_\mu\partial_\nu B_\rho = 0. \tag{24}$$

Inserting Eq. (3) to this equation and arranging terms, we have

$$g^{\mu\nu}\partial_\mu\partial_\nu b_\rho = -ic^\lambda\partial_\lambda(g^{\mu\nu}\partial_\mu\partial_\nu\bar{c}_\rho) = 0. \tag{25}$$

In other words, setting $X^M = \{x^\mu, b_\mu, \sigma, B, c^\mu, \bar{c}_\mu, c, \bar{c}\}$,[6] X^M turns out to obey the d'Alembert-like equation:

$$g^{\mu\nu}\partial_\mu\partial_\nu X^M = 0. \tag{26}$$

This fact, together with the gauge condition $\partial_\mu(\tilde{g}^{\mu\nu}\phi^2) = 0$, produces the two kinds of conserved currents

$$\mathcal{P}^{\mu M} \equiv \tilde{g}^{\mu\nu}\phi^2\partial_\nu X^M = \tilde{g}^{\mu\nu}\phi^2\left(1\overset{\leftrightarrow}{\partial}_\nu X^M\right),$$
$$\mathcal{M}^{\mu MN} \equiv \tilde{g}^{\mu\nu}\phi^2\left(X^M\overset{\leftrightarrow}{\partial}_\nu Y^N\right), \tag{27}$$

where we have defined $X^M\overset{\leftrightarrow}{\partial}_\mu Y^N \equiv X^M\partial_\mu Y^N - (\partial_\mu X^M)Y^N$.

[5] This field equation has been previously obtained from the Einstein's equation [8,9,19], but this time we use the GCT BRST transformation since we have no concrete expression of the Einstein's equation owing to a generic gravitational action.

[6] In the present formalism, it is important to incorporate the mere space-time coordinates x^μ into a set of fields X^M. This might suggest that x^μ should be promoted to a quantum field in a more complete formalism.

3 Canonical quantization and equal-time commutation relations

In this section, after introducing the canonical (anti) commutation relations (CCRs), we will evaluate various equal-time (anti)commutation relations (ETCRs) among fundamental variables. To simplify various expressions, we will obey the following abbreviations adopted in the textbook of Nakanishi and Ojima [19,20]

$$[A, B'] = [A(x), B(x')]|_{x^0=x'^0}, \qquad \delta^3 = \delta(\vec{x}-\vec{x}'),$$
$$\tilde{f} = \frac{1}{\tilde{g}^{00}} = \frac{1}{\sqrt{-g}g^{00}}, \tag{28}$$

where we assume that $\tilde{g}^{00}$ is invertible.

The canonical quantization formalism of quantum gravity is a non-perturbative approach and is not directly related to perturbation theory. At present we do not know a completely satisfying method of approximation which is free of divergences, but we should note that it is not the problem of the formalism itself but the one which should be studied in the next stage. As long as the formalism of quantum gravity is concerned, the formalism we will present in what follows seems to be the most natural and satisfactory one of quantum gravity.

Now, in order to carry out the canonical quantization, we perform the integration by parts once and rewrite the Lagrangian (13) as

$$\mathcal{L}_q' = \mathcal{L}_c + \partial_\mu(\tilde{g}^{\mu\nu}\phi^2)b_\nu - i\tilde{g}^{\mu\nu}\phi^2\partial_\mu\bar{c}_\lambda\partial_\nu c^\lambda$$
$$+\tilde{g}^{\mu\nu}\phi\partial_\mu B\partial_\nu\phi - i\tilde{g}^{\mu\nu}\phi^2\partial_\mu\bar{c}\partial_\nu c + \partial_\mu\mathcal{V}^\mu, \tag{29}$$

where a surface term $\mathcal{V}^\mu$ is defined as $\mathcal{V}^\mu \equiv -\tilde{g}^{\mu\nu}\phi^2 b_\nu$. Next, let us set up the canonical (anti)commutation relations (CCRs)

$$[g_{\mu\nu}, \pi_g^{\rho\lambda\prime}] = i\frac{1}{2}(\delta_\mu^\rho\delta_\nu^\lambda + \delta_\mu^\lambda\delta_\nu^\rho)\delta^3, \quad [\phi, \pi_\phi'] = i\delta^3,$$
$$[B, \pi_B'] = i\delta^3,$$
$$\{c^\sigma, \pi_{c\lambda}'\} = \{\bar{c}_\lambda, \pi_{\bar{c}}^{\sigma\prime}\} = i\delta_\lambda^\sigma\delta^3, \quad \{c, \pi_c'\} = \{\bar{c}, \pi_{\bar{c}}'\} = i\delta^3, \tag{30}$$

where the other (anti)commutation relations vanish. Here the canonical variables are $g_{\mu\nu}, \phi, B, c^\mu, \bar{c}_\mu, c, \bar{c}$ and the corresponding canonical conjugate momenta are $\pi_g^{\mu\nu}, \pi_\phi, \pi_B, \pi_{c\mu}, \pi_{\bar{c}}^\mu, \pi_c, \pi_{\bar{c}}$, respectively and the b_μ field is regarded as not a canonical variable but a conjugate momentum of $\tilde{g}^{0\mu}$.

From the Lagrangian (29), it is easy to derive canonical conjugate momenta

$$\pi_g^{\mu\nu} = \frac{\partial\mathcal{L}_q'}{\partial\dot{g}_{\mu\nu}} = \frac{\partial\mathcal{L}_c}{\partial\dot{g}_{\mu\nu}} - \frac{1}{2}\sqrt{-g}\phi^2(g^{0\mu}g^{\nu\rho}$$
$$+g^{0\nu}g^{\mu\rho} - g^{0\rho}g^{\mu\nu})b_\rho,$$
$$\pi_\phi = \frac{\partial\mathcal{L}_q'}{\partial\dot{\phi}} = \frac{\partial\mathcal{L}_c}{\partial\dot{\phi}} + 2\tilde{g}^{0\mu}\phi b_\mu + \tilde{g}^{0\mu}\phi\partial_\mu B,$$

$$
\begin{aligned}
\pi_B &= \frac{\partial \mathcal{L}_q'}{\partial \dot{B}} = \tilde{g}^{0\mu}\phi\partial_\mu\phi,\\
\pi_{c\sigma} &= \frac{\partial \mathcal{L}_q'}{\partial \dot{c}^\sigma} = -i\tilde{g}^{0\mu}\phi^2\partial_\mu\bar{c}_\sigma, \quad \pi_{\bar{c}}^{\sigma} = \frac{\partial \mathcal{L}_q'}{\partial \dot{\bar{c}}_\sigma} = i\tilde{g}^{0\mu}\phi^2\partial_\mu c^\sigma,\\
\pi_c &= \frac{\partial \mathcal{L}_q'}{\partial \dot{c}} = -i\tilde{g}^{0\mu}\phi^2\partial_\mu\bar{c}, \quad \pi_{\bar{c}} = \frac{\partial \mathcal{L}_q'}{\partial \dot{\bar{c}}} = i\tilde{g}^{0\mu}\phi^2\partial_\mu c,
\end{aligned} \tag{31}
$$

where we have defined the time derivative such as $\dot{g}_{\mu\nu} \equiv \frac{\partial g_{\mu\nu}}{\partial t} \equiv \partial_0 g_{\mu\nu}$, and differentiation of ghosts is taken from the right. At this point, it is worthwhile to point out that although we have not specified the classical Lagrangian $\mathcal{L}_c$ and cannot write down a concrete expression of the conjugate momentum $\pi_g^{\mu\nu}$, the canonical conjugate momentum $\pi_g^{0\mu}$ generally has the following expression

$$
\pi_g^{0\mu} = A^\mu + B^{\mu\nu}\partial_\nu\phi + C^{\mu\nu}b_\nu, \tag{32}
$$

where A^μ, $B^{\mu\nu}$ and $C^{\mu\nu} \equiv -\frac{1}{2}\tilde{g}^{00}g^{\mu\nu}\phi^2$ have no $\dot{g}_{\mu\nu}$, and $B^{\mu\nu}\partial_\nu\phi$ does not have $\dot{\phi}$ since $\pi_g^{0\mu}$ does not include the dynamics of the metric and the scalar field.

Henceforth, we will evaluate various nontrivial equal-time (anti)commutation relations (ETCRs) by using the canonical (anti)commutation relations (CCRs), field equations and two kinds of BRST transformations. In particular, use of the GCT and Weyl BRST transformations makes it possible to derive nontrivial ETCRs without appealing to the Einstein's equation.

Let us first calculate ETCRs, which can be directly derived from the CCRs in Eq. (30) and the canonical conjugate momenta in Eq. (31). The CCR, $[B, \pi_B'] = i\delta^3$ leads to the ETCR:

$$
[B, \dot{\phi}'] = i\tilde{f}\phi^{-1}\delta^3. \tag{33}
$$

The other CCRs, $[\Phi, \pi_B'] = 0$ produce

$$
[\Phi, \dot{\phi}'] = 0, \tag{34}
$$

where Φ denotes a set of canonical variables except B, i.e, $\Phi = \{g_{\mu\nu}, \phi, c^\mu, \bar{c}_\mu, c, \bar{c}\}$. The antiCCRs, $\{c^\nu, \pi_{c\mu}'\} = \{\bar{c}_\mu, \pi_{\bar{c}}^{\nu\prime}\} = i\delta_\mu^\nu\delta^3$ yield

$$
\{c^\nu, \dot{\bar{c}}_\mu'\} = -\{\dot{c}^\nu, \bar{c}_\mu'\} = -\tilde{f}\phi^{-2}\delta_\mu^\nu\delta^3, \tag{35}
$$

where we have used a useful identity for generic variables Φ and Ψ

$$
[\Phi, \dot{\Psi}'] = \partial_0[\Phi, \Psi'] - [\dot{\Phi}, \Psi'], \tag{36}
$$

which holds for the anticommutation relation as well. Similarly, the antiCCRs, $\{c, \pi_c'\} = \{\bar{c}, \pi_{\bar{c}}'\} = i\delta^3$ produce

$$
\{c, \dot{\bar{c}}'\} = -\{\dot{c}, \bar{c}'\} = -\tilde{f}\phi^{-2}\delta^3. \tag{37}
$$

Now we are willing to derive a form of the (anti)ETCRs, $[\Phi, b_\rho']$ with Φ denoting a generic field, which are important ETCRs since the Nakanishi–Lautrup field b_μ in essence generates a translation. Particularly, the ETCR, $[g_{\mu\nu}, b_\rho']$ enables us to derive more complicated ETCRs later, so we present two different derivations, one of which is based on Eq. (32) while the other derivation relies on the GCT BRST transformation.

Let us first work with the CCR:

$$
[\pi_g^{\alpha 0}, g_{\mu\nu}'] = -i\frac{1}{2}(\delta_\mu^\alpha\delta_\nu^0 + \delta_\mu^0\delta_\nu^\alpha)\delta^3. \tag{38}
$$

Then, using Eq. (32), we find that this CCR produces

$$
[g_{\mu\nu}, b_\rho'] = -i\tilde{f}\phi^{-2}(\delta_\mu^0 g_{\rho\nu} + \delta_\nu^0 g_{\rho\mu})\delta^3. \tag{39}
$$

Incidentally, this ETCR gives us the similar ETCRs:

$$
\begin{aligned}
&[g^{\mu\nu}, b_\rho'] = i\tilde{f}\phi^{-2}(g^{\mu 0}\delta_\rho^\nu + g^{\nu 0}\delta_\rho^\mu)\delta^3,\\
&[\tilde{g}^{\mu\nu}, b_\rho'] = i\tilde{f}\phi^{-2}(\tilde{g}^{\mu 0}\delta_\rho^\nu + \tilde{g}^{\nu 0}\delta_\rho^\mu - \tilde{g}^{\mu\nu}\delta_\rho^0)\delta^3.
\end{aligned} \tag{40}
$$

Here we have used the following fact; since a commutator works as a derivation, we can have formulae

$$
\begin{aligned}
&[g^{\mu\nu}, \Phi'] = -g^{\mu\alpha}g^{\nu\beta}[g_{\alpha\beta}, \Phi'],\\
&[\tilde{g}^{\mu\nu}, \Phi'] = -\left(\tilde{g}^{\mu\alpha}g^{\nu\beta} - \frac{1}{2}\tilde{g}^{\mu\nu}g^{\alpha\beta}\right)[g_{\alpha\beta}, \Phi'],
\end{aligned} \tag{41}
$$

where Φ is a generic field.

As an alternative derivation of Eq. (39), we make use of the GCT BRST transformation. For this purpose, let us start with the CCR, $[g_{\mu\nu}, \bar{c}_\rho'] = 0$, and take its GCT BRST transformation:

$$
-\{c^\alpha\partial_\alpha g_{\mu\nu} + \partial_\mu c^\alpha g_{\alpha\nu} + \partial_\nu c^\alpha g_{\alpha\mu}, \bar{c}_\rho'\} + [g_{\mu\nu}, iB_\rho'] = 0. \tag{42}
$$

Then, we find that the CCR, $[g_{\mu\nu}, \pi_{c\rho}'] = 0$ produces

$$
[\dot{g}_{\mu\nu}, \bar{c}_\rho'] = 0, \tag{43}
$$

where Eqs. (31) and (36) were used. Using this ETCR twice, the relation (3), and (35) in Eq. (42), we can arrive at our desired ETCR (39). Incidentally, the CCRs, $[g_{\mu\nu}, \pi_{\bar{c}}^{\rho\prime}] = [g_{\mu\nu}, \pi_c'] = [g_{\mu\nu}, \pi_{\bar{c}}'] = 0$ provide the ETCRs between $g_{\mu\nu}$ and FP ghosts:

$$
[g_{\mu\nu}, \dot{c}^{\rho\prime}] = [g_{\mu\nu}, \dot{c}'] = [g_{\mu\nu}, \dot{\bar{c}}'] = 0. \tag{44}
$$

Moreover, with the help of the CCR, $[\pi_g^{\alpha 0}, \Phi'] = 0$, it is easy to derive the remaining ETCRs with the form of $[\Phi, b_\rho']$ except for $\Phi = b_\mu$, which will be treated later:

$$
\begin{aligned}
[\phi, b_\rho'] &= [B, b_\rho'] = [c^\mu, b_\rho'] = [\bar{c}_\mu, b_\rho'] = [c, b_\rho']\\
&= [\bar{c}, b_\rho'] = 0.
\end{aligned} \tag{45}
$$

Next, let us attempt to derive a form of the (anti)ETCRs, $[\dot{\Phi}, b_\rho']$ with Φ denoting a generic field. Let us begin by showing the following ETCRs:

$$
[\phi, \dot{\bar{c}}_\rho'] = [\phi, \ddot{\bar{c}}_\rho'] = [\dot{\phi}, \dot{\bar{c}}_\rho'] = 0. \tag{46}
$$

The first equality can be obtained from either Eq. (34) or the CCR, $[\phi, \pi'_{c\rho}] = 0$. A proof of the second equality needs the field equation for $\bar{c}_\rho$ in Eq. (22), from which we have

$$\ddot{\bar{c}}_\rho = -\tilde{f}(2\tilde{g}^{0i}\partial_i\dot{\bar{c}}_\rho + \tilde{g}^{ij}\partial_i\partial_j\bar{c}_\rho). \tag{47}$$

Together with the first equality in (46), this equation makes it possible to show the second equality, i.e., $[\phi, \ddot{\bar{c}}'_\rho] = 0$. And the last equality in (46) can be obtained from the first and second equalities. Then, taking the GCT BRST transformation of the first equality in (46), we have

$$\begin{aligned} 0 &= \{-c^\alpha\partial_\alpha\phi, \dot{\bar{c}}'_\rho\} + [\phi, i\partial_0(b'_\rho + ic^{\alpha\prime}\partial_\alpha\bar{c}'_\rho)] \\ &= -\{c^\alpha, \dot{\bar{c}}'_\rho\}\partial_\alpha\phi + i[\phi, \dot{b}'_\rho], \end{aligned} \tag{48}$$

where Eq. (46) was used again. Using Eqs. (35), (36) and (45), we are able to obtain

$$[\dot{\phi}, b'_\rho] = -i\tilde{f}\phi^{-2}\partial_\rho\phi\delta^3. \tag{49}$$

It turns out that the CCRs, $[\pi_{c\mu}, \pi_g^{\alpha 0\prime}] = [\pi_{\bar{c}}^\mu, \pi_g^{\alpha 0\prime}] = 0$ give rise to

$$\begin{aligned} &[\dot{\bar{c}}_\mu, b'_\rho] = -i\tilde{f}\phi^{-2}\partial_\rho\bar{c}_\mu\delta^3, \\ &[\dot{c}^\mu, b'_\rho] = -i\tilde{f}\phi^{-2}\partial_\rho c^\mu\delta^3. \end{aligned} \tag{50}$$

Similarly, the CCRs, $[\pi_c, \pi_g^{\alpha 0\prime}] = [\pi_{\bar{c}}, \pi_g^{\alpha 0\prime}] = 0$ give us

$$[\dot{\bar{c}}, b'_\rho] = -i\tilde{f}\phi^{-2}\partial_\rho\bar{c}\delta^3, \qquad [\dot{c}, b'_\rho] = -i\tilde{f}\phi^{-2}\partial_\rho c\delta^3. \tag{51}$$

In order to evaluate $[\dot{B}, b'_\rho]$, we start with the first equality in Eq. (51), and take its BRST transformation for the Weyl transformation, which immediately leads to the equation:

$$[\dot{B}, b'_\rho] = -i\tilde{f}\phi^{-2}\partial_\rho B\delta^3. \tag{52}$$

Furthermore, the ETCR, $[\dot{g}_{\mu\nu}, b'_\rho]$ (or equivalently, $[g_{\mu\nu}, \dot{b}'_\rho]$) is derived in Appendix B, whose result is written out as

$$\begin{aligned} [\dot{g}_{\mu\nu}, b'_\rho] = -i\Big\{&\tilde{f}\phi^{-2}(\partial_\rho g_{\mu\nu} + \delta^0_\mu\dot{g}_{\rho\nu} + \delta^0_\nu\dot{g}_{\rho\mu})\delta^3 \\ &+[(\delta^k_\mu - 2\delta^0_\mu\tilde{f}\tilde{g}^{0k})g_{\rho\nu} + (\mu \leftrightarrow \nu)] \\ &\times\partial_k(\tilde{f}\phi^{-2}\delta^3)\Big\}. \end{aligned} \tag{53}$$

Similarly, we can obtain that

$$\begin{aligned} &[g_{\mu\nu}, \dot{b}'_\rho] = i\Big\{[\tilde{f}\phi^{-2}\partial_\rho g_{\mu\nu} - \partial_0(\tilde{f}\phi^{-2}) \\ &\times(\delta^0_\mu g_{\rho\nu} + \delta^0_\nu g_{\rho\mu})]\delta^3 \\ &+[(\delta^k_\mu - 2\delta^0_\mu\tilde{f}\tilde{g}^{0k})g_{\rho\nu} + (\mu \leftrightarrow \nu)]\partial_k(\tilde{f}\phi^{-2}\delta^3)\Big\}, \\ &\times[\tilde{g}^{\mu\nu}, \dot{b}'_\rho] = i\Big\{[\tilde{f}\phi^{-2}\partial_\rho\tilde{g}^{\mu\nu} + \partial_0(\tilde{f}\phi^{-2})(\tilde{g}^{0\mu}\delta^\nu_\rho \\ &+\tilde{g}^{0\nu}\delta^\mu_\rho - \tilde{g}^{\mu\nu}\delta^0_\rho)]\delta^3 \\ &-[(\tilde{g}^{\mu k} - 2\tilde{g}^{0\mu}\tilde{f}\tilde{g}^{0k})\delta^\nu_\rho - \frac{1}{2}\tilde{g}^{\mu\nu}(\delta^k_\rho - 2\delta^0_\rho\tilde{f}\tilde{g}^{0k}) \\ &+(\mu \leftrightarrow \nu)]\partial_k(\tilde{f}\phi^{-2}\delta^3)\Big\}. \end{aligned} \tag{54}$$

In particular, the last equality will be useful in computing an algebra of an extended conformal algebra later.

At this stage, we would like to prove very important ETCRs:

$$\begin{aligned} &[b_\mu, b'_\nu] = 0, \\ &[\dot{b}_\mu, b'_\nu] = -i\tilde{f}\phi^{-2}(\partial_\mu b_\nu + \partial_\nu b_\mu)\delta^3. \end{aligned} \tag{55}$$

The first equality can be easily shown by taking the GCT BRST transformation of the ETCR, $[b_\mu, \bar{c}'_\nu] = 0$. It is a proof of the second equality that we have to rely on the Einstein's equation [19]. In the formalism under consideration, we have not specified the classical action, so we cannot utilize the Einstein's equation to prove this equality. However, we have found that instead of using the Einstein's equation, we can prove the second equality in (55) by appealing to the BRST transformation whose proof is given in Appendix C.

To close this section, we should comment on the ETCRs involving the canonical variable B since the B field is essentially the generator of a global scale transformation. As seen in Eq. (33), $\dot{\phi}$ is the canonical conjugate momentum of the B field, and the other ETCRs, which include $\dot{B}$, can be calculated from the CCRs and the Weyl BRST transformation. For instance, the CCRs, $[B, \pi'_{c\mu}] = [B, \pi_{\bar{c}}^{\mu\prime}] = [B, \pi'_c] = [B, \pi'_{\bar{c}}] = 0$ give us the ETCRs:

$$[\dot{B}, c'_\mu] = [\dot{B}, \bar{c}^{\mu\prime}] = [\dot{B}, c'] = [\dot{B}, \bar{c}'] = 0, \tag{56}$$

where the formula (36) was used. Moreover, taking the Weyl BRST transformation of the last equality in (56) produces

$$[\dot{B}, B'] = 0. \tag{57}$$

Similarly, taking the Weyl BRST transformation of $[\dot{\bar{c}}, g'_{\mu\nu}] = 0$, which can be obtained from the CCR, $[\pi_c, g'_{\mu\nu}] = 0$, gives rise to the ETCR:

$$[\dot{B}, g'_{\mu\nu}] = 2i\tilde{f}\phi^{-2}g_{\mu\nu}\delta^3. \tag{58}$$

It is worthwhile to emphasize that via the CCRs, field equations and BRST transformations, we can calculate all the ECTRs except for the ones relevant to the metric and its derivatives such as $[\dot{g}_{\mu\nu}, g'_{\rho\sigma}]$ etc.

4 Poincaré-like $IOSp(10|10)$ global symmetry

As mentioned in Sect. 2, a set of fields (including the spacetime coordinates x^μ) $X^M \equiv \{x^\mu, b_\mu, \sigma, B, c^\mu, \bar{c}_\mu, c, \bar{c}\}$ obeys a very simple d'Alembert-like equation:

$$g^{\mu\nu}\partial_\mu\partial_\nu X^M = 0. \tag{59}$$

This equation holds if and only if we adopt the extended de Donder gauge and the scalar gauge as gauge-fixing conditions for the GCT and the Weyl transformation, respectively. The existence of this simple equation suggests that there could be many of conserved currents as defined in Eq. (27). In this section, we shall show explicitly that there exist such currents and consequently we have a Poincaré-like $IOSp(10|10)$ global symmetry.

Let us start with the Lagrangian (17), which can be cast to the form:

$$\mathcal{L}_q = \mathcal{L}_c - \frac{1}{2}\tilde{g}^{\mu\nu}\phi^2 \hat{E}_{\mu\nu}. \tag{60}$$

We can further rewrite it into the form

$$\begin{aligned}\mathcal{L}_q &= \mathcal{L}_c - \frac{1}{2}\tilde{g}^{\mu\nu}\phi^2 \eta_{NM}\partial_\mu X^M \partial_\nu X^N \\ &= \mathcal{L}_c - \frac{1}{2}\tilde{g}^{\mu\nu}\phi^2 \partial_\mu X^M \tilde{\eta}_{MN}\partial_\nu X^N,\end{aligned} \tag{61}$$

where we have introduced an $IOSp(10|10)$ metric $\eta_{NM} = \eta^T_{MN} \equiv \tilde{\eta}_{MN}$ defined as [21]

$$\eta_{NM} = \tilde{\eta}_{MN} = \left(\begin{array}{cc|cc|cc|c} & \delta_\mu^\nu & & & & & \\ \delta_\nu^\mu & & & & & & \\ \hline & & 0 & -1 & & & \\ & & -1 & 0 & & & \\ \hline & & & & & -i\delta_\mu^\nu & \\ & & & & i\delta_\nu^\mu & & \\ \hline & & & & & & -i \\ & & & & & & i \end{array}\right). \tag{62}$$

Let us note that this $IOSp(10|10)$ metric η_{NM} has the symmetry property such that

$$\eta_{MN} = (-)^{|M|\cdot|N|}\eta_{NM} = (-)^{|M|}\eta_{NM} = (-)^{|N|}\eta_{NM}, \tag{63}$$

where the statistics index $|M|$ is 0 or 1 when X^M is Grassmann-even or Grassmann-odd, respectively. This property comes from the fact that η_{MN} is 'diagonal' in the sense that its off-diagonal, Grassmann-even and Grassmann-odd, and vice versa, matrix elements vanish, i.e., $\eta_{MN} = 0$ when $|M| \neq |N|$, thereby being $|M| = |N| = |M|\cdot|N|$ in front of η_{MN} [21].

Now we would like to show that the quantum Lagrangian (61) is invariant under a Poincaré-like $IOSp(10|10)$ global transformation. First, let us focus on the infinitesimal OSp rotation, which is defined by

$$\delta X^M = \eta^{ML}\varepsilon_{LN}X^N \equiv \varepsilon^M{}_N X^N, \tag{64}$$

where η^{MN} is the inverse matrix of η_{MN},[7] and the infinitesimal parameter ε_{MN} has the following properties:

$$\begin{aligned}&\varepsilon_{MN} = (-)^{1+|M|\cdot|N|}\varepsilon_{NM}, \\ &\varepsilon_{MN}X^L = (-)^{|L|(|M|+|N|)}X^L\varepsilon_{MN}.\end{aligned} \tag{65}$$

Moreover, in order to find the conserved current, we assume that the infinitesimal parameter ε_{MN} depends on the spacetime coordinates x^μ, i.e., $\varepsilon_{MN} = \varepsilon_{MN}(x^\mu)$.

[7] Note that η_{MN} is a usual c-number quantity and satisfies the relation, $\eta_{MN} = \eta^{MN}$.

Before delving into the invariance of the quantum Lagrangian (61), we should notice that under the infinitesimal OSp rotation (64), the dilaton $\sigma(x)$, which is defined as $\phi = e^\sigma$, transforms as

$$\delta\sigma = \eta^{\sigma L}\varepsilon_{LN}X^N = -\varepsilon_{BN}X^N, \tag{66}$$

where we have used (62) and $\eta_{MN} = \eta^{MN}$. As for the scalar field $\phi(x)$, this transformation for the dilaton amounts to a Weyl transformation

$$\phi \to \phi' = e^{\epsilon(x)}\phi, \tag{67}$$

where the infinitesimal parameter is defined as $\epsilon(x) = -\varepsilon_{BN}X^N$.

To make the classical Lagrangian $\mathcal{L}_c$ and the compound metric $\tilde{g}^{\mu\nu}\phi^2$ be invariant under the OSp rotation (64), the transformation (67) should accompany a Weyl transformation for the metric tensor field:

$$g_{\mu\nu} \to g'_{\mu\nu} = e^{-2\epsilon(x)}g_{\mu\nu}. \tag{68}$$

To put it differently, it is necessary to perform the Weyl transformation (68) for the metric when we perform the OSp rotation (64).

With the proviso that the Weyl transformation (68) is made, we can prove an invariance of the quantum Lagrangian (61) under the OSp rotation (64), which is given by

$$\delta\mathcal{L}_q = -\tilde{g}^{\mu\nu}\phi^2(\partial_\mu\varepsilon_{NM}X^M\partial_\nu X^N + \varepsilon_{NM}\partial_\mu X^M\partial_\nu X^N), \tag{69}$$

where we have used the fact that both the classical Lagrangian $\mathcal{L}_c$ and the metric $\tilde{g}^{\mu\nu}\phi^2$ are invariant under the Weyl transformation. It is easy to prove that the second term on the RHS vanishes owing to the first property in Eq. (65). Thus, $\mathcal{L}_q$ is invariant under the infinitesimal OSp rotation. The conserved current is then calculated as

$$\begin{aligned}\delta\mathcal{L}_q &= -\tilde{g}^{\mu\nu}\phi^2\partial_\mu\varepsilon_{NM}X^M\partial_\nu X^N \\ &= -\frac{1}{2}\tilde{g}^{\mu\nu}\phi^2\partial_\mu\varepsilon_{NM}\left[X^M\partial_\nu X^N - (-)^{|M|\cdot|N|}X^N\partial_\nu X^M\right] \\ &= -\frac{1}{2}\tilde{g}^{\mu\nu}\phi^2\partial_\mu\varepsilon_{NM}\left(X^M\partial_\nu X^N - \partial_\nu X^M X^N\right) \\ &= -\frac{1}{2}\tilde{g}^{\mu\nu}\phi^2\partial_\mu\varepsilon_{NM}X^M\overset{\leftrightarrow}{\partial}_\nu X^N \\ &\equiv -\frac{1}{2}\partial_\mu\varepsilon_{NM}\mathcal{M}^{\mu MN},\end{aligned} \tag{70}$$

from which the conserved current $\mathcal{M}^{\mu MN}$ for the OSp rotation takes the form:

$$\mathcal{M}^{\mu MN} = \tilde{g}^{\mu\nu}\phi^2 X^M\overset{\leftrightarrow}{\partial}_\nu X^N. \tag{71}$$

In a similar way, we can derive the conserved current for the infinitesimal translation

$$\delta X^M = \varepsilon^M, \tag{72}$$

where ε^M is the infinitesimal parameter and assume that it is a local one for deriving the corresponding conserved current. Indeed, we can show that $\mathcal{L}_q$ is invariant under the infinitesimal translation as follows

$$\begin{aligned}\delta\mathcal{L}_q &= -\tilde{g}^{\mu\nu}\phi^2\eta_{NM}\partial_\mu\varepsilon^M\partial_\nu X^N\\ &= -\tilde{g}^{\mu\nu}\phi^2\partial_\mu\varepsilon_N\partial_\nu X^N\\ &\equiv -\partial_\mu\varepsilon_M\mathcal{P}^{\mu M},\end{aligned} \tag{73}$$

which implies that the conserved current $\mathcal{P}^{\mu M}$ for the translation reads

$$\mathcal{P}^{\mu M} = \tilde{g}^{\mu\nu}\phi^2\partial_\nu X^M = \tilde{g}^{\mu\nu}\phi^2\left(1\overleftrightarrow{\partial}_\nu X^M\right). \tag{74}$$

In this case as well, we need to perform an appropriate Weyl transformation for the metric since the dilaton transforms as $\delta\sigma = \varepsilon^\sigma(x)$ in Eq. (72).

An important remark is relevant to the expression of the conserved currents (71) and (74). To make the quantum Lagrangian $\mathcal{L}_q$ be invariant under the $IOSp(10|10)$ transformation, it is necessary to perform a Weyl transformation for the metric. Then, there could be a possibility that the expression of the currents might be modified because of this associated Weyl transformation. The good news is that as shown in Refs. [22–27], the current for the Weyl transformation identically vanishes since the Weyl transformation does not involve the derivative of the transformation parameter as in $\delta g_{\mu\nu} = 2\Lambda g_{\mu\nu}$ unlike the conventional gauge transformation as in $\delta A_\mu = \partial_\mu\Lambda$. Thus, although we perform the Weyl transformation, the conserved currents (71) and (74) remain unchanged. From the conserved currents (71) and (74), the corresponding conserved charges become

$$\begin{aligned}M^{MN} &\equiv \int d^3x\,\mathcal{M}^{0MN} = \int d^3x\,\tilde{g}^{0\nu}\phi^2 X^M\overleftrightarrow{\partial}_\nu X^N,\\ P^M &\equiv \int d^3x\,\mathcal{P}^{0M} = \int d^3x\,\tilde{g}^{0\nu}\phi^2\partial_\nu X^M.\end{aligned} \tag{75}$$

It then turns out that using various ETCRs obtained so far, the $IOSp(10|10)$ generators $\{M^{MN}, P^M\}$ generate an $IOSp(10|10)$ algebra

$$\begin{aligned}[P^M, P^N\} &= 0,\\ [M^{MN}, P^R\} &= i\left[P^M\tilde{\eta}^{NR} - (-)^{|N||R|}P^N\tilde{\eta}^{MR}\right],\\ [M^{MN}, M^{RS}\} &= i\left[M^{MS}\tilde{\eta}^{NR} - (-)^{|N||R|}M^{MR}\tilde{\eta}^{NS}\right.\\ &\quad -(-)^{|N||R|}M^{NS}\tilde{\eta}^{MR}\\ &\quad \left.+(-)^{|M||R|+|N||S|}M^{NR}\tilde{\eta}^{MS}\right],\end{aligned} \tag{76}$$

where the graded bracket is defined as $[A, B\} = AB - (-)^{|A||B|}BA$.

Actually, it is a little tedious to derive the $IOSp(10|10)$ algebra (76) from the ETCRs in a direct manner, so we present a different and more concise derivation here. The key observation is that as mentioned in the proof of the $IOSp(10|10)$ symmetry of the quantum Lagrangian $\mathcal{L}_q$, in order to make the classical Lagrangian $\mathcal{L}_c$ and the composite metric $\tilde{g}^{\mu\nu}\phi^2$ be invariant under the $IOSp(10|10)$ transformation, we simultaneously need to perform an associated Weyl transformation for the metric. But, then this Weyl transformation makes no effect on the conserved currents. This fact implies that as long as the $IOSp(10|10)$ charges and their algebra are concerned, one does not have to care about such Weyl invariant quantities and can ignore any contribution from them.

Under such a situation, one can regard X^M in the Lagrangian (61) as only a canonical variable and introduce its corresponding canonical conjugate momentum π_M as

$$\pi_M \equiv \frac{\partial\mathcal{L}_q}{\partial\dot{X}^M} = -\tilde{g}^{0\mu}\phi^2\eta_{MN}\partial_\mu X^N, \tag{77}$$

where the differentiation is taken from the right. The graded CCR, $[X^M, \pi'_N\} = i\delta^M_N\delta^3$ leads to

$$[X^M, \dot{X}^{N\prime}\} = -[\dot{X}^M, X^{N\prime}\} = -i\tilde{f}\phi^{-2}\tilde{\eta}^{MN}\delta^3. \tag{78}$$

From this CCR, it is easy to obtain the following algebra:

$$\begin{aligned}[M^{MN}, X^R\} &= i\left(X^M\tilde{\eta}^{NR} - (-)^{|N||R|}X^N\tilde{\eta}^{MR}\right),\\ [M^{MN}, \partial_\nu X^R\} &= i\left(\partial_\nu X^M\tilde{\eta}^{NR} - (-)^{|N||R|}\partial_\nu X^N\tilde{\eta}^{MR}\right),\\ [P^M, X^R\} &= i\tilde{\eta}^{MR}, \qquad [P^M, \partial_\nu X^R\} = 0.\end{aligned} \tag{79}$$

Then, on the basis of this algebra, it is straightforward to show the $IOSp(10|10)$ algebra (76). For instance, $[M^{MN}, M^{RS}\}$ can be calculated as follows:

$$\begin{aligned}[M^{MN}, M^{RS}\} &= [M^{MN}, \int d^3x\,\tilde{g}^{0\nu}\phi^2 X^R\overleftrightarrow{\partial}_\nu X^S\}\\ &= \int d^3x\,\tilde{g}^{0\nu}\phi^2[M^{MN}, X^R\partial_\nu X^S - \partial_\nu X^R X^S\}\\ &= \int d^3x\,\tilde{g}^{0\nu}\phi^2\Big([M^{MN}, X^R\}\partial_\nu X^S\\ &\quad +(-)^{(|M|+|N|)|R|}X^R[M^{MN}, \partial_\nu X^S\}\\ &\quad -[M^{MN}, \partial_\nu X^R\}X^S - (-)^{(|M|+|N|)|R|}\partial_\nu\\ &\quad X^R[M^{MN}, X^S\}\Big)\\ &= i\int d^3x\,\tilde{g}^{0\nu}\phi^2\Big[(X^M\tilde{\eta}^{NR} - (-)^{|N||R|}\\ &\quad X^N\tilde{\eta}^{MR})\partial_\nu X^S\\ &\quad +(-)^{(|M|+|N|)|R|}X^R(\partial_\nu X^M\tilde{\eta}^{NS}\\ &\quad -(-)^{|N||S|}\partial_\nu X^N\tilde{\eta}^{MS})\\ &\quad -(\partial_\nu X^M\tilde{\eta}^{NR} - (-)^{|N||R|}\partial_\nu X^N\tilde{\eta}^{MR})X^S\\ &\quad -(-)^{(|M|+|N|)|R|}\partial_\nu X^R(X^M\tilde{\eta}^{NS}\\ &\quad -(-)^{|N||S|}X^N\tilde{\eta}^{MS})\Big]\\ &= i\int d^3x\,\tilde{g}^{0\nu}\phi^2\Big[X^M\overleftrightarrow{\partial}_\nu X^S\tilde{\eta}^{NR}\\ &\quad +(-)^{(|M|+|N|)|R|}X^R\overleftrightarrow{\partial}_\nu X^M\tilde{\eta}^{NS}\end{aligned}$$

$$
\begin{aligned}
&-(-)^{|N||R|}X^N \overleftrightarrow{\partial}_\nu X^S \tilde{\eta}^{MR} \\
&-(-)^{(|M|+|N|)|R|+|N||S|}X^R \overleftrightarrow{\partial}_\nu X^N \tilde{\eta}^{MS}\Big] \\
= i\Big[&M^{MS}\tilde{\eta}^{NR} + (-)^{(|M|+|N|)|R|}M^{RM}\tilde{\eta}^{NS} \\
&-(-)^{|N||R|}M^{NS}\tilde{\eta}^{MR} \\
&-(-)^{(|M|+|N|)|R|+|N||S|}M^{RN}\tilde{\eta}^{MS}\Big]. \\
= i\Big[&M^{MS}\tilde{\eta}^{NR} - (-)^{|N||R|}M^{MR}\tilde{\eta}^{NS} \\
&-(-)^{|N||R|}M^{NS}\tilde{\eta}^{MR} \\
&+(-)^{|M||R|+|N||S|}M^{NR}\tilde{\eta}^{MS}\Big]. \qquad (80)
\end{aligned}
$$

At the last equality, we have used the relation

$$M^{MN} = -(-)^{|M||N|}M^{NM}. \qquad (81)$$

As a final remark, it is worthwhile to point out that all the global symmetries existing in the present theory are expressed in terms of the generators of the Poincaré-like $IOSp(10|10)$ global symmetry. For instance, the BRST charges for the GCT and Weyl transformation are respectively described as

$$
\begin{aligned}
Q_B &\equiv M(b_\rho, c^\rho) = \int d^3x\, \tilde{g}^{0\nu}\phi^2 b_\rho \overleftrightarrow{\partial}_\nu c^\rho, \\
\bar{Q}_B &\equiv M(B, c) = \int d^3x\, \tilde{g}^{0\nu}\phi^2 B \overleftrightarrow{\partial}_\nu c. \qquad (82)
\end{aligned}
$$

5 Conformal symmetry

In this section, we wish to explain that the Poincaré-like $IOSp(10|10)$ global symmetry, which was established in the previous section, includes conformal symmetry as a subgroup when the background is fixed to be a flat Minkowski space-time.

Before doing that, let us review conformal symmetry briefly. Conformal transformation can be defined as the general coordinate transformation which can be undone by a Weyl transformation when the space-time metric is the flat Minkowski one [5,6]. With this definition, the conformal transformation is described by the equation

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = 2\Lambda(x)\eta_{\mu\nu}, \qquad (83)$$

where $\epsilon_\mu(x)$ and $\Lambda(x)$ are the infinitesimal transformation parameters of the GCT and the Weyl transformation, respectively. Taking the trace of Eq. (83) enables us to determine $\Lambda(x)$ to be

$$\Lambda = \frac{1}{4}\partial^\rho \epsilon_\rho. \qquad (84)$$

Inserting this Λ to Eq. (83) yields

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{1}{2}\partial^\rho \epsilon_\rho \eta_{\mu\nu}, \qquad (85)$$

which is often called the "conformal Killing equation" in the Minkowski space-time. It is well-known that a general solution to the conformal Killing equation reads

$$\epsilon^\mu = a^\mu + \omega^{\mu\nu}x_\nu + \lambda x^\mu + k^\mu x^2 - 2x^\mu k_\rho x^\rho, \qquad (86)$$

where a^μ, $\omega^{\mu\nu} = -\omega^{\nu\mu}$, λ and k^μ are all constant parameters and they correspond to the translation, the Lorentz transformation, the dilatation and the special conformal transformation, respectively. Substituting Eq. (86) into Eq. (84) gives us

$$\Lambda = \lambda - 2k_\mu x^\mu. \qquad (87)$$

We are now ready to clarify the presence of conformal symmetry in the formalism at hand. To do so, let us note that although we have already fixed general coordinate invariance and Weyl one by the extended de Donder gauge condition (10) and the scalar gauge one (11), respectively, we are still left with its linearized, residual symmetries. In order to look for such residual symmetries, it is convenient to begin by the extended de Donder gauge (10) and take its variation under the GCT.[8] As seen in Eq. (2), the GCT is defined as

$$
\begin{aligned}
\delta g_{\mu\nu} &= -(\varepsilon^\alpha \partial_\alpha g_{\mu\nu} + \partial_\mu \varepsilon^\alpha g_{\alpha\nu} + \partial_\nu \varepsilon^\alpha g_{\mu\alpha}), \\
\delta \tilde{g}^{\mu\nu} &= -\partial_\alpha(\varepsilon^\alpha \tilde{g}^{\mu\nu}) + \partial_\alpha \varepsilon^\mu \tilde{g}^{\alpha\nu} + \partial_\alpha \varepsilon^\nu \tilde{g}^{\mu\alpha}, \\
\delta\phi &= -\varepsilon^\alpha \partial_\alpha \phi, \qquad (88)
\end{aligned}
$$

where ε^α is an infinitesimal transformation parameter since replacing ε^α with the ghost c^α produces the GCT BRST in (2). By using Eq. (88) and the extended de Donder gauge condition (10), the variation of the extended de Donder gauge condition (10) takes the form

$$\delta\left[\partial_\mu(\tilde{g}^{\mu\nu}\phi^2)\right] = \partial_\mu \partial_\alpha \varepsilon^\nu \tilde{g}^{\mu\alpha}\phi^2. \qquad (89)$$

Thus, a residual symmetry for the GCT exists when the following equation is obeyed:

$$\partial_\mu \partial_\alpha \varepsilon^\nu = 0. \qquad (90)$$

It is obvious that with a flat Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$, there is a zero-mode solution to Eq. (90) which is linear in x^μ

$$\varepsilon^\mu = A^\mu{}_\nu x^\nu + a^\mu, \qquad (91)$$

where $A^\mu{}_\nu$ and a^μ are constant quantities and here indices are raised and lowered with the flat Minkowski metric $\eta_{\mu\nu}$. This equation implies that we have a global $GL(4)$ symmetry and a translation symmetry associated with $A^\mu{}_\nu$ and a^μ, respectively, as residual symmetries. To put it differently, the extended de Donder gauge condition (10) leaves invariances

[8] A similar strategy has been adopted in different theories in Refs. [28–31].

under the GCT with the transformation parameter $\varepsilon^\mu(x)$ linear in the coordinates x^μ, which precisely correspond to the global $GL(4)$ and translational invariances.

Next, along the similar line of arguments, let us start with the scalar gauge condition (11) and take its variation under Weyl transformation, which is defined as

$$\bar{\delta} g_{\mu\nu} = 2\Lambda g_{\mu\nu}, \qquad \bar{\delta} \tilde{g}^{\mu\nu} = 2\Lambda \tilde{g}^{\mu\nu}, \\ \bar{\delta}\phi = -\Lambda\phi, \qquad \bar{\delta}\sigma = -\Lambda, \tag{92}$$

where Λ is an infinitesimal transformation parameter. Note that the replacement of Λ with the ghost c in Eq. (92) leads to the Weyl BRST transformation in Eq. (5) as required. The result reads

$$\bar{\delta}\left[\partial_\mu(\tilde{g}^{\mu\nu}\phi\partial_\nu\phi)\right] = \bar{\delta}\left[\partial_\mu(\tilde{g}^{\mu\nu}\phi^2\partial_\nu\sigma)\right] = -\tilde{g}^{\mu\nu}\phi^2\partial_\mu\partial_\nu\Lambda, \tag{93}$$

where the extended de Donder gauge condition (10) was used. We therefore find that a residual symmetry for the Weyl transformation exists when the equation

$$g^{\mu\nu}\partial_\mu\partial_\nu\Lambda = 0, \tag{94}$$

is satisfied. With the flat Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$, this equation has a linearized zero-mode solution

$$\Lambda = \lambda - 2k_\mu x^\mu, \tag{95}$$

where λ and k_μ are constants. Here the coefficient -2 in front of the last term is chosen for convenience. Comparing (95) with (87), we find that the transformations associated with the parameters λ and k_μ, respectively, correspond to dilatation and special conformal transformation in a flat Minkowski background.[9] In other words, in this case, finding the residual symmetries is equivalent to solving the conformal Killing equation.

We can also verify the invariance of the quantum Lagrangian under the residual symmetries more directly. As an illustration, let us take the residual symmetries (95) stemmed from the Weyl transformation into consideration. It is easy to check that the quantum Lagrangian (17) is invariant under the residual symmetries

$$\delta g_{\mu\nu} = 2(\lambda - 2k_\rho x^\rho) g_{\mu\nu}, \\ \delta\sigma = -(\lambda - 2k_\rho x^\rho), \qquad \delta b_\mu = 2k_\mu B, \tag{96}$$

where the other fields are unchanged. Then, generators corresponding to the transformation parameters λ and k_μ are respectively constructed out of those of the Poincaré-like $IOSp(10|10)$ symmetry as

$$D_0 \equiv -P^B \equiv -P(B) = -\int d^3x\, \tilde{g}^{0\nu}\phi^2\partial_\nu B, \\ K^\mu \equiv 2M^{x^\mu B} \equiv 2M^\mu(x, B) = 2\int d^3x\, \tilde{g}^{0\nu}\phi^2 x^\mu \overset{\leftrightarrow}{\partial}_\nu B. \tag{97}$$

For instance, we can explicitly show that these generators generate the former transformation in Eq. (96) as follows

$$\begin{aligned} \delta g_{\mu\nu}(x) &= [i(\lambda D_0 + k_\rho K^\rho), g_{\mu\nu}(x)] \\ &= i\int d^3y [\tilde{g}^{0\sigma}\phi^2(y)(-\lambda\partial_\sigma B(y) \\ &\quad + 2k_\rho y^\rho \overset{\leftrightarrow}{\partial}_\sigma B(y)), g_{\mu\nu}(x)] \\ &= i\int d^3y\, \tilde{g}^{00}\phi^2(y)(-\lambda + 2k_\rho y^\rho)[\dot{B}(y), g_{\mu\nu}(x)] \\ &= 2(\lambda - 2k_\rho x^\rho) g_{\mu\nu}(x), \end{aligned} \tag{98}$$

where the ETCR (58) was used in the last equality.

In addition to the generators D_0 and K^μ, one can also construct the translation generator P_μ and $GL(4)$ generator $G^\mu{}_\nu$ from those of the Poincaré-like $IOSp(10|10)$ symmetry as

$$\begin{aligned} P_\mu &\equiv P_{b_\mu} \equiv P_\mu(b) = \int d^3x\, \tilde{g}^{0\nu}\phi^2\partial_\nu b_\mu, \\ G^\mu{}_\nu &\equiv M^{x^\mu}{}_{b_\nu} - iM^{c^\mu}{}_{\bar{c}_\nu} \equiv M^\mu{}_\nu(x, b) - iM^\mu{}_\nu(c^\tau, \bar{c}_\lambda) \\ &= \int d^3x\, \tilde{g}^{0\lambda}\phi^2(x^\mu \overset{\leftrightarrow}{\partial}_\lambda b_\nu - ic^\mu \overset{\leftrightarrow}{\partial}_\lambda \bar{c}_\nu). \end{aligned} \tag{99}$$

At this point, let us elaborate a bit on the special conformal generator K^μ. It is well-known that under the special conformal transformation an arbitrary "*primary*" operators $\mathcal{O}_i(x)$ of conformal dimension Δ_i transforms as [5,6]

$$[iK_\mu, \mathcal{O}_i(x)] = (2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)\mathcal{O}_i(x) \\ +2\Delta_i x_\mu \mathcal{O}_i(x) - 2x^\nu (S_{\mu\nu})_i{}^j \mathcal{O}_j(x), \tag{100}$$

where $(S_{\mu\nu})_i{}^j$ is a spin matrix for the Lorentz transformation. In our formalism, it is easy to show that under the special conformal transformation the fundamental variables $\Phi \equiv \{g_{\mu\nu}, \phi, b_\mu, B, c_\mu, \bar{c}^\mu, c, \bar{c}\}$ transform as

$$\begin{aligned} &[iK^\rho, g_{\mu\nu}] = -4x^\rho g_{\mu\nu}, \qquad [iK^\rho, \phi] = 2x^\rho\phi, \\ &[iK^\rho, b_\mu] = 2\delta^\rho_\mu B, \\ &[iK^\rho, B] = [iK^\rho, c_\mu] = [iK^\rho, \bar{c}^\mu] = [iK^\rho, c] \\ &\quad = [iK^\rho, \bar{c}] = 0. \end{aligned} \tag{101}$$

Comparing Eq. (101) to Eq. (100), we can immediately notice three things: Firstly, $g_{\mu\nu}$ and ϕ are primary fields of conformal dimension -2 and 1, respectively, and B, c_μ, $\bar{c}^\mu$, c and $\bar{c}$ are all primary fields of conformal dimension 0 whereas b_μ is not a primary field. Secondly, there is no term involving the terms quadratic in x^μ in Eq. (101) since we have confined

[9] For clarity, we will call a global scale transformation in a flat Minkowski space-time "dilatation". Dilatation is usually interpreted as a subgroup of the general coordinate transformation in a such way that the space-time coordinates are transformed as $x^\mu \to \Omega x^\mu$ in the flat space-time where Ω is a constant scale factor, whereas the global scale transformation is a rescaling of all lengths by the same Ω by $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$. The two viewpoints are completely equivalent since all the lengths are defined via the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$.

ourselves to linearized symmetries in x^μ. Finally, there is not the term including the spin matrix $(S_{\mu\nu})_i{}^j$ in Eq. (101) since we do not consider spinor fields.

Now we would like to show that in our theory there is an extended conformal algebra in a flat Minkowski space-time where the Lorentz symmetry is replaced with the $GL(4)$ symmetry. For this aim, let us consider a set of generators, $\{P_\mu, G^\mu{}_\nu, K^\mu, D_0\}$, which has 25 independent generators and consists of a larger symmetry group than conformal symmetry constructed in terms of 15 independent generators, $\{P_\mu, M^\mu{}_\nu, K^\mu, D\}$.

Let us begin by making the generator D for dilatation. Recall that in conformal field theory in the four-dimensional Minkowski space-time, the dilatation generator obeys the following algebra for a local primary operator $O_i(x)$ of conformal dimension Δ_i [5,6]:

$$[iD, O_i(x)] = x^\mu \partial_\mu O_i(x) + \Delta_i O_i(x). \tag{102}$$

Since the scalar field $\phi(x)$ has conformal dimension 1 as shown in (101), it must satisfy the equation:

$$[iD, \phi(x)] = x^\mu \partial_\mu \phi(x) + \phi(x). \tag{103}$$

To be consistent with this equation, we shall make a generator D for the dilatation. From the definitions (97) and (99), we find

$$[iG^\mu{}_\mu, \phi(x)] = x^\mu \partial_\mu \phi(x), \qquad [iD_0, \phi(x)] = -\phi(x), \tag{104}$$

where Eqs. (33) and (49) were used. The following linear combination of $G^\mu{}_\nu$ and D_0 does the job:

$$D \equiv G^\mu{}_\mu - D_0. \tag{105}$$

As a consistency check, it is valuable to see how this operator D acts on the metric field whose result reads

$$\begin{aligned}[iD, g_{\sigma\tau}] &= [iG^\mu{}_\mu, g_{\sigma\tau}] - [iD_0, g_{\sigma\tau}] \\ &= (x^\mu \partial_\mu g_{\sigma\tau} + 2g_{\sigma\tau}) - 2g_{\sigma\tau} \\ &= x^\mu \partial_\mu g_{\sigma\tau},\end{aligned} \tag{106}$$

where we have used Eqs. (10), (39), (54) and (58). This equation implies that the metric field is a conformal field of conformal dimension 0 as expected in the conventional conformal field theory.

Next, let us calculate an algebra among the generators $\{P_\mu, G^\mu{}_\nu, K^\mu, D\}$. After some calculations, we find that the algebra closes and takes the form:

$$\begin{aligned}&[P_\mu, P_\nu] = 0, \quad [P_\mu, G^\rho{}_\sigma] = iP_\sigma \delta^\rho_\mu, \\ &[P_\mu, K^\nu] = -2i(G^\rho{}_\rho - D)\delta^\nu_\mu, \\ &[P_\mu, D] = iP_\mu, \quad [G^\mu{}_\nu, G^\rho{}_\sigma] = i(G^\mu{}_\sigma \delta^\rho_\nu - G^\rho{}_\nu \delta^\mu_\sigma), \\ &[G^\mu{}_\nu, K^\rho] = iK^\mu \delta^\rho_\nu, \quad [G^\mu{}_\nu, D] = [K^\mu, K^\nu] = 0, \\ &[K^\mu, D] = -iK^\mu, \quad [D, D] = 0.\end{aligned} \tag{107}$$

In this way, we have succeeded in showing that we have an extended conformal algebra in the formalism at hand. As mentioned above, the essential difference between the extended conformal symmetry and the conventional conformal symmetry is the presence of the $GL(4)$ symmetry instead of the $SO(1, 3)$ Lorentz symmetry, and this fact provides us with an important proof of exact masslessness of the graviton [32].

We mention two remarks, one of which is that to show this closure we do not have to use the gravitational ETCRs such as $[\dot{g}_{\mu\nu}, g'_{\rho\sigma}]$ which is missing in the formalism under consideration owing to an unspecified gravitational classical action. As the second remark, let us note that we have two different ways for deriving the algebra (107). The direct way is to use almost all the ETCRs obtained in this article. In addition to it, we need some nontrivial ETCRs which are not written out explicitly but can be obtained through the ETCRs appeared thus far. For instance, we sometimes have to make use of the ETCR:

$$\begin{aligned}[\dot{b}_\mu, \dot{b}'_\nu] = {}& i\tilde{f}\phi^{-2}(\delta^0_\nu \ddot{b}_\mu + 2\delta^k_\nu \partial_k \dot{b}_\mu)\delta^3 \\ & -2i\tilde{f}\tilde{g}^{0k}\partial_k[\tilde{f}\phi^{-2}(\partial_\mu b_\nu + \partial_\nu b_\mu)\delta^3] \\ & +i\tilde{f}^2\phi^{-2}(2\tilde{g}^{0k}\delta^l_\nu - \tilde{g}^{kl}\delta^0_\nu)\partial_k\partial_l b_\mu \delta^3 \\ & -i\partial_0[\tilde{f}\phi^{-2}(\partial_\mu b_\nu + \partial_\nu b_\mu)]\delta^3.\end{aligned} \tag{108}$$

This ETCR is obtained by taking time-derivative of the latter equation in (55) and then using the field equation (25). The other easier way of the derivation of (107) is to appeal to the $IOSp(10|10)$ algebra in Eq. (76). To illustrate the two different ways of derivation, we will explicitly evaluate the algebra $[P_\mu, K^\nu] = -2i(G^\rho{}_\rho - D)\delta^\nu_\mu$ in Appendix D.

To extract conformal algebra from the extended conformal algebra (107), it is necessary to introduce the "Lorentz generator", which can be constructed from the $GL(4)$ generator as

$$M_{\mu\nu} \equiv -\eta_{\mu\rho} G^\rho{}_\nu + \eta_{\nu\rho} G^\rho{}_\mu. \tag{109}$$

In terms of the generator $M_{\mu\nu}$, the algebra (107) can be cast to the form:

$$\begin{aligned}&[P_\mu, P_\nu] = 0, \quad [P_\mu, M_{\rho\sigma}] = i(P_\rho \eta_{\mu\sigma} - P_\sigma \eta_{\mu\rho}), \\ &[P_\mu, K^\nu] = -2i(G^\rho{}_\rho - D)\delta^\nu_\mu, \quad [P_\mu, D] = iP_\mu, \\ &[M_{\mu\nu}, M_{\rho\sigma}] = -i(M_{\mu\sigma}\eta_{\nu\rho} - M_{\nu\sigma}\eta_{\mu\rho} \\ &+M_{\rho\mu}\eta_{\sigma\nu} - M_{\rho\nu}\eta_{\sigma\mu}), \\ &[M_{\mu\nu}, K^\rho] = i(-K_\mu \delta^\rho_\nu + K_\nu \delta^\rho_\mu), \\ &[M_{\mu\nu}, D] = [K^\mu, K^\nu] = 0, \\ &[K^\mu, D] = -iK^\mu, \quad [D, D] = 0.\end{aligned} \tag{110}$$

where we have defined $K_\mu \equiv \eta_{\mu\nu} K^\nu$. It is worthwhile to point out that the the algebra (110) in quantum gravity coincides with the well-known conformal algebra except for the expression of $[P_\mu, K^\nu]$, which takes the form, $[P_\mu, K^\nu] =$

$-2i(\delta^\nu_\mu D + M_\mu{}^\nu)$ in the conformal algebra. However, this difference reflects only the difference of the definition of conformal dimension in both gravity and conformal field theory, for which the metric tensor field $g_{\mu\nu}$ has -2 in gravity as seen in Eq. (92) or Eq. (101) while it has 0 in conformal field theory as seen in Eq. (106). Moreover, the absence of the term $M_\mu{}^\nu$ on the RHS of $[P_\mu, K^\nu]$ in (110) can be understood from the fact that this term comes from the quadratic terms in x^μ in Eq. (100), which is now missing at present owing to our linear approximation as seen in Eq. (101). Accordingly, in this sense the algebra (110) is essentially equivalent to the conventional conformal algebra, and thus we have succeeded in obtaining the conformal symmetry within the framework of quantum gravity.

6 Spontaneous breakdown of symmetries

In the theory under consideration, there is a huge global symmetry called the Poincaré-like $IOSp(10|10)$ symmetry where we have 20 P^M generators and 200 M^{MN} generators so totally 220 symmetry generators. Then, it is valuable to investigate which symmetries are spontaneously broken or survive even in quantum regime. In particular, since our world has a built-in scale in it, a global scale symmetry or dilatation must be spontaneously broken if our theory makes sense.

In order to clarify the mechanism of symmetry breakdown, let us first postulate the existence of a unique vacuum $|0\rangle$, which is normalized to be the unity:

$$\langle 0|0\rangle = 1. \tag{111}$$

It is also natural to assume that the vacuum is translation invariant

$$P_\mu|0\rangle \equiv P_\mu(b)|0\rangle = 0. \tag{112}$$

Furthermore, it is supposed that the vacuum expectation values (VEVs) of the metric tensor $g_{\mu\nu}$ and the scalar field ϕ are respectively the Minkowski metric $\eta_{\mu\nu}$ and a non-zero constant $\phi_0 \neq 0$:

$$\langle 0|g_{\mu\nu}|0\rangle = \eta_{\mu\nu}, \qquad \langle 0|\phi|0\rangle = \phi_0. \tag{113}$$

With these assumptions, a straightforward calculation reveals the following nonvanishing VEVs for P^M

$$\begin{aligned}
&\langle 0|[iP^\mu(x), b_\rho]|0\rangle = -\delta^\mu_\rho,\\
&\langle 0|[iP(\sigma), B]|0\rangle = -\langle 0|[iP(B), \sigma]|0\rangle = 1,\\
&\langle 0|\{iP^\mu(c^\tau), \bar{c}_\rho\}|0\rangle = i\delta^\mu_\rho, \quad \langle 0|\{iP_\mu(\bar{c}_\tau), c^\rho\}|0\rangle = -i\delta^\rho_\mu,\\
&\langle 0|\{iP(c), \bar{c}\}|0\rangle = -\langle 0|\{iP(\bar{c}), c\}|0\rangle = i,
\end{aligned} \tag{114}$$

and the ones for M^{MN}

$$\begin{aligned}
&\langle 0|[iM^{\mu\nu}(x,x), \frac{1}{2}(\partial_\lambda b_\rho - \partial_\rho b_\lambda)]|0\rangle = -(\delta^\mu_\lambda\delta^\nu_\rho - \delta^\nu_\lambda\delta^\mu_\rho),\\
&\langle 0|[iM^\mu{}_\nu(x,b), g_{\sigma\tau}]|0\rangle = \delta^\mu_\sigma\eta_{\nu\tau} + \delta^\mu_\tau\eta_{\nu\sigma},\\
&\langle 0|[iM^\mu(x,\sigma), \partial_\lambda B]|0\rangle = \langle 0|[iM^\mu(x,B), \partial_\lambda\sigma]|0\rangle = \delta^\mu_\lambda,\\
&\langle 0|\{iM^{\mu\nu}(x,c^\tau), \partial_\lambda\bar{c}_\rho\}|0\rangle = i\delta^\mu_\lambda\delta^\nu_\rho,\\
&\langle 0|\{iM^\mu{}_\nu(x,\bar{c}_\tau), \partial_\lambda c^\rho\}|0\rangle = -i\delta^\mu_\lambda\delta^\rho_\nu,\\
&\langle 0|\{iM^\mu(x,c), \partial_\lambda\bar{c}\}|0\rangle = i\delta^\mu_\lambda,\\
&\langle 0|\{iM^\mu(x,\bar{c}), \partial_\lambda c\}|0\rangle = -i\delta^\mu_\lambda,\\
&\langle 0|[iM(\sigma,B), \sigma]|0\rangle = \sigma_0,\\
&\langle 0|\{iM^\mu(\sigma,c^\tau), \bar{c}_\rho\}|0\rangle = i\sigma_0\delta^\mu_\rho,\\
&\langle 0|\{iM_\mu(\sigma,\bar{c}_\tau), c^\rho\}|0\rangle = -i\sigma_0\delta^\rho_\mu,\\
&\langle 0|\{iM(\sigma,c), \bar{c}\}|0\rangle = -\langle 0|\{iM(\sigma,\bar{c}), c\}|0\rangle = i\sigma_0,
\end{aligned} \tag{115}$$

where $\langle 0|\sigma(x)|0\rangle \equiv \sigma_0$. These VEVs clearly show that the symmetries generated by the conserved charges

$$\begin{aligned}
&\{P^\mu(x), P(\sigma), P(B), P^\mu(c^\tau), P_\mu(\bar{c}_\tau), P(c),\\
&P(\bar{c}), M^{\mu\nu}(x,x), M^\mu{}_\nu(x,b),\\
&M^\mu(x,\sigma), M^\mu(x,B), M^{\mu\nu}(x,c^\tau), M^\mu{}_\nu(x,\bar{c}_\tau),\\
&M^\mu(x,c), M^\mu(x,\bar{c}),\\
&M(\sigma,B), M^\mu(\sigma,c^\tau), M_\mu(\sigma,\bar{c}_\tau), M(\sigma,c), M(\sigma,\bar{c})\}
\end{aligned}$$

are necessarily broken spontaneously, thereby $g_{\mu\nu}$, b_μ, σ, B, c^μ, $\bar{c}_\mu$, c and $\bar{c}$ acquiring massless Nambu–Goldstone modes. On the other hand, the remaining generators

$$\begin{aligned}
&\{M_{\mu\nu}(b,b), M_\mu(b,\sigma), M_\mu(b,B), M_\mu{}^\nu(b,c^\tau), M_{\mu\nu}(b,\bar{c}_\tau),\\
&M_\mu(b,c), M_\mu(b,\bar{c}),\\
&M^\mu(B,c^\tau), M_\mu(B,\bar{c}_\tau), M(B,c), M(B,\bar{c}),\\
&M^{\mu\nu}(c^\tau,c^\lambda), M^\mu{}_\nu(c^\tau,\bar{c}_\lambda),\\
&M^\mu(c^\tau,c), M^\mu(c^\tau,\bar{c}), M_{\mu\nu}(\bar{c}_\tau,\bar{c}_\lambda),\\
&M_\mu(\bar{c}_\tau,c), M_\mu(\bar{c}_\tau,\bar{c}),\\
&M(c,c), M(c,\bar{c}), M(\bar{c},\bar{c})\},
\end{aligned}$$

are found to be unbroken.

We are now ready to show that $GL(4)$, special conformal symmetry and dilatation are spontaneously broken down to the Poincaré symmetry. For this aim, it is more convenient to utilize the extended conformal algebra (107) rather than the Poincaré-like $IOSp(10|10)$ algebra (76) and the conformal algebra (110). It is then easy to see that the VEV of a commutator between the $GL(4)$ generator and the metric field reads

$$\langle 0|[iG^\mu{}_\nu, g_{\sigma\tau}]|0\rangle = \delta^\mu_\sigma\eta_{\nu\tau} + \delta^\mu_\tau\eta_{\nu\sigma}. \tag{116}$$

Thus, the Lorentz generator, which is defined in Eq. (109), has the vanishing VEV:

$$\langle 0|[iM_{\mu\nu}, g_{\sigma\tau}]|0\rangle = 0. \tag{117}$$

On the other hand, the symmetric part, which is defined as $\bar{M}_{\mu\nu} \equiv \eta_{\mu\rho} G^\rho{}_\nu + \eta_{\nu\rho} G^\rho{}_\mu$, has the non-vanishing VEV:

$$\langle 0|[i\bar{M}_{\mu\nu}, g_{\sigma\tau}]|0\rangle = 2(\eta_{\mu\sigma}\eta_{\nu\tau} + \eta_{\mu\tau}\eta_{\nu\sigma}). \tag{118}$$

Thus, the $GL(4)$ symmetry is spontaneously broken to the Lorentz symmetry where the corresponding Nambu–Goldstone boson with ten independent components is nothing but the massless graviton [32]. Here it is interesting that in a sector of the scalar field, the $GL(4)$ symmetry and the Lorentz symmetry as well, do not give rise to a symmetry breaking as can be checked in the commutators:

$$\begin{aligned} \langle 0|[iG^\mu{}_\nu, \phi]|0\rangle &= \langle 0|[iM_{\mu\nu}, \phi]|0\rangle = \langle 0|[i\bar{M}_{\mu\nu}, \phi]|0\rangle \\ &= 0. \end{aligned} \tag{119}$$

Next, we wish to clarify how the dilatation and special conformal symmetry are spontaneously broken and what the corresponding Nambu–Goldstone bosons are. As for the dilatation, it is not the gravitational field but the dilaton that triggers a spontaneous symmetry breaking since for the metric Eq. (106) provides us with

$$\langle 0|[iD, g_{\sigma\tau}]|0\rangle = 0, \tag{120}$$

while, for the dilaton, from Eq. (103) we have

$$\langle 0|[iD, \sigma]|0\rangle = 1. \tag{121}$$

This fact elucidates the spontaneous symmetry breakdown of the dilatation whose Nambu–Goldstone boson is just the massless dilaton $\sigma(x)$.

Regarding the special conformal symmetry, we find

$$\langle 0|[iK^\mu, \partial_\nu\sigma]|0\rangle = 2\delta^\mu_\nu. \tag{122}$$

This equation means that the special conformal symmetry is certainly broken spontaneously and its Nambu–Goldstone boson is the derivative of the dilaton. This interpretation can be also verified from the extended conformal algebra. In the algebra (107), we have a commutator between P_μ and K^ν:

$$[P_\mu, K^\nu] = -2i(G^\rho{}_\rho - D)\delta^\nu_\mu. \tag{123}$$

Let us consider the Jacobi identity:

$$[[P_\mu, K^\nu], \sigma] + [[K^\nu, \sigma], P_\mu] + [[\sigma, P_\mu], K^\nu] = 0. \tag{124}$$

Using the translational invariance of the vacuum in Eq. (112) and the equation

$$[P_\mu, \sigma] = -i\partial_\mu\sigma, \tag{125}$$

and taking the VEV of the Jacobi identity (124), we can obtain the VEV

$$\begin{aligned} \langle 0|[K^\nu, \partial_\mu\sigma]|0\rangle &= -2\delta^\nu_\mu\langle 0|[G^\rho{}_\rho - D, \sigma]|0\rangle \\ &= -2i\delta^\nu_\mu, \end{aligned} \tag{126}$$

which coincides with Eq. (122) as promised. In other words, the $GL(4)$ symmetry is spontaneously broken to the Poincaré symmetry whose Nambu–Goldstone boson is the graviton, the dilatation and the special conformal symmetry are at the same time spontaneously broken and the corresponding Nambu–Goldstone bosons are the dilaton and the derivative of the dilaton, respectively. It is not surprising that this pattern of the symmetry breaking for the dilatation and special conformal symmetry is the same as that of the conventional conformal field theory [33] since we share the same conformal algebra.

7 Conclusion

In this article, we have performed a manifestly covariant quantization and constructed a quantum theory of a general gravitational theory, which is invariant under both general coordinate transformation (GCT) and Weyl transformation, within the framework of the BRST formalism. In the present formulation, since we have started with such a general action, it is impossible to derive a concrete expression of the Einstein's equation as well as equal-time commutation relations (ETCRs) relevant to the metric tensor and its derivatives such as $[\dot{g}_{\mu\nu}, g'_{\rho\sigma}]$. The key point is, however, that without the Einstein's equation and those ETCRs one can derive the field equation for b_μ and the other nontrivial ETCRs, in particular, $[\dot{b}_\mu, b'_\nu]$, which have been so far derived from the Einstein's equation and the metric's ETCRs [19], by the help of two kinds of BRST transformations and field equations except for the Einstein's equation.

Although we do not fix the classical gravitational action and consider a general one, it is remarkable that we have a huge global symmetry, called a Poincaré-like $IOSp(10|10)$ global transformation with 220 symmetry generators, which is larger than an $ISOp(8|8)$ with 144 symmetry generators in case of general relativity [19], owing to the presence of the Weyl symmetry in our formulation. In a flat Minkowski background, the Poincaré-like $IOSp(10|10)$ symmetry naturally includes the extended conformal algebra where the usual Lorentz symmetry is extended to a larger $GL(4)$ symmetry. Then, it is obvious that this extended conformal algebra includes conformal algebra as a subgroup. In other words, in our formulation of quantum gravity, the conformal symmetry emerges from not the classical action but the gauge-fixing and FP ghost actions.

According to the Zumino theorem [7], the theories which are invariant under the GCT and Weyl transformation, have conformal invariance in the flat Minkowski background at the classical level. The present study partially supports a conjecture that the Zumino theorem could be valid even in quantum gravity. However, there is a loophole in our argument. In this article, we have assumed that a classical action does not involve more than first order derivatives of the metric and matter fields. If the classical theory includes the higher-derivative terms as in quadratic gravity, we have to rewrite the theory in order not to include more than first order derivatives by introducing suitable auxiliary fields when we perform the canonical quantization. Then, the introduction of the auxiliary fields sometimes gives rise to a new local symmetry such as the Stückelberg symmetry [11,34–38]. In such a situation, our argument is not valid. Thus, we can address a generalized Zumino theorem in quantum gravity as follows: The Zumino theorem is certainly valid in quantum gravity when a classical action does not involve more than first order derivatives.

As future works, we would like to list up two important issues. First, it is well-known that Weyl symmetry is explicitly broken by conformal anomaly at the quantum level [39]. However, since a scalar field is now at our disposal, it is not obvious that the Weyl symmetry is anomalous or not in our theory. Actually, in a Weyl invariant scalar-tensor gravity, there is no conformal anomaly [40]. It has been recently pointed out in Ref. [41] that there could be an alternative conformal anomaly which comes from the topological Gauss-Bonnet term. Moreover, it has been also shown that this new conformal anomaly does not appear in the Weyl geometry [41]. We wish to apply this study for the present theory by introducing a new gauge field associated with the Weyl symmetry and verify that our formalism is also free from the conformal anomaly. Incidentally, the canonical quantization formalism of the Weyl invariant scalar-tensor gravity in Weyl geometry has been already constructed in [10].

Another important issue is related to the non-renormalizability of general relativity. To remedy the problem, we are accustomed to adding quadratic terms in the Riemann curvatures and consider the Einstein-Hilbert term supplemented with the terms containing R^2 and $R_{\mu\nu}R^{\mu\nu}$, but then we meet another serious problem of non-unitarity. We conjecture that the non-unitarity problem in the higher-derivative gravities might be solved by a mechanism of ghost confinement [42] and then the Poincaré-like global symmetry would play a critical role, as in the confinement of quarks and gluons by the BRST symmetry in QCD [43]. We shall return these two issues in near future.

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Appendix

A $GL(4)$ invariance of the extended de Donder gauge condition

In this appendix, we present a proof of the global $GL(4)$ invariance of the extended de Donder gauge condition (10). To do that, let us first recall the $GL(4)$ transformation:

$$x'^\mu = A^\mu{}_\nu x^\nu, \tag{A.1}$$

where $A^\mu{}_\nu$ is a constant and invertible 4×4 matrix belonging to a $GL(4, R)$. From this definition, $A^\mu{}_\nu$ and its inverse $A^{-1\mu}{}_\nu$ can be expressed as

$$A^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}, \qquad A^{-1\mu}{}_\nu = \frac{\partial x^\mu}{\partial x'^\nu}. \tag{A.2}$$

Under the $GL(4)$ transformation, the metric and the scalar field transform as

$$g'_{\mu\nu}(x') = A^{-1\rho}{}_\mu A^{-1\sigma}{}_\nu g_{\rho\sigma}(x),$$
$$g'^{\mu\nu}(x') = A^\mu{}_\rho A^\nu{}_\sigma g^{\rho\sigma}(x),$$
$$\sqrt{-g'(x')} = (\det A)^{-1}\sqrt{-g(x)}, \qquad \phi'(x') = \phi(x). \tag{A.3}$$

Using the transformation law and $\partial'_\mu = A^{-1\nu}{}_\mu \partial_\nu$, we can show that

$$\partial'_\mu(\sqrt{-g'(x')}g'^{\mu\nu}(x')\phi'^2(x')) = (\det A)^{-1} A^\nu{}_\sigma \partial_\rho (\sqrt{-g(x)}g^{\rho\sigma}(x)\phi^2(x)). \tag{A.4}$$

Thus, the extended de Donder gauge condition (10) is invariant under the $GL(4)$ transformation in the sense that when the extended de Donder gauge condition is imposed in a frame it holds true in a $GL(4)$ transformed frame as well. In a similar manner, the $GL(4)$ invariance of the unitary gauge $\phi =$ constant and the Lorenz gauge $\nabla_\mu S^\mu = 0$ can be shown. Precisely speaking, the extended de Donder gauge condition is transformed as a vector density under the $GL(4)$ transformation as seen in Eq. (A.4).

Indeed, it is a gauge-fixing action that is exactly invariant under the $GL(4)$ transformation. To see this fact, let us consider a gauge-fixed action corresponding to the extended de

Donder gauge condition (10):

$$S_{GF} = -\int d^4x \sqrt{-g} g^{\mu\nu} \phi^2 \partial_\mu b_\nu. \tag{A.5}$$

Under the $GL(4)$ transformation, this action transforms as

$$\begin{aligned} S'_{GF} &= -\int d^4x' \sqrt{-g'} g'^{\mu\nu} \phi'^2 \partial'_\mu b'_\nu \\ &= -\int d^4x \det A (\det A)^{-1} A^\mu{}_\rho A^\nu{}_\sigma \\ &\quad \sqrt{-g} g^{\rho\sigma} \phi^2 A^{-1\tau}{}_\mu \partial_\tau (A^{-1\lambda}{}_\nu b_\lambda) \\ &= S_{GF}, \end{aligned} \tag{A.6}$$

where we have used

$$b'_\mu(x') = A^{-1\nu}{}_\mu b_\nu(x), \qquad d^4x' = \det A \, d^4x. \tag{A.7}$$

Hence, the gauge-fixed action is precisely invariant under the $GL(4)$ transformation.

B Proof of Eqs. (53) and (54)

In this Appendix, we derive Eqs. (53) and (54) by following the method developed previously in case of the de Donder gauge condition [8].

The translational invariance in general requires the validity of the following equation for a generic field $\Phi(x)$:

$$[\Phi(x), P_\rho] = i\partial_\rho \Phi(x), \tag{B.1}$$

where P_ρ is the generator of the translation which is now given by

$$P_\rho = \int d^3x \, \tilde{g}^{0\lambda} \phi^2 \partial_\lambda b_\rho. \tag{B.2}$$

Now let us consider the specific case $\Phi(x) = g_{\mu\nu}(x)$

$$\begin{aligned} [g_{\mu\nu}(x), P_\rho] &= [g_{\mu\nu}(x), \int d^3x' \, \tilde{g}^{0\lambda\prime} \phi^{2\prime} \partial_\lambda b'_\rho] \\ &= i\partial_\rho g_{\mu\nu}(x). \end{aligned} \tag{B.3}$$

Taking $x^0 = x'^0$ and using $[g_{\mu\nu}, \tilde{g}^{0\lambda\prime}\phi^{2\prime}] = 0$, we have

$$\int d^3x' \, \tilde{g}^{0\lambda}(x')\phi^2(x')[g_{\mu\nu}, \partial_\lambda b'_\rho] = i\partial_\rho g_{\mu\nu}(x). \tag{B.4}$$

Using the extended de Donder gauge condition (10) and Eq. (39), this equation can be rewritten as

$$\begin{aligned} &\int d^3x' \, \tilde{g}^{00}(x')\phi^2(x')[g_{\mu\nu}, \dot{b}'_\rho] \\ &= i\left[\partial_\rho g_{\mu\nu} + \partial_0(\tilde{g}^{00}\phi^2)\tilde{f}\phi^{-2}(\delta^0_\mu g_{\rho\nu} + \delta^0_\nu g_{\rho\mu})\right], \end{aligned} \tag{B.5}$$

which is easily solved for $[g_{\mu\nu}, \dot{b}'_\rho]$ to be

$$\begin{aligned} [g_{\mu\nu}, \dot{b}'_\rho] = i\tilde{f}\phi^{-2}\Big[\partial_\rho g_{\mu\nu} + \partial_0(\tilde{g}^{00}\phi^2)\tilde{f}\phi^{-2}(\delta^0_\mu g_{\rho\nu} \\ + \delta^0_\nu g_{\rho\mu})\Big]\delta^3 + F_{(\mu\nu)\rho}{}^k \partial_k(\tilde{f}\phi^{-2}\delta^3), \end{aligned} \tag{B.6}$$

where $F_{(\mu\nu)\rho}{}^k$ is an arbitrary function which is symmetric under the exchange of $\mu \leftrightarrow \nu$.

Next, to fix the function $F_{(\mu\nu)\rho}{}^k$, let us take account of the consistency with the extended de Donder gauge condition (10):

$$[\partial_\mu(\tilde{g}^{\mu\nu}\phi^2), b'_\rho] = 0. \tag{B.7}$$

An explicit calculation reveals that Eq. (B.7) leads to an equation for $F_{(\mu\nu)\rho}{}^k$:

$$\begin{aligned} \left(\tilde{g}^{0\alpha}g^{\nu\beta} - \frac{1}{2}\tilde{g}^{0\nu}g^{\alpha\beta}\right) F_{(\alpha\beta)\rho}{}^k = -i(\tilde{g}^{0k}\delta^\nu_\rho + \tilde{g}^{0\nu}\delta^k_\rho \\ -\tilde{g}^{k\nu}\delta^0_\rho). \end{aligned} \tag{B.8}$$

This equation has the unique solution given by

$$F_{(\mu\nu)\rho}{}^k = i[(\delta^k_\mu - 2\delta^0_\mu \tilde{f}\tilde{g}^{0k})g_{\rho\nu} + (\mu \leftrightarrow \nu)]. \tag{B.9}$$

We can therefore obtain

$$\begin{aligned} [g_{\mu\nu}, \dot{b}'_\rho] = i\Big\{&[\tilde{f}\phi^{-2}\partial_\rho g_{\mu\nu} - \partial_0(\tilde{f}\phi^{-2})(\delta^0_\mu g_{\rho\nu} + \delta^0_\nu g_{\rho\mu})]\delta^3 \\ &+[(\delta^k_\mu - 2\delta^0_\mu \tilde{f}\tilde{g}^{0k})g_{\rho\nu} \\ &+(\mu \leftrightarrow \nu)]\partial_k(\tilde{f}\phi^{-2}\delta^3)\Big\}, \end{aligned} \tag{B.10}$$

which is nothing but the former equation in Eq. (54). The latter equation in Eq. (54) can be easily obtained from Eq. (B.10). Finally, using Eqs. (36) and (39), we can arrive at another equation (53). It is of interest that Eq. (53) can be derived from only the translational invariance and the extended de Donder gauge condition without reference to the classical Lagrangian $\mathcal{L}_c$ which knows information of the dynamics of the gravitational field $g_{\mu\nu}$ and the scalar field ϕ.

C Proof of $[\dot{b}_\mu, b'_\nu] = -i\tilde{f}\phi^{-2}(\partial_\mu b_\nu + \partial_\nu b_\mu)\delta^3$

In this Appendix, we present a proof of the ETCR, $[\dot{b}_\mu, b'_\nu] = -i\tilde{f}\phi^{-2}(\partial_\mu b_\nu + \partial_\nu b_\mu)\delta^3$ on the basis of the BRST transformation without using the Einstein's equation. This calculation exhibits that the BRST transformation offers a very powerful method for deriving a nontrivial ETCR.

Let us begin by the ETCR in Eq. (50):

$$[\dot{\bar{c}}_\mu, b'_\nu] = -i\tilde{f}\phi^{-2}\partial_\nu \bar{c}_\mu \delta^3. \tag{C.1}$$

Then, taking its GCT BRST transformation yields

$$\begin{aligned} [i\dot{B}_\mu, b'_\nu] - \{\dot{\bar{c}}_\mu, -c^{\alpha\prime}\partial_\alpha b'_\nu\} = -i(\delta_B\tilde{f}\phi^{-2} \\ -2\tilde{f}\phi^{-3}\delta_B\phi)\partial_\nu\bar{c}_\mu\delta^3 \\ -i\tilde{f}\phi^{-2}\partial_\nu(iB_\mu)\delta^3. \end{aligned} \tag{C.2}$$

Using Eq. (3), the first term on the left-hand side (LHS) and the last term on the right-hand side (RHS) can be rewritten as

$$[i\dot{B}_\mu, b'_\nu] = i[\dot{b}_\mu, b'_\nu] - [\partial_0(c^\alpha \partial_\alpha \bar{c}_\mu), b'_\nu],$$
$$-i\tilde{f}\phi^{-2}\partial_\nu(iB_\mu)\delta^3 = f\phi^{-2}\partial_\nu b_\mu \delta^3 + i\tilde{f}\phi^{-2}\partial_\nu(c^\alpha \partial_\alpha \bar{c}_\mu)\delta^3. \tag{C.3}$$

Furthermore, we can calculate $\delta_B \tilde{f}$ to be

$$\delta_B \tilde{f} = -\tilde{f}^2 \delta_B \tilde{g}^{00} = \tilde{f}\partial_\rho c^\rho - \partial_\rho \tilde{f} c^\rho - 2\tilde{f}^2 \tilde{g}^{0\rho}\partial_\rho c^0. \tag{C.4}$$

Using these equations and $\delta_B \phi = -c^\rho \partial_\rho \phi$, Eq. (C.2) can be further cast to the form:

$$\begin{aligned}[\dot{b}_\mu, b'_\nu] = &-i[\partial_0(c^\alpha \partial_\alpha \bar{c}_\mu), b'_\nu] \\ &+i\{\dot{\bar{c}}_\mu, c^{\alpha\prime}\partial_\alpha b'_\nu\} - if\phi^{-2}\partial_\nu b_\mu \delta^3 \\ &+\tilde{f}\phi^{-2}\partial_\nu(c^\alpha \partial_\alpha \bar{c}_\mu)\delta^3 - \tilde{f}\phi^{-2}(\partial_\rho c^\rho \\ &-\tilde{f}^{-1}\partial_\rho \tilde{f} c^\rho - 2\tilde{f}\tilde{g}^{0\rho}\partial_\rho c^0 \\ &+2\phi^{-1}c^\rho \partial_\rho \phi)\partial_\nu \bar{c}_\mu \delta^3. \end{aligned} \tag{C.5}$$

The remaining work is to calculate two (anti)commutators on the RHS. It is straightforward to carry out such calculations by using various ETCRs obtained thus far. In particular, we make use of the following ETCRs

$$\begin{aligned}[\ddot{\bar{c}}_\mu, b'_\nu] = &-i\tilde{f}\phi^{-2}\delta^0_\nu \ddot{\bar{c}}_\mu \delta^3 \\ &-2i\tilde{f}[\phi^{-2}\delta^k_\nu \partial_k \dot{\bar{c}}_\mu \delta^3 - \tilde{g}^{0k}\partial_k(\tilde{f}\phi^{-2}\partial_\nu \bar{c}_\mu \delta^3)] \\ &-i\tilde{f}^2\phi^{-2}(2\tilde{g}^{0k}\delta^l_\nu - \tilde{g}^{kl}\delta^0_\nu)\partial_k \partial_l \bar{c}_\mu \delta^3, \end{aligned} \tag{C.6}$$

and

$$[\dot{\bar{c}}_\mu, \dot{b}'_\nu] = -[\ddot{\bar{c}}_\mu, b'_\nu] - i\partial_0(\tilde{f}\phi^{-2}\partial_\nu \bar{c}_\mu)\delta^3. \tag{C.7}$$

Eq. (C.6) can be obtained from Eqs. (40), (45), (47) and (50).

As a result, the two (anti)commutators on the RHS in Eq. (C.5) are given by

$$\begin{aligned} -i[\partial_0(c^\alpha \partial_\alpha \bar{c}_\mu), b'_\nu] = &-\tilde{f}\phi^{-2}(\partial_\nu c^\alpha \partial_\alpha \bar{c}_\mu \\ &+\dot{c}^0 \partial_\nu \bar{c}_\mu + \delta^0_\nu c^0 \ddot{\bar{c}}_\mu)\delta^3 \\ &-2\tilde{f}c^0[\phi^{-2}\delta^k_\nu \partial_k \dot{\bar{c}}_\mu \delta^3 - \tilde{g}^{0k}\partial_k(\tilde{f}\phi^{-2}\partial_\nu \bar{c}_\mu \delta^3)] \\ &-\tilde{f}^2\phi^{-2}c^0(2\tilde{g}^{0k}\delta^l_\nu - \tilde{g}^{kl}\delta^0_\nu)\partial_k \partial_l \bar{c}_\mu \delta^3 \\ &-c^k \partial_k(\tilde{f}\phi^{-2}\partial_\nu \bar{c}_\mu \delta^3), \end{aligned} \tag{C.8}$$

and

$$\begin{aligned} &i\{\dot{\bar{c}}_\mu, c^{\alpha\prime}\partial_\alpha b'_\nu\} = -i\tilde{f}\phi^{-2}\partial_\mu b_\nu \delta^3 + \tilde{f}\phi^{-2}\delta^0_\nu c^0 \ddot{\bar{c}}_\mu \delta^3 \\ &+2\tilde{f}[\phi^{-2}\delta^k_\nu c^0 \partial_k \dot{\bar{c}}_\mu \delta^3 - \tilde{g}^{0k}\partial_k(\tilde{f}\phi^{-2}c^0 \partial_\nu \bar{c}_\mu \delta^3)] \\ &+\tilde{f}^2\phi^{-2}(2\tilde{g}^{0k}\delta^l_\nu - \tilde{g}^{kl}\delta^0_\nu)c^0 \partial_k \partial_l \bar{c}_\mu \delta^3 \\ &-c^0 \partial_0(\tilde{f}\phi^{-2}\partial_\nu \bar{c}_\mu)\delta^3 + \tilde{f}\phi^{-2}\partial_k(c^k \delta^3)\partial_\nu \bar{c}_\mu. \end{aligned} \tag{C.9}$$

Substituting Eqs. (C.8) and (C.9) into Eq. (C.5), we can reach the desired equation:

$$[\dot{b}_\mu, b'_\nu] = -i\tilde{f}\phi^{-2}(\partial_\mu b_\nu + \partial_\nu b_\mu)\delta^3. \tag{C.10}$$

D Two derivations of $[P_\mu, K^\nu] = -2i(G^\rho{}_\rho - D)\delta^\nu_\mu$

For illustrative purposes, in this appendix we present two different derivations for one of conformal algebra, $[P_\mu, K^\nu] = -2i(G^\rho{}_\rho - D)\delta^\nu_\mu$, one of which relies on the $IOSp(10|10)$ algebra in Eq. (76). Another derivation is a direct calculation based on the ETCRs obtained in this article. We find that the former derivation method is much easier than that of the latter and this fact elucidates the power of the Poincaré-like $IOSp(10|10)$ algebra.

For the translation generator P_μ in (99) and the special conformal generator K^μ in (97), the $IOSp(10|10)$ algebra Eq. (76) provides us with

$$\begin{aligned}[P_\mu, K^\nu] &= -[K^\nu, P_\mu] = -2[M^\nu(x, B), P_\mu(b)] \\ &= -2[M^{x^\nu B}, P_{b_\mu}] \\ &= -2i[P^{x^\nu}\tilde{\eta}^B{}_{b_\mu} - (-)^{|B||b_\mu|}P^B \tilde{\eta}^{x^\nu}{}_{b_\mu}] = 2iP^B \delta^\nu_\mu \\ &= 2iP(B)\delta^\nu_\mu = -2iD_0 \delta^\nu_\mu = -2i(G^\rho{}_\rho - D)\delta^\nu_\mu, \end{aligned} \tag{D.1}$$

where we have used $\tilde{\eta}^B{}_{b_\mu} = 0$ and $\tilde{\eta}^{x^\nu}{}_{b_\mu} = \delta^\nu_\mu$ at the fifth equality. Note that this derivation is very transparent.

As an alternative derivation of this algebra, one can rely on the ETCRs obtained thus far. This derivation method is very direct but it is a bit complicated since we have to use the extended de Donder gauge condition (10), field equations and the integration by parts etc. repeatedly.

From Eqs. (97) and (99), we have

$$\begin{aligned}[P_\mu, K^\nu] &= \int d^3x d^3x' [\tilde{g}^{0\lambda}\phi^2 \partial_\lambda b_\mu, 2\tilde{g}^{0\rho\prime}\phi^{2\prime}x^{\nu\prime}\overleftrightarrow{\partial}_\rho B'] \\ &= 2\int d^3x d^3x' \Big(x^{\nu\prime}[\tilde{g}^{0\lambda}\phi^2 \partial_\lambda b_\mu, \tilde{g}^{0\rho\prime}\phi^{2\prime}\partial_\rho B'] \\ &\quad -[\tilde{g}^{0\lambda}\phi^2 \partial_\lambda b_\mu, \tilde{g}^{0\nu\prime}\phi^{2\prime}B']\Big) \\ &\equiv 2(I + J). \end{aligned} \tag{D.2}$$

It is straightforward to compute J whose final result takes the form

$$J = i\int d^3x\, \delta^0_\mu \tilde{g}^{\lambda\nu}\phi^2 \partial_\lambda B. \tag{D.3}$$

Here, as well as the extended de Donder gauge condition (10), we have used the following ETCRs:

$$\begin{aligned}[\dot{b}_\mu, \tilde{g}^{0\nu\prime}] &= -i\Big\{[\tilde{f}\phi^{-2}\partial_\mu \tilde{g}^{0\nu} + \partial_0(\tilde{f}\phi^{-2})\tilde{g}^{00}\delta^\nu_\mu]\delta^3 \\ &\quad +(\tilde{g}^{0k\prime}\delta^\nu_\mu - \tilde{g}^{\nu k\prime}\delta^0_\mu + \tilde{g}^{0\nu\prime}\delta^k_\mu)\partial'_k(\tilde{f}\phi^{-2}\delta^3)\Big\}, \\ [\dot{b}_\mu, \phi^{2\prime}] &= -2i\tilde{f}\phi^{-1}\partial_\mu \phi \delta^3, \\ [\dot{b}_\mu, B'] &= -i\tilde{f}\phi^{-2}\partial_\mu B \delta^3. \end{aligned} \tag{D.4}$$

Next, let us pay our attention to an evaluation of I, which can be divided into four parts:

$$\begin{aligned} I = & \int d^3x d^3x' x^{\nu\prime} \Big([\tilde{g}^{00} \phi^2 \dot{b}_\mu, \tilde{g}^{00\prime} \phi^{2\prime} \dot{B}'] \\ & + [\tilde{g}^{0i} \phi^2 \partial_i b_\mu, \tilde{g}^{00\prime} \phi^{2\prime} \dot{B}'] \\ & + [\tilde{g}^{00} \phi^2 \dot{b}_\mu, \tilde{g}^{0j\prime} \phi^{2\prime} \partial_j B'] + [\tilde{g}^{0i} \phi^2 \partial_i b_\mu, \tilde{g}^{0j\prime} \phi^{2\prime} \partial_j B'] \Big) \\ \equiv & I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{D.5}$$

The calculation of I_4 is easiest since it involves only one ETCR as follows:

$$\begin{aligned} I_4 = & \int d^3x d^3x' x^{\nu\prime} \tilde{g}^{0i} \phi^2 \phi^{2\prime} \partial_j B' [\partial_i b_\mu, \tilde{g}^{0j\prime}] \\ = & -i \int d^3x \delta^i_\mu x^\nu \partial_0 (\tilde{g}^{00} \phi^2) \partial_i B. \end{aligned} \tag{D.6}$$

In a similar manner, it is straightforward to carry out the calculations of I_2 and I_3, which are summarized as

$$\begin{aligned} I_2 = & i \int d^3x \tilde{g}^{0i} \phi^2 \delta^j_\mu (\delta^\nu_i \partial_j B + x^\nu \partial_i \partial_j B), \\ I_3 = & i \int d^3x \Big[x^\nu \phi^2 (\tilde{g}^{00} \ddot{B} \delta^0_\mu + \tilde{g}^{0i} \partial_i \partial_\mu B) - \delta^\nu_i \phi^2 (-\tilde{g}^{0i} \delta^j_\mu \\ & + \tilde{g}^{ij} \delta^0_\mu - \tilde{g}^{0j} \delta^i_\mu) \partial_j B \Big]. \end{aligned} \tag{D.7}$$

The evaluation of I_1 is more difficult than those of the other $I_i (i = 2, 3, 4)$, but is straightforward whose result is given by

$$I_1 = -i \int d^3x x^\nu \Big[\partial_\mu (\tilde{g}^{00} \phi^2 \dot{B}) + \delta^i_\mu \partial_0 (\tilde{g}^{00} \phi^2) \partial_i B \Big], \tag{D.8}$$

where in particular we have used the ETCR:

$$[\dot{b}_\mu, \dot{B}'] = -i \partial_0 (\tilde{f} \phi^{-2} \partial_\mu B) \delta^3 - 2i \tilde{f} \tilde{g}^{0i} \partial_i (\tilde{f} \phi^{-2} \partial_\mu B \delta^3). \tag{D.9}$$

Then, summing up I and J, and multiplying it by 2, we can obtain

$$\begin{aligned} [P_\mu, K^\nu] \equiv & 2(I + J) \\ = & 2i \int d^3x \, \tilde{g}^{0\lambda} \phi^2 \partial_\lambda B \delta^\nu_\mu \\ = & -2i D_0 \delta^\nu_\mu \\ = & -2i (G^\rho{}_\rho - D) \delta^\nu_\mu. \end{aligned} \tag{D.10}$$

References

1. C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation* (W H Freeman and Co (Sd), 1973)
2. T. Banks, N. Seiberg, Symmetries and strings in field theory and gravity. Phys. Rev. D **83**, 084019 (2011)
3. T. Kugo, I. Ojima, Local covariant operator formalism of non-abelian gauge theories and quark confinement problem. Prog. Theor. Phys. Suppl. **66**, 1 (1979)
4. M.E. Peskin, *Introduction to String and Superstring Theory II, in from the Planck Scale to the Weak Scale*, TASI 1986, ed. by H.E. Haber, (World Scientific, 1987), p. 277
5. D.J. Gross, J. Wess, Scale invariance, conformal invariance, and the high-energy behavior of scattering amplitudes. Phys. Rev. D **2**, 753 (1970). (**And references therein**)
6. Y. Nakayama, Scale invariance vs conformal invariance. Phys. Rep. **569**, 1 (2015). (**And references therein**)
7. B. Zumino, *Effective Lagrangian and broken symmetries, Lectures on Elementary Particles and Quantum Field Theory* vol. 2 (Brandeis Univ., Cambridge, 1970), p. 437–500
8. I. Oda, Quantum scale invariant gravity in de Donder Gauge. Phys. Rev. D **105**, 066001 (2022)
9. I. Oda, Quantum theory of Weyl invariant scalar-tensor gravity. Phys. Rev. D **105**, 120618 (2022)
10. I. Oda, P. Saake, BRST formalism of Weyl conformal gravity. Phys. Rev. D **106**, 106007 (2022)
11. I. Oda, M. Ohta, Quantum conformal gravity. JHEP **02**, 213 (2024)
12. P.D. Mannheim, Making the case for conformal gravity. Found. Phys. **42**, 388 (2012)
13. G. 't Hooft, Local conformal symmetry: the missing symmetry component for space and time. Int. J. Phys. D **24**, 1543001 (2015)
14. L. Modesto, L. Rachwal, Finite conformal quantum gravity and spacetime singularities. J. Phys. Conf. Ser. **942**, 012015 (2017)
15. G. de Berredo-Peixoto, I.L. Shapiro, Conformal quantum gravity with the Gauss–Bonnet term. Phys. Rev. D **70**, 044024 (2004)
16. A. Salvio, A. Strumia, A gravity up to infinite energy. Eur. Phys. J. C **78**, 124 (2018)
17. D.M. Ghilencea, Spontaneous breaking of Weyl quadratic gravity to Einstein action and Higgs potential. JHEP **03**, 049 (2019)
18. L. Rachwal, Introduction to quantization of conformal gravity. Universe **8**, 225 (2022)
19. N. Nakanishi, I. Ojima, *Covariant Operator Formalism of Gauge Theories and Quantum Gravity* (World Scientific Publishing, 1990) (**And references therein**)
20. N. Nakanishi, Indefinite metric quantum field theory of general gravity. Prog. Theor. Phys. **59**, 972 (1978)
21. T. Kugo, Noether currents and maxwell-type equations of motion in higher derivative gravity theories. Symmetry **13**, 1408 (2021)
22. R. Jackiw, S.-Y. Pi, Fake conformal symmetry in conformal cosmological models. Phys. Rev. D **91**, 067501 (2015)
23. M. Campigotto, L. Fatibene, Gauge natural formulation of conformal gravity. Ann. Phys. **354**, 328 (2015)
24. I. Oda, Fake conformal symmetry in unimodular gravity. Phys. Rev. D **94**, 044032 (2016)
25. I. Oda, Classical Weyl transverse gravity. Eur. Phys. J. C **77**, 284 (2017)
26. A. Alonso-Serrano, L.J. Garay, M. Liška, Noether charge formalism for Weyl transverse gravity. Class. Quantum Gravity **40**, 025012 (2023)
27. I. Oda, Vanishing noether current in Weyl invariant gravities. Int. J. Mod. Phys. A **37**, 2250213 (2022)
28. R. Ferrari, L.E. Picasso, Spontaneous breakdown in quantum electrodynamics. Nucl. Phys. B **31**, 316 (1971)
29. I. Oda, Restricted Weyl symmetry. Phys. Rev. D **102**, 045008 (2020)
30. A. Kamimura, I. Oda, Quadratic gravity and restricted Weyl symmetry. Mod. Phys. Lett. A **36**, 2150139 (2021)
31. I. Oda, Restricted Weyl symmetry and spontaneous symmetry breakdown of conformal symmetry. Mod. Phys. Lett. A **36**, 2150203 (2021)
32. N. Nakanishi, I. Ojima, Proof of the exact masslessness of gravitons. Phys. Rev. Lett. **43**, 91 (1979)
33. K. Kobayashi, T. Uematsu, Non-linear realization of superconformal symmetry. Nucl. Phys. B **263**, 309 (1986). (**And references therein**)

34. S. Kawasaki, T. Kimura, K. Kitago, Canonical quantum theory of gravitational field with higher derivatives. Prog. Theor. Phys. **66**, 2085 (1981)
35. S. Kawasaki, T. Kimura, Canonical quantum theory of gravitational field with higher derivatives. II. Prog. Theor. Phys. **68**, 1749 (1982)
36. S. Kawasaki, T. Kimura, Canonical quantum theory of gravitational field with higher derivatives. III. Prog. Theor. Phys. **69**, 1015 (1983)
37. J. Kubo, J. Kuntz, Analysis of unitarity in conformal quantum gravity. Class. Quantum Gravity **39**, 175010 (2022)
38. J. Kubo, J. Kuntz, Spontaneous conformal symmetry breaking and quantum quadratic gravity. Phys. Rev. D **106**, 126015 (2022)
39. M.J. Duff, Twenty years of the Weyl anomaly. Class. Quantum Gravity **11**, 1387 (1994)
40. F. Englert, C. Truffin, R. Gastmans, Conformal invariance in quantum gravity. Nucl. Phys. B **117**, 407 (1976)
41. D.M. Ghilencea, Weyl conformal geometry vs Weyl anomaly. JHEP **10**, 113 (2023)
42. S. Kawasaki, T. Kimura, A possible mechanism of ghost confinement in a renormalizable quantum gravity. Prog. Theor. Phys. **65**, 1767 (1981)
43. T. Kugo, Gauge invariance and a possible mechanism of color confinement. Phys. Lett. B **83**, 93 (1979)