



# Notes on time entanglement and pseudo-entropy

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**Abstract** Following [arXiv:2210.12963](https://arxiv.org/abs/2210.12963) [hep-th], we investigate aspects of the time evolution operator regarded as a density operator and associated entanglement-like structures in various quantum systems. These involve timelike separations and generically lead to complex-valued entropy, although there are interesting real subfamilies. There are many parallels and close relations with reduced transition matrices and pseudo-entropy, which we discuss and clarify. For instance, a related quantity involves the time evolution operator along with a projection onto some initial state, which amounts to analysing pseudo-entropy for the initial state and its time-evolved final state.

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## 1 Introduction

Generalizations of the Ryu–Takayanagi formulation of holographic entanglement [1–3] in *AdS/CFT* [4–6] to de Sitter space reveal new fascinating structures. These are based on taking the future boundary  $I^+$  of de Sitter space as the anchoring surface for extremal surfaces, along the lines of *dS/CFT* [7–10]. Most recently these appear in [11, 12], refining previous investigations of extremal surfaces and holographic entanglement in de Sitter space [13–20] (see also [21, 22]).

In the present work, we explore aspects of “time-entanglement”, or timelike entanglement, in various quantum mechanical systems, towards understanding entanglement-like structures involving timelike-separations, following [12]. There are close parallels with pseudo-entropy [23] (and [11]), as we will describe. Related investigations appear in *e.g.* [24–34] (also [35]).

To summarize the de Sitter studies (from [12]), extremal surfaces anchored at  $I^+$  turn out to not return to  $I^+$  (unlike those in *AdS* where the surfaces possess turning points). Since such surfaces do not return, they require extra data or boundary conditions in the past (interior). In entirely Lorentzian de Sitter spacetime, this leads to future-past timelike surfaces stretching between  $I^\pm$ . Apart from an overall  $-i$  factor (relative to spacelike surfaces in *AdS*) their areas are real and positive. With a no-boundary type boundary condition, the top half of these timelike surfaces joins with a spacelike part on the hemisphere giving a complex-valued area. Since these surfaces necessarily have a timelike component (or run along a complex time contour), they have complex areas. Two aspects of “time-entanglement” in simple toy models in quantum mechanics were described in [12]. One is based on a future-past thermofield double type state entangling timelike separated states, which leads to entirely positive structures. Another is based on the time evolution operator and reduced transition amplitudes, which leads to complex-valued entropy.

In the present paper we discuss various aspects of the time evolution operator regarded as a density operator and its entanglement structures which involve timelike separations. There are many parallels and close relations with pseudo-entropy [23]: we summarize some central points on time evolution and pseudo-entropy in Sect. 2, including a general map in Sect. 2.1. We then study various classes of finite quantum mechanical examples in Sect. 3, including qubit systems and harmonic oscillators (some detailed in Appendices A, B and C). In Sect. 4 we study entanglement structures for the time evolution operator along with a projection operator onto some state towards isolating components of the time evolution operator. This ends up amounting to pseudo-entropy for this state and its time-evolution: in Sect. 4.1 we study thermofield-double type states and find that some general features emerge. In Sect. 5 we study the time evolution operator normalized at  $t = 0$  (rather than at general time  $t$ ): this gives rise to various detailed differences in the entanglement structures that emerge. In Sect. 6 we describe some aspects of entanglement entropy in 2-dim CFT for timelike intervals, elaborating on that in [12]. Some of the discussions here have partial overlap with [11,32]. In these time-independent situations so far, the structure of time-entanglement shows parallels with ordinary finite temperature entanglement, but with analytic continuation to imaginary temperature  $\beta = it$ . In Sect. 7, we study time-dependent interactions focussing on simple 2-qubit systems with  $\delta$ -function potentials, and the resulting time entanglement.

Overall, pseudo-entropy [23] is a generalization of entanglement entropy involving two arbitrary states (without necessarily specifying dynamical information): this does not need to pertain to timelike separations per se. The notions of time entanglement are designed to deal with timelike separations, involving entanglement structures based on the time evolution operator, as well as projection onto specific initial states: so in particular we require specifying a Hamiltonian that dictates time evolution. However the calculations involved in studying time entanglement entropy are closely related to those in evaluating pseudo-entropy [23]. Our goal in these notes is more an exploration of time entanglement and how it dovetails with pseudo-entropy, rather than a detailed classification (which already appears for pseudo-entropy of various quantum systems in [23] and subsequent work).

## 2 Summary: time evolution and pseudo-entropy

Our investigations, following [12], are based on regarding the time evolution operator as a density operator, performing partial traces over subsystems and evaluating the corresponding von Neumann entropy. The time evolution operator  $\mathcal{U}(t) = e^{-iHt}$  for a system with Hamiltonian  $H$  can be

written in terms of (time-independent) Hamiltonian eigenstates  $|i\rangle$  (which are defined on some past time slice  $P$ ). Then the time evolution operator normalized at an arbitrary time  $t$  gives

$$\begin{aligned} \mathcal{U}(t) &= e^{-iHt} = \sum_i e^{-iE_i t} |i\rangle\langle i| \\ &= \sum_i |i\rangle_t \langle i|_P, \quad |i(t)\rangle \equiv |i\rangle_t = e^{-iE_i t} |i\rangle_P; \\ \rho_t(t) &\equiv \frac{\mathcal{U}(t)}{\text{Tr} \mathcal{U}(t)} \Rightarrow \\ \rho_t(t) &= \sum_i p_i |i\rangle_P \langle i|_P, \quad p_i = \frac{e^{-iE_i t}}{\sum_j e^{-iE_j t}} \\ \rightarrow \rho_t^A &= \text{Tr}_B \rho_t \\ &= \sum_i p'_i |i'\rangle_P \langle i'|_P \rightarrow S_A = - \sum_i p'_i \log p'_i. \end{aligned} \tag{2.1}$$

As is clear, there are sharp parallels with ordinary finite temperature entanglement structures, except with imaginary temperature  $\beta = it$ : this will be seen explicitly as a recurring theme throughout much of what follows.

A related quantity involves the time evolution operator with projection onto some state  $|i\rangle$ ,

$$\begin{aligned} \rho_t^{(i)} &= \frac{\rho_t |i\rangle\langle i|}{\text{Tr}(\rho_t |i\rangle\langle i|)} \\ &= \frac{|f[i](t)\rangle\langle i|}{\text{Tr}(|f[i](t)\rangle\langle i|)}, \quad |f[i](t)\rangle = e^{-iHt} |i\rangle; \\ \rho_t^{(i),A} &= \text{Tr}_B \rho_t^{(i)}. \end{aligned} \tag{2.2}$$

The state  $|f[i]\rangle$  is the final state obtained by time-evolving the initial state  $|i\rangle$ . We obtain

$$\begin{aligned} |i\rangle &= \sum_n c_n |n\rangle \\ \rho_t^{(i)} &= \frac{1}{\sum_k e^{-iE_k t} |c_k|^2} \sum_{k,m} e^{-iE_k t} c_k c_m^* |k\rangle\langle m| \end{aligned} \tag{2.3}$$

for a general (non-eigen)state  $|i\rangle$ . At  $t = 0$ , the time evolution operator is just the identity operator, a sum over all the eigenstate projection operators, while the time evolution operator with projection becomes simply the density matrix for the initial state  $|i\rangle$ . For any nonzero time  $t$ , there is timelike separation between the initial states  $|\psi\rangle_P$  and the eventual states  $|\psi\rangle_t$ . These entanglement structures involving timelike separations and time evolution have close parallels with pseudo-entropy [23] obtained from the reduced transition matrix for two arbitrary states  $|i\rangle, |f\rangle$ :

$$\mathcal{T}_{f|i}^A = \text{Tr}_B \left( \frac{|f\rangle\langle i|}{\text{Tr}(|f\rangle\langle i|)} \right). \tag{2.4}$$

To summarise in generality, consider a bipartite system, the Hilbert space being characterized by Hamiltonian eigenstates  $|i, i'\rangle$  with energies  $E_{i,i'}$ . The normalized time evolu-

tion operator (2.1) and its partial trace over  $B \equiv \{i'\}$  are

$$\begin{aligned} \rho_t &= \frac{1}{\sum_{i,i'} e^{-iE_{i,i'}t}} \sum_{i,i'} e^{-iE_{i,i'}t} |i, i'\rangle \langle i, i'| \\ \rightarrow \rho_t^A &= \frac{1}{\sum_{i,i'} e^{-iE_{i,i'}t}} \left( \sum_{i'} e^{-iE_{i,i'}t} \right) |i\rangle \langle i|. \end{aligned} \tag{2.5}$$

The time evolution operator with projection onto state  $|I\rangle$  is

$$\begin{aligned} |I\rangle &= \sum_{k,k'} c_{k,k'} |k, k'\rangle, \\ \rho_t^{(I)} &= \frac{1}{\sum_{i,i'} |c_{i,i'}|^2 e^{-iE_{i,i'}t}} \\ &\quad \times \sum_{i,i',j,j'} c_{i,i'} c_{j,j'}^* e^{-iE_{i,i'}t} |i, i'\rangle \langle j, j'|, \\ \rho_t^{(I),A} &= \frac{1}{\sum_{i,i'} |c_{i,i'}|^2 e^{-iE_{i,i'}t}} \\ &\quad \times \sum_{i,j} \left( \sum_{i'} c_{i,i'} c_{j,i'}^* e^{-iE_{i,i'}t} \right) |i\rangle \langle j|. \end{aligned} \tag{2.6}$$

The reduced transition matrix for pseudo-entropy is obtained as

$$\begin{aligned} |I\rangle &= c_{i,i'} |i, i'\rangle, |F\rangle = c'_{i,i'} |i, i'\rangle; \\ \mathcal{T}_{F|I} &= \frac{1}{\sum_{i,i'} c'_{i,i'} c_{i,i}^*} \\ &\quad \times \sum_{i,i',j,j'} c'_{i,i'} c_{j,j'}^* |i, i'\rangle \langle j, j'| \\ \mathcal{T}_{F|I}^A &= \frac{1}{\sum_{i,i'} c'_{i,i'} c_{i,i}^*} \\ &\quad \times \left( \sum_{i'} c'_{i,i'} c_{j,i'}^* \right) |i\rangle \langle j|. \end{aligned} \tag{2.7}$$

It is clear that the time evolution operator with projection (2.6) is obtained from the pseudo-entropy reduced transition matrix (2.7) by restricting to the final state being that obtained by time-evolving the initial state, *i.e.*  $|F\rangle = \mathcal{U}(t)|I\rangle$ .

### 2.1 The time evolution operator and the transition matrix

With a single Hilbert space, the structure of the reduced transition matrix appears different in detail from that of the reduced time evolution operator: this is clear in bipartite systems from (2.5), (2.6), (2.7). However it would seem that there should be close connections between the time evolution operator and the transition matrix since both pertain to time evolution if we focus on final states as time-evolved initial states.

Towards studying this, let us first recall that a special class of states comprises thermofield-double type states  $|I\rangle_{TFD} = \sum_k c_{k,\{k\}} |k, \{k\}\rangle$ , with only diagonal components (a further

special subclass comprises maximally entangled TFD states, with all  $c_{k,\{k\}}$  equal).

Towards mapping time evolution and the transition matrix, consider doubling the Hilbert space at both initial and final times: *i.e.* extend the Hilbert state  $\mathcal{H} \equiv \mathcal{H}_1$  to  $\mathcal{H}_1 \otimes \mathcal{H}_2$ , where the Hilbert space  $\mathcal{H}_2$  is an identical copy of  $\mathcal{H}_1$ . Now consider thermofield-double type initial and final states:

$$\begin{aligned} |\psi_I\rangle &= \sum_i c_i^I |i\rangle_1 |i\rangle_2, \\ |\psi_F\rangle &= \sum_i c_i^F |i\rangle_1 |i\rangle_2, \end{aligned} \tag{2.8}$$

where  $\{|i\rangle\}$  is a basis of states. The (un-normalized) transition matrix is

$$\begin{aligned} \mathcal{T}_{F|I} &= |\psi_F\rangle \langle \psi_I| \\ &= \sum_{i,j} c_i^F c_j^{I*} |i\rangle_1 |i\rangle_2 \langle j|_1 \langle j|_2. \end{aligned} \tag{2.9}$$

Performing a partial trace over copy-2 gives

$$\text{Tr}_2 \mathcal{T}_{F|I} = \sum_i c_i^F c_i^{I*} |i\rangle_1 \langle i|_1. \tag{2.10}$$

For this to equal the time evolution operator, we require

$$\begin{aligned} \text{Tr}_2 \mathcal{T}_{F|I} &= \mathcal{U}(t) \\ &= \sum_i e^{-iE_i t} |i\rangle \langle i| \Rightarrow c_i^F c_i^{I*} = e^{-iE_i t}. \end{aligned} \tag{2.11}$$

A ‘‘symmetric’’ solution is

$$\begin{aligned} c_i^I &= e^{iE_i t/2} : |\psi_I\rangle = \sum_i e^{iE_i t/2} |i\rangle_1 |i\rangle_2, \\ c_i^F &= e^{-iE_i t/2} : |\psi_F\rangle = \sum_i e^{-iE_i t/2} |i\rangle_1 |i\rangle_2. \end{aligned} \tag{2.12}$$

These can be regarded as obtained from a continuation  $\beta \rightarrow it$  of the usual finite temperature thermofield-double type states  $e^{-\beta E_i/2} |i\rangle |i\rangle$ . There are of course less symmetric solutions  $c_i^I, c_i^F$ , describing the initial and final states. However the symmetric solution reduces to ordinary entanglement when the initial and final states are the same, *i.e.*  $|\psi_I\rangle = |\psi_F\rangle$  (*i.e.* at  $t = 0$ ), the transition matrix becomes the usual density matrix  $\mathcal{T}_{F|I} = |\psi_I\rangle \langle \psi_I| = \rho_I$  for the state  $|\psi_I\rangle$ . Thus the time evolution operator can be regarded as a particular reorganization of the transition matrix appearing in pseudo-entropy.

It is worth noting that for systems with infinite towers of states, the trace of the time evolution operator contains highly oscillatory terms and thus requires a regulator to be well-defined: we will see this explicitly for the harmonic oscillator later; see (3.18).

*Single qubit:* This simple case serves to illustrate the above. In this case (described by (3.1)), we have  $H|1\rangle =$

$E_1|1\rangle, H|2\rangle = E_2|2\rangle$ , with  $H$  the Hamiltonian. Let us take

$$|\psi_F\rangle = \sum_{n=1,2} e^{-\frac{iE_n t}{2}} |n\rangle_1 \otimes |n\rangle_2,$$

$$|\psi_I\rangle = \sum_{m=1,2} e^{\frac{iE_m t}{2}} |m\rangle_1 \otimes |m\rangle_2. \tag{2.13}$$

Here the subscript 2 stands for the second auxiliary system with the identical Hilbert space  $\mathcal{H}_2$ . Then the unnormalised transition matrix  $T = |\psi_F\rangle\langle\psi_I|$  is

$$T_{F|I} = |\psi_F\rangle\langle\psi_I| = \sum_{n,m=1,2} e^{-\frac{i(E_n+E_m)t}{2}} |n\rangle_1|n\rangle_2\langle m|_1\langle m|_2. \tag{2.14}$$

Taking a partial trace over the second component gives

$$T_{F|I}^1 = Tr_2(T_{F|I}) = \sum_{n=1,2} e^{-iE_n t} |n\rangle_1\langle n|_1 = e^{-iHt}, \tag{2.15}$$

thus obtaining the time evolution operator. This illustrates the general discussion earlier in this simple case.

### 3 Time evolution operator and entanglement: examples

In this section we will study various examples of finite quantum systems to explore the entanglement structure of the time evolution operator.

#### 3.1 2-qubit systems

For a 2-state system,

$$H|k\rangle = E_k|k\rangle, k = 1, 2$$

$$|k\rangle_F \equiv |k(t)\rangle = e^{-iE_k t} |k\rangle_P. \quad [ \langle 1|2\rangle = 0 ] \tag{3.1}$$

we obtain  $\rho_t(t)$  using (2.1). Now, imagining a 2-spin analogy  $|1\rangle \equiv |+\rangle, |2\rangle \equiv |-\rangle$ , performing a partial trace over the second spins gives

$$\rho_t^A = \frac{1}{1 + e^{i\theta}} (|+\rangle_P\langle +|_P + e^{i\theta} |-\rangle_P\langle -|_P), \theta = -(E_2 - E_1)t,$$

$$S_A = -\text{tr}(\rho_t^A \log \rho_t^A) = -\frac{1}{1 + e^{i\theta}} \log \frac{1}{1 + e^{i\theta}} - \frac{1}{1 + e^{-i\theta}} \log \frac{1}{1 + e^{-i\theta}}, \tag{3.2}$$

so the von Neumann entropy, recast as  $\alpha + \alpha^*$ , is real-valued in this special case. We see that  $S_t^A$  grows large as  $\theta \rightarrow (2n + 1)\pi$ . Further  $\rho_t^A$  and  $S_t^A$  are periodic in  $\theta$  and so in time  $t$  (simplifying  $S_t^A$  shows terms containing  $\log(e^{i\theta/2})$  which we retain as it is, rather than  $\frac{i\theta}{2}$ , so as to avoid picking specific branches of the logarithm, thereby losing manifest

periodicity; within one  $\theta$ -cell the simplified expression for  $S_t^A$  coincides with the corresponding one in [23]).

Now consider two qubits, each being  $|1\rangle, |2\rangle$ , with a more general Hamiltonian

$$H = E_{11}|11\rangle\langle 11| + E_{22}|22\rangle\langle 22| + E_{12}(|12\rangle\langle 12| + |21\rangle\langle 21|) \tag{3.3}$$

that is diagonal in this basis. It is reasonable to take  $E_{12} = E_{21}$ . So the normalized time evolution operator (2.1) becomes

$$\rho_t = \frac{\sum_{i,j} e^{-iE_{ij}t}}{\sum_{kl} e^{-iE_{kl}t}} |ij\rangle\langle ij| = \frac{(|11\rangle\langle 11| + e^{i\theta_1}|22\rangle\langle 22| + e^{i\theta_2}(|12\rangle\langle 12| + |21\rangle\langle 21|))}{1 + e^{i\theta_1} + 2e^{i\theta_2}};$$

$$\theta_1 \equiv -(E_{22} - E_{11})t, \quad \theta_2 \equiv -(E_{12} - E_{11})t. \tag{3.4}$$

(At  $t = 0$ , the  $\theta_i$  vanish and this is the normalized identity operator.) A partial trace over the 2nd component gives the reduced time evolution operator,

$$\rho_t^A = \frac{1}{1 + e^{i\theta_1} + 2e^{i\theta_2}} \times \left( (1 + e^{i\theta_2})|1\rangle\langle 1| + (e^{i\theta_1} + e^{i\theta_2})|2\rangle\langle 2| \right) \tag{3.5}$$

which generically has complex-valued von Neumann entropy. It is clear that this matches ordinary finite temperature entanglement, except with imaginary temperature  $\beta = it$ .

Now let us impose an exchange symmetry  $|1\rangle \leftrightarrow |2\rangle$ : this occurs for instance if we consider two spins  $|\pm\rangle$  with nearest neighbour interaction  $H = -J s_z^1 s_z^2$ . This restriction now implies  $E_{22} = E_{11}$  thereby reducing (3.5) to (3.2) earlier, with just one nontrivial phase, giving real entropy.

*Qubit chains:* In Appendix B, we study finite and infinite chains of qubits with nearest neighbour interactions, towards understanding the reduced time evolution operator for a single qubit, after partial trace over all other qubits. This also reveals interesting complex-valued entropy in general, obtainable as a finite temperature system but with imaginary temperature. We also find a real-valued slice when the system enjoys  $|1\rangle \leftrightarrow |2\rangle$  exchange symmetry.

To illustrate obtaining the time evolution operator (3.4) from the doubled transition matrix as in (2.11), (2.12), we write

$$|\psi_F\rangle = \sum_{n,m=1,2} e^{-\frac{iE_{nm}t}{2}} |nm\rangle_1 \otimes |nm\rangle_2,$$

$$|\psi_I\rangle = \sum_{n,m=1,2} e^{\frac{iE_{nm}t}{2}} |nm\rangle_1 \otimes |nm\rangle_2. \tag{3.6}$$

Then the unnormalized transition matrix  $T = |\psi_F\rangle\langle\psi_I|$  after partial trace over the second component gives

$$T_{F|I}^1 = \text{Tr}_2 \left( \sum_{n,m,p,q=1,2} e^{-\frac{iE_{nm}t}{2}} e^{-\frac{iE_{pq}t}{2}} \right)$$

$$\begin{aligned}
 & |nm\rangle_1 |nm\rangle_2 \langle pq|_1 \langle pq|_2 \\
 &= \sum_{n,m=1,2} e^{-i E_{nm}t} |nm\rangle_1 \langle nm|_1, \tag{3.7}
 \end{aligned}$$

so this reduced transition matrix is the same as the unnormalized time evolution operator.

### 3.1.1 Mutual information

Mutual information defined as  $I[A, B] = S[A] + S[B] - S[A \cup B]$  can be studied for the time evolution operator as well. In the general 2-qubit case (3.3), (3.4), above, we can calculate  $\rho_t^1 = \text{Tr}_2 \rho_t$  and  $\rho_t^2 = \text{Tr}_1 \rho_t$ , which then leads to the von Neumann entropies  $S_t^1$  and  $S_t^2$  respectively. The time evolution operator  $\rho_t$  itself leads to  $S_t = -\text{tr}(\rho_t \log \rho_t)$ . It is straightforward to see that  $\rho_t^{1,2}$  are of the same form as  $\rho_t^A$  in (3.5), which along with  $\rho_t$  in (3.4) gives

$$\begin{aligned}
 H &= \sum E_{ij} |ij\rangle \langle ij|, E_{ij} = \{E_{00}, E_{11}, E_{22}, E_{01}, E_{02}, E_{12}\} \\
 \rho_t &= \frac{e^{-i E_{ij}t}}{\sum_{ij} e^{-i E_{ij}t}} |ij\rangle \langle ij| = \frac{e^{-i E_{ij}t}}{e^{-i E_{00}t} + e^{-i E_{11}t} + e^{-i E_{22}t} + 2e^{-i E_{01}t} + 2e^{-i E_{02}t} + 2e^{-i E_{12}t}} |ij\rangle \langle ij|, \tag{3.11}
 \end{aligned}$$

$$\begin{aligned}
 S_t^{1,2} &= -\frac{1 + e^{i\theta_2}}{1 + e^{i\theta_1} + 2e^{i\theta_2}} \log \frac{1 + e^{i\theta_2}}{1 + e^{i\theta_1} + 2e^{i\theta_2}} \\
 &\quad - \frac{e^{i\theta_1} + e^{i\theta_2}}{1 + e^{i\theta_1} + 2e^{i\theta_2}} \log \frac{e^{i\theta_1} + e^{i\theta_2}}{1 + e^{i\theta_1} + 2e^{i\theta_2}} \\
 S_t &= -\frac{1}{1 + e^{i\theta_1} + 2e^{i\theta_2}} \log \frac{1}{1 + e^{i\theta_1} + 2e^{i\theta_2}} \\
 &\quad - \frac{e^{i\theta_1}}{1 + e^{i\theta_1} + 2e^{i\theta_2}} \log \frac{e^{i\theta_1}}{1 + e^{i\theta_1} + 2e^{i\theta_2}} \\
 &\quad - \frac{2e^{i\theta_2}}{1 + e^{i\theta_1} + 2e^{i\theta_2}} \log \frac{e^{i\theta_2}}{1 + e^{i\theta_1} + 2e^{i\theta_2}}, \tag{3.8}
 \end{aligned}$$

so the mutual information is

$$I[A, B] = S_t^1 + S_t^2 - S_t. \tag{3.9}$$

In general this is nonzero and complex since the entropies are complex in general. However there are special cases: for instance if all energy eigenvalues are identical, then

$$\theta_{1,2} = 0 : S_t^{1,2} = \log 2, \quad S_t = 2 \log 2 \Rightarrow I[A, B] = 0 \tag{3.10}$$

although the time evolution is nontrivial since each phase  $e^{-iEt}$  is nonzero.

Likewise the 2-state subcase (3.1) is obtained by setting  $e^{i\theta_2} = 0$  which gives  $S_t^{1,2}, S_t$  of the same real-valued form as in (3.2), so  $I[A, B] = S_t^1$ .

These expressions above can also be viewed as arising from the finite temperature results for inverse temperature  $\beta$  continued to  $\beta = it$ . From that point of view, the high temperature limit  $\beta \rightarrow 0$  gives vanishing mutual information: this limit has  $\beta E_i \rightarrow 0$  which is mathematically equivalent to the  $\theta_{1,2} = 0$  subcase earlier, with  $I[A, B] \rightarrow 0$ . In the present context, this is  $t \rightarrow 0$ , and we again obtain vanishing mutual information,  $I[A, B] \rightarrow 0$ .

### 3.2 2-qutrit systems

Consider now two qutrits,  $|i\rangle, i = 0, 1, 2$ : the Hamiltonian (in eigenstate basis) and the normalized time evolution operator are

again with  $E_{ij} = E_{ji}$ . The reduced time evolution operator tracing over the second qutrit is

$$\begin{aligned}
 (\rho_t^A)_{ij} &= (\rho_t)_{ijkl} \delta^{kl}; \\
 \rho_t^A &= \frac{1}{\sum_{ij} e^{-i E_{ij}t}} \\
 &\quad \times \sum_{i=0,1,2} \left( \sum_j e^{-i E_{ij}t} \right) |i\rangle \langle i|. \tag{3.12}
 \end{aligned}$$

In general this leads to complex-valued entropy as before, with multiple distinct phases. Imposing exchange symmetry between the qutrits, *i.e.*  $|0\rangle \leftrightarrow |1\rangle \leftrightarrow |2\rangle$ , this reduces to a single independent phase controlled by  $-(E_{01} - E_{00})t$  which then gives real entropy.

### 3.3 Two uncoupled oscillators

We consider two uncoupled harmonic oscillators: the Hamiltonian is

$$\begin{aligned}
 H &= \sum E_{n_1 n_2} |n_1, n_2\rangle \langle n_1, n_2|, \\
 E_{n_1 n_2} &= \omega(n_1 + n_2 + 1). \tag{3.13}
 \end{aligned}$$

The normalized time evolution operator then becomes

$$\rho_t = \sum \frac{e^{-i E_{n_1 n_2}t}}{\sum e^{-i E_{n_1 n_2}t}} |n_1, n_2\rangle \langle n_1, n_2| \tag{3.14}$$

The normalization evaluates to

$$\begin{aligned} \sum_{1,2} e^{-iE_{n_1 n_2} t} &= e^{-i\omega t} \sum_{1,2} e^{-i\omega n_1 t} e^{-i\omega n_2 t} \\ &= \frac{e^{-i\omega t}}{(1 - e^{-i\omega t})^2}. \end{aligned} \tag{3.15}$$

Now, tracing over the second oscillator, we obtain

$$\begin{aligned} \rho_t^A &= \sum_{n_2=0}^{\infty} \rho_t \\ &= \sum_{n_1} \frac{e^{-i\omega n_1 t}}{1/(1 - e^{-i\omega t})} |n_1\rangle\langle n_1| \end{aligned} \tag{3.16}$$

with the von Neumann entropy

$$\begin{aligned} S_t^A &= - \sum_n \frac{e^{-i\omega n t}}{1/(1 - e^{-i\omega t})} \log \frac{e^{-i\omega n t}}{1/(1 - e^{-i\omega t})} \\ &= - \log(1 - e^{-i\omega t}) + \frac{i\omega t e^{-i\omega t}}{1 - e^{-i\omega t}} \end{aligned} \tag{3.17}$$

which is the usual entropy for a single oscillator at finite temperature with  $\beta = it$ . In general this is complex-valued. The zero temperature limit gives  $S \sim \beta E e^{-\beta E}$  which here gives  $S \sim it\omega e^{-i\omega t}$ .

In evaluating the normalization (3.15), it is important to note that this sum over the infinite tower of states (and similar quantities involving any infinite tower of states) is not strictly convergent as an infinite series since this complex expression is highly oscillatory for high energy states, although the sum and its closed form expression are formally true. This is also true for the single oscillator expression (3.16) obtained as the reduced time evolution operator, whose normalization is  $\sum_{n_1} e^{-i\omega n_1 t} = 1/(1 - e^{-i\omega t})$ . Towards rendering this well-defined as a series, one can introduce a small regulator either in  $\omega$  or in  $t$  (giving time a tiny regulating Euclidean component) which then makes it converge: e.g. a small Euclidean time component gives

$$\begin{aligned} \sum_{n_1} e^{-i\omega n_1(t-i\epsilon)} &= \sum_{n_1} e^{-i\omega n_1 t} e^{-n_1 \omega \epsilon} \\ &= \frac{1}{1 - e^{-i\omega(t-i\epsilon)}}, \end{aligned} \tag{3.18}$$

which defines the sum. An alternative way to view it is to start with the (convergent) finite temperature partition function  $\sum_n e^{-\beta E_n}$  and then perform analytic continuation to imaginary temperature  $\beta = it$ .

It is interesting to also study two coupled harmonic oscillators with Hamiltonian

$$H = \frac{1}{2} (p_A^2 + p_B^2) + \frac{k_1}{2} (x_A^2 + x_B^2) + \frac{k_2}{2} (x_A - x_B)^2. \tag{3.19}$$

We describe this in detail in Appendix C. The resulting entropy from the time evolution operator can be realized as following from imaginary temperature.

#### 4 The time evolution operator with projections

As we have seen, the entanglement structures arising from the time evolution operator involve the entire space of states since the time evolution operator is like a full density matrix. It is desirable to isolate a ‘‘part’’ of the time evolution operator, to understand various components of the latter. This suggests appending projections onto individual states.

With this in mind, we now consider the time evolution operator along with a projection operator onto some state  $|i\rangle$ , as in (2.2):

$$\begin{aligned} \rho_t^{[i]} &= \frac{\rho_t |i\rangle\langle i|}{\text{Tr}(\rho_t |i\rangle\langle i|)} \\ &= \frac{|f[i]\rangle\langle i|}{\text{Tr}(|f[i]\rangle\langle i|)} \quad |f[i]\rangle = e^{-iHt} |i\rangle. \end{aligned} \tag{4.1}$$

(The projection here is from the right: at the calculational level, projecting from the left is similar but leads to complex conjugate expressions in general.) The state  $|f[i]\rangle$  is the final state obtained by time-evolving the initial state  $|i\rangle$ . If  $|i\rangle$  is a Hamiltonian eigenstate, then  $\rho_t^{[i]}$  reduces to just a single component  $|i\rangle\langle i|$  (the phase coefficient cancels upon normalizing), i.e. the usual density matrix for  $|i\rangle$ . This is also true at  $t = 0$  for a generic state  $|i\rangle$ : here  $\rho_t^{[i]}|_{t=0} = \frac{|i\rangle\langle i|}{\text{Tr}(|i\rangle\langle i|)}$  which gives ordinary entanglement structures at  $t = 0$ .

For a generic state  $|i\rangle$ , we obtain (2.6). As a simple concrete example, consider the 2-state system (3.1) earlier with a generic initial state:

$$\begin{aligned} |i\rangle &= c_1|1\rangle + c_2|2\rangle \quad (|c_1|^2 + |c_2|^2 = 1) \rightarrow \\ |f[i]\rangle &= c_1 e^{-iE_1 t} |1\rangle + c_2 e^{-iE_2 t} |2\rangle; \\ \rho_t^{[i]} &= \mathcal{N}^{-1} \left( |c_1|^2 e^{-iE_1 t} |1\rangle\langle 1| \right. \\ &\quad \left. + |c_2|^2 e^{-iE_2 t} |2\rangle\langle 2| + c_1 c_2^* e^{-iE_1 t} |1\rangle\langle 2| \right. \\ &\quad \left. + c_2 c_1^* e^{-iE_2 t} |2\rangle\langle 1| \right), \end{aligned} \tag{4.2}$$

where  $\mathcal{N} = \text{Tr}(|f\rangle\langle i|)$  is the normalization. Now taking  $|1\rangle \equiv |+\rangle$  and  $|2\rangle \equiv |-\rangle$  and performing a partial trace over the second component gives

$$\begin{aligned} \rho_t^{[i],A} &= \frac{1}{|c_1|^2 + |c_2|^2 e^{i\theta}} \left( |c_1|^2 |+\rangle\langle +| \right. \\ &\quad \left. + |c_2|^2 e^{i\theta} |-\rangle\langle -| \right), \quad \theta = -(E_2 - E_1)t, \\ S_t^{[i],A} &= - \frac{|c_1|^2}{|c_1|^2 + |c_2|^2 e^{i\theta}} \log \frac{|c_1|^2}{|c_1|^2 + |c_2|^2 e^{i\theta}} \\ &\quad - \frac{|c_2|^2 e^{i\theta}}{|c_1|^2 + |c_2|^2 e^{i\theta}} \log \frac{|c_2|^2 e^{i\theta}}{|c_1|^2 + |c_2|^2 e^{i\theta}}. \end{aligned} \tag{4.3}$$



At  $t = 0$ , the von Neumann entropy above is ordinary entanglement entropy for the generic state  $|i\rangle$  (obtained from  $\rho_A = \text{Tr}_B |i\rangle\langle i|$ ). For general timelike separation  $t$ , the entropy  $S_A$  is real-valued only if  $|c_1|^2 = |c_2|^2$ , i.e. maximal entanglement at  $t = 0$  (or  $\theta = 0$ ).

Consider now two qubits, each  $|1\rangle, |2\rangle$ , with a general Hamiltonian (3.3) as before. For a generic state

$$|I\rangle = \sum_{ij} c_{ij} |ij\rangle \tag{4.4}$$

with the basis  $|ij\rangle = \{|11\rangle, |22\rangle, |12\rangle, |21\rangle\}$ , and the time evolution operator with projection can be evaluated as (2.6). Performing a partial trace over the second component here gives

$$\begin{aligned} \rho_t^{I,A} &= \frac{1}{\sum_{ij} |c_{ij}|^2 e^{-iE_{ij}t}} \\ &\times \sum_{i,k=1}^2 \left( \sum_j c_{ij} c_{kj}^* e^{-iE_{ij}t} \right) |i\rangle\langle k| \\ &= \frac{1}{\sum_{ij} |c_{ij}|^2 e^{-iE_{ij}t}} \left( (|c_{11}|^2 e^{-iE_{11}t} + |c_{12}|^2 e^{-iE_{12}t}) |1\rangle\langle 1| \right. \\ &\quad + (c_{11} c_{21}^* e^{-iE_{11}t} + c_{12} c_{22}^* e^{-iE_{12}t}) |1\rangle\langle 2| \\ &\quad + (c_{21} c_{11}^* e^{-iE_{12}t} + c_{22} c_{12}^* e^{-iE_{22}t}) |2\rangle\langle 1| \\ &\quad \left. + (|c_{21}|^2 e^{-iE_{12}t} + |c_{22}|^2 e^{-iE_{22}t}) |2\rangle\langle 2| \right) \end{aligned} \tag{4.5}$$

At  $t = 0$ , this is ordinary entanglement for the generic state  $|I\rangle$ . There are special subcases with interesting structure, some of which we will discuss soon.

For 3-qubits with Hamiltonian (B.2) with energies  $E_{ijk}$  for eigenstates  $|ijk\rangle$  (along with the symmetry-based simplifications there), we obtain

$$\begin{aligned} |I\rangle &= \sum_{i,j,k=1}^2 c_{ijk} |ijk\rangle : \rho_t^{I,A} = \frac{1}{\sum_{ijk} |c_{ijk}|^2 e^{-iE_{ijk}t}} \\ &\times \sum_{i,j,k,l,m,n=1}^2 c_{ijk} c_{lmn}^* e^{-iE_{ijk}t} |ijk\rangle\langle lmn|, \\ \rho_t^{I,A} &= \frac{1}{\sum_{ijk} |c_{ijk}|^2 e^{-iE_{ijk}t}} \\ &\times \sum_{j,m=1}^2 \left( \sum_i \sum_k c_{ijk} c_{imk}^* e^{-iE_{ijk}t} \right) |j\rangle\langle m|, \end{aligned} \tag{4.6}$$

where the last line is the reduced transition matrix for the middle qubit, arising after a partial trace over the 1st and 3rd components  $(\rho_t^A)_{jm} = (\rho_t)_{ijk,lmn} \delta^{il} \delta^{kn}$ .

### 4.1 Thermofield-double type states

It is interesting to focus on thermofield-double type initial states with only ‘‘diagonal’’ components: then for 2-qubits,

using (4.5) we obtain

$$\begin{aligned} |I\rangle &= \sum_{i=1,2} c_{ii} |ii\rangle : \\ \rho_t^{I,A} &= \frac{1}{\sum_i |c_{ii}|^2 e^{-iE_{ii}t}} \sum_{i,k=1}^2 c_{ii} c_{kk}^* e^{-iE_{ii}t} |ii\rangle\langle kk|, \\ \rho_t^{I,A} &= \frac{1}{|c_{11}|^2 e^{-iE_{11}t} + |c_{22}|^2 e^{-iE_{22}t}} \left( |c_{11}|^2 e^{-iE_{11}t} |1\rangle\langle 1| \right. \\ &\quad \left. + |c_{22}|^2 e^{-iE_{22}t} |2\rangle\langle 2| \right). \end{aligned} \tag{4.7}$$

This is identical to (4.3). To elaborate a little, the initial state is  $|I\rangle = c_{11}|11\rangle + c_{22}|22\rangle$  and its time-evolved final state is  $|F\rangle = c_{11}e^{-iE_{11}t}|11\rangle + c_{22}e^{-iE_{22}t}|22\rangle$ , and the reduced time evolution operator with projection,  $\rho_t^{I,A}$  above, is the normalized reduced transition matrix for  $|I\rangle, |F\rangle$ , with the corresponding (in general complex-valued) pseudo-entropy (2.4).

Now restricting further to maximally entangled states with  $|c_{11}|^2 = |c_{22}|^2 = \frac{1}{2}$  simplifies this to just a single non-trivial phase  $e^{i\theta} = e^{-i\Delta E t}$  where  $\Delta E = E_{22} - E_{11}$ , thereby leading to the entanglement structure (3.2) of the time evolution operator for the 2-state case, i.e.  $S_t^{I,A} = -\frac{1}{1+e^{i\theta}} \log \frac{1}{1+e^{i\theta}} - \frac{1}{1+e^{-i\theta}} \log \frac{1}{1+e^{-i\theta}}$ . The states in question here can be regarded as maximally entangled Bell pairs and the entropy can be regarded as pseudo-entropy for the Bell pair initial state  $|I\rangle$  and its time-evolved final state  $|F\rangle$ . As noted there, this is a real-valued entropy, oscillating in time with periodicity set by  $\Delta E$ , growing unbounded at specific time values where  $t = \frac{(2n+1)\pi}{\Delta E}$ . Note also that specific time values  $t = \frac{2n\pi}{\Delta E}$  lead to the minimum value  $S_A = \log 2$ , which is simply the ordinary entanglement entropy of the maximally entangled initial state. The fact that this time entanglement entropy can be unbounded is a novel feature compared with ordinary entanglement entropy for ordinary quantum systems.

For an  $n$ -qubit system comprising basis states  $\{|i_1, \dots, i_n\rangle\}$ , with  $i_k = 1, 2$ , the time evolution operator with projection onto generic initial states gives complicated entanglement structure. However projecting onto thermofield double type initial states, we obtain

$$\begin{aligned} |I\rangle &= \sum_{i=1,2} c_{ii\dots i} |ii\dots i\rangle : \\ \rho_t^{I,A} &= \frac{1}{\sum_i |c_{ii\dots i}|^2 e^{-iE_{ii\dots i}t}} \\ &\times \sum_{i=1}^2 |c_{ii\dots i}|^2 e^{-iE_{ii\dots i}t} |i\rangle\langle i|, \end{aligned} \tag{4.8}$$

which is identical to the 2-qubit case. It is clear that any qubit system has identical entanglement structure for the time evolution operator with projection onto thermofield double type states. Now if we additionally restrict to maximal entangle-

ment, we have both  $|c_{ii\dots i}|^2$  equal so  $|c_{ii\dots i}|^2 = \frac{1}{2}$ . This again contains just one nontrivial phase thereby leading to the entanglement structure of the time evolution operator for the 2-state case, *i.e.* (3.2).

### 5 Time evolution operator, normalized at $t = 0$

In this section, we will discuss aspects of the time evolution operator with normalization at  $t = 0$  (rather than at general time  $t$ ), following [12]. This gives

$$\begin{aligned} \rho_t^0(t) &\equiv \frac{\mathcal{U}(t)}{\text{Tr}\mathcal{U}(0)} \\ \rightarrow \rho_t^{0,A} &= \text{tr}_B \rho_t \rightarrow S_A = -\text{tr}(\rho_t^A \log \rho_t^A). \end{aligned} \tag{5.1}$$

The normalization ensures that we obtain ordinary entanglement structures at  $t = 0$ . In this case  $\text{Tr} \rho_t(t) = 1$  at  $t = 0$  but not at general  $t$ . This gives quite different entanglement structures, as we will see.

Since  $\mathcal{U}(0) = \sum_I |I\rangle\langle I| = \mathbf{1}$  *i.e.* the identity operator made up as a sum over all eigenstate projection operators, the normalization factor is  $\text{Tr}\mathcal{U}(0) = N$ , the dimension of the Hilbert space, constant in time. Thus for a general bipartite system we obtain

$$\begin{aligned} \rho_t^0(t) &= \frac{1}{N} \sum_{i,i'} e^{-iE_{i,i'}t} |i, i'\rangle\langle i, i'| \rightarrow \rho_t^{0,A} \\ &= \frac{1}{N} \sum_i \left( \sum_{i'} e^{-iE_{i,i'}t} \right) |i\rangle\langle i|, \end{aligned} \tag{5.2}$$

differing from (2.5) only in the normalization. A general 2-qubit system (3.3) now gives

$$\rho_t^0(t) = \frac{1}{4} \sum_{ij} e^{-iE_{ij}t} |ij\rangle\langle ij| \tag{5.3}$$

and taking a partial trace over the second component gives

$$\begin{aligned} \rho_t^{0,A} &= \frac{1}{4} \left( (e^{-iE_{11}t} + e^{-iE_{12}t}) |1\rangle\langle 1| \right. \\ &\quad \left. + (e^{-iE_{21}t} + e^{-iE_{22}t}) |2\rangle\langle 2| \right) \\ S_t^{0,A} &= -\frac{1}{4} (e^{-iE_{11}t} + e^{-iE_{12}t}) \\ &\quad \times \log \left( \frac{1}{4} (e^{-iE_{11}t} + e^{-iE_{12}t}) \right) \\ &\quad - \frac{1}{4} (e^{-iE_{21}t} + e^{-iE_{22}t}) \\ &\quad \times \log \left( \frac{1}{4} (e^{-iE_{21}t} + e^{-iE_{22}t}) \right). \end{aligned} \tag{5.4}$$

In general  $S_t^{0,A}$  is a complicated complex entropy. However there are special cases. If all energy values are the same, this

simplifies to

$$\begin{aligned} E_{ij} = E_0 : \rho_t &= \frac{e^{-iE_0t}}{4} \sum_{ij} |ij\rangle\langle ij|, \\ \rho_t^{0,A} &= \frac{e^{-iE_0t}}{2} \sum_{i=1,2} |i\rangle\langle i|, \\ S_t^{0,A} &= -e^{-iE_0t} \log \left( \frac{1}{2} e^{-iE_0t} \right) \\ &= (\log 2 + iE_0t) e^{-iE_0t}. \end{aligned} \tag{5.5}$$

Appending a projection operator for a state  $|i\rangle$  as in Sect. 4, we obtain

$$\begin{aligned} \rho_t^{0,|i\rangle} &= \frac{\rho_t^0 |i\rangle\langle i|}{\text{Tr}(\rho_t^0 |i\rangle\langle i|)} = \frac{|f[i](t)\rangle\langle i|}{\text{Tr}(\mathcal{U}(0)|i\rangle\langle i|)} \\ &= \frac{|f[i](t)\rangle\langle i|}{\text{Tr}(|i\rangle\langle i|)}, \end{aligned} \tag{5.6}$$

since  $\mathcal{U}(0)$  is the identity operator. This is similar to (2.2), but differs in normalization. So if the initial state is unit-normalized, the normalization factor is a trivial 1. This is not ordinary entanglement even if the state is an eigenstate since the nontrivial time evolution phase remains. For instance a 2-qubit system (3.3) gives

$$\begin{aligned} |i\rangle = |11\rangle : \rho_t^{0,|i\rangle} &= \frac{\mathcal{U}(t)|11\rangle\langle 11|}{\text{Tr}(|11\rangle\langle 11|)} \\ &= e^{-iE_{11}t} |11\rangle\langle 11|, \end{aligned} \tag{5.7}$$

after projecting onto a simple eigenstate  $|11\rangle$ . The partial trace then gives

$$\begin{aligned} \rho_t^{0,|i\rangle,A} &= \text{Tr}_2 \rho_t^{0,|i\rangle} = e^{-iE_{11}t} |1\rangle\langle 1| \Rightarrow S_t^{0,|i\rangle,A} \\ &= -e^{-iE_{11}t} \log(e^{-iE_{11}t}) = iE_{11}t e^{-iE_{11}t}. \end{aligned} \tag{5.8}$$

The normalization at  $t = 0$  makes this different from ordinary mixed state entanglement structures at finite temperature, although these still resemble imaginary temperature structures. Although it might seem natural to normalize at general  $t$ , part of the motivation here, following [12], is that the time evolution only enters via the final state in (5.6), which apart from this is akin to the pseudo-entropy (2.2), (4.1). This appears to help isolate the timelike characteristics, as in (5.8) where the leading time-dependence is manifestly pure imaginary: it would be interesting to explore this further.

### 6 2-dim CFTs and timelike intervals

The studies of  $dS_3$  extremal surfaces in [11, 12, 19, 20], led to studies of timelike entanglement in ordinary 2-dim CFT (in particular (6.8)): we now elaborate on this (there are parallels with some discussions in [32] which appeared as we were finalizing this paper).



We want to consider the time evolution operator as a density operator towards exploring entanglement-like structures: towards this we define

$$\rho_t[\{\phi(x)\}|\{\phi(x')'\}] = \frac{1}{Z_t} \langle \{\phi(x)\} | e^{-itH} | \{\phi(x')'\} \rangle \quad (6.1)$$

with  $Z_t = \text{Tr } e^{-itH}$ . However rendering this well-defined is best done in the Euclidean path integral formulation, defining the ground state wavefunction for the configuration  $\phi(x')$  as

$$\begin{aligned} \Psi[\{\phi(x')\}] &= \int_{t_E=-\infty}^{\phi(t_E=0,x)=\phi(x')} D\phi e^{-S_E} \\ &= \int_{t_E=-\infty}^{t_E=0} D\phi e^{-S_E} \prod_x \delta(\phi(t_E=0, x) - \phi(x')) \end{aligned} \quad (6.2)$$

with  $S_E$  the Euclidean action for the field  $\phi(t_E, x)$  (we model this discussion along the lines of [36–38], and [2]). Now the reduced density matrix for the interval  $A$  is obtained from  $\rho_t[\phi_0(x)|\phi'_0(x')]$  above by performing a partial trace over the environment  $B$  setting  $\phi_0(x) = \phi'_0(x)$ . This becomes

$$\begin{aligned} \rho[\phi(x)_{0+}|\phi(x)_{0-}] &= \frac{1}{Z} \int_{t_E=-\infty}^{t_E=\infty} D\phi e^{-S_E(\phi)} \prod_{x \in A} \delta(\phi(0^+, x) \\ &\quad - \phi(x)_{0+}) \delta(\phi(0^-, x) - \phi(x)_{0-}) \end{aligned} \quad (6.3)$$

In this form there is no sacrosanct meaning to what we define as Euclidean time: the differences for a timelike interval only enter in the analytic continuation to Lorentzian signature eventually. For a free massless 2-dim scalar, the action is  $S_E = \int dt_E dx ((\partial_{t_E}\phi)^2 + (\partial_x\phi)^2)$  and Euclidean evolution appears symmetric between  $t_E, x$ . For the usual spacelike interval, the reduced density matrix involves Euclidean time evolution along  $t_E$ : for a timelike interval on the other hand, the reduced density matrix involves Euclidean time evolution along  $x$  which is regarded as Euclidean time now calculationally. So we have

$$\begin{aligned} \rho_t[\phi(t_E)_{0+}|\phi(t_E)_{0-}] &= \frac{1}{Z_{t_E}} \int_{x=-\infty}^{x=\infty} D\phi e^{-S_E(\phi)} \prod_{t_E \in A} \delta(\phi(t_E, 0^+) \\ &\quad - \phi(t_E)_{0+}) \delta(\phi(t_E, 0^-) - \phi(t_E)_{0-}) \end{aligned} \quad (6.4)$$

Apart from  $x \leftrightarrow t_E$ , this is equivalent to (6.3).

Let us now discuss this in terms of Hamiltonians for a free massless scalar: note that Euclidean and Lorentzian times are related as  $t_E = it$ . For the usual time coordinate  $t$ , the Hamiltonian is  $H_t^+ = \int dx ((\partial_t\phi)^2 + (\partial_x\phi)^2) = \int dx (-\partial_{t_E}\phi)^2 + (\partial_x\phi)^2$ : this is positive definite. Now compactifying  $t_E$  can be used to obtain the reduced density matrix  $\text{Tr}_B e^{-\beta_t H}$  at finite temperature for an interval with width  $\Delta x$ . With  $x$  taken as Euclidean time, we obtain the Hamiltonian  $H_x = \int dt_E ((\partial_{t_E}\phi)^2 - (\partial_x\phi)^2)$ . Now compactifying  $x$  with

periodicity  $\beta_x$  and considering a timelike interval with width  $\Delta t$ , the reduced density matrix becomes

$$\begin{aligned} H_x &= \int dt_E (-\partial_x\phi)^2 + (\partial_{t_E}\phi)^2 \\ &= -i \int dt ((\partial_x\phi)^2 + (\partial_t\phi)^2) \equiv -i H_x^+; \\ \rho_t^A &= \text{Tr}_B e^{-\beta_x H_x} = \text{Tr}_B e^{i\beta_x H_x^+}, \end{aligned} \quad (6.5)$$

so that in terms of the positive definite Hamiltonian  $H_x^+$ , this resembles a thermal reduced density matrix but with imaginary temperature.

The usual replica formulation of entanglement entropy for a single interval proceeds by picking some Euclidean time direction  $\tau_E$  and the interval  $\Delta x \equiv [u, v]$  on that slice, then constructing  $n$  replica copies of the space glued at the interval endpoints and evaluating  $\text{Tr} \rho_A^n$ . The reduced density matrix for the ground state is formulated as above, via Euclidean time evolution, with appropriate boundary conditions for the fields on the replica sheets. Then  $\text{Tr} \rho_A^n$  in the replica space can be mapped to the twist operator 2-point function at the interval endpoints which implement the boundary conditions across the sheets. This finally leads to

$$S_A = - \lim_{n \rightarrow 1} \partial_n \text{Tr} \rho_A^n \rightarrow \frac{c}{6} \log \frac{(\Delta x)^2}{\epsilon^2}. \quad (6.6)$$

The only data that enters this is the central charge of the CFT and the interval in question. When we consider a timelike interval, the above formulation goes through with the only change being that the Euclidean time slice we pick is the spatial slice  $x = \text{const}$  with the interval being  $\Delta t \equiv [u_t, v_t]$ . However now when we continue back to Lorentzian time, we must rotate  $u_t, v_t$  accordingly, so the spacetime interval is

$$\Delta^2 = -(\Delta t)^2 = -(v_t - u_t)^2 \quad (6.7)$$

and the entanglement entropy becomes

$$\begin{aligned} S_A &= \frac{c}{6} \log \frac{\Delta^2}{\epsilon^2} \\ &= \frac{c}{6} \log \frac{-(\Delta t)^2}{\epsilon^2} = \frac{c}{3} \log \frac{\Delta t}{\epsilon} + \frac{c}{6}(i\pi) \end{aligned} \quad (6.8)$$

with the imaginary part arising as  $i\pi = \log(-1)$ . Note that imaginary values also arise in studies of quantum extremal surfaces in de Sitter with regard to the future boundary [39, 40], stemming from timelike-separations.

The discussions above are formulated in terms of Euclidean path integrals with an eventual analytic continuation to obtain timelike interval entanglement. Along the lines of our finite quantum system descriptions, one could consider Lorentzian time evolution explicitly. Towards this consider a CFT on a cylinder, with time running along the axis. The Hamiltonian is  $H_{\text{cyl}} = \frac{\pi}{l}(L_0 + \bar{L}_0 - \frac{c+\bar{c}}{24})$  and the unnormalized time evolution operator becomes  $e^{-iH_{\text{cyl}}t} \sim q^{\sum_n n N_n} |N_n\rangle \langle N_n|$

with  $q = e^{-2it/l}$  for both left/right modes, and the normalization becomes  $\text{Tr} q^{\sum_n n N_n} = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$  (the  $\frac{c+\bar{c}}{24}$  factor cancels with normalization). In the momentum basis, the time evolution operator is an infinite sum of decoupled oscillators. Recalling the case of two uncoupled oscillators (3.14), tracing out all higher mode oscillators leaving only the lowest frequency  $n = 1$  oscillator mode naively gives  $\rho_t^A = \sum_n \frac{q^n}{1-(1-q)} |n\rangle\langle n|$  and  $S_t^A = -\log(1-q) - \frac{q \log q}{1-q}$ , with appropriate limits as described after (3.16). Also, along the lines of Sect. 4, we can study aspects of the time evolution operator along with projection onto initial states. We leave these and related investigations for the future.

### 7 Time entanglement, time-dependent interactions

So far we have considered time-independent Hamiltonians. In these cases we can relate the time evolution operator to the thermal density matrix by the analytic continuation  $\beta \rightarrow it$ , consistent with the expectation that time independence maps to thermal equilibrium. In this section, we consider some special simple examples of time-dependent Hamiltonians: we expect that the time evolution operator will not admit any simple map to some thermal density matrix in such cases (no thermal equilibrium).

We obtain the time evolution operator in the interaction picture by solving the Schrodinger time evolution equations, evolving the state by the time evolution operator

$$|\alpha, t; t_0\rangle_I = U_I(t, t_0) |\alpha, t_0; t_0\rangle_I = \sum c_{ij}(t) |ij\rangle. \tag{7.1}$$

This enables to determine the time evolution operator, where  $|ij\rangle$  are the eigenstates of  $H_0$  (and  $t_0 = 0$ ). Our conventions are those of [41], with the interaction picture time evolution equations of the form  $i\hbar \frac{d}{dt} c_N(t) = \sum_M V_{NM} e^{i\omega_{NM}t} c_M(t)$  with  $\omega_{NM} = E_N - E_M$ .

As a toy example, consider a 2-state system with states  $|1\rangle, |2\rangle$ , and energies  $E_1, E_2$ : then a  $\delta$ -function interaction  $V_{12} = V\delta(t-\epsilon)$  (with  $\epsilon > 0$  an infinitesimal regulator) gives the interaction picture evolution equations (with  $\dot{c}_i = \frac{d}{dt} c_i$ )

$$\begin{aligned} i\hbar \dot{c}_1 &= V_{12} e^{i\omega_{12}t} c_2, & i\hbar \dot{c}_2 &= V_{21} e^{i\omega_{21}t} c_1; \\ i\hbar c_1(t) &= V c_2(\epsilon) + i\hbar c_1(0), \\ i\hbar c_2(t) &= V c_1(\epsilon) + i\hbar c_2(0), \end{aligned} \tag{7.2}$$

where the second line is obtained by integrating across the interaction support at  $t = \epsilon$  (and the phases  $e^{i\omega_{12}t}$  are trivial). Since the time dependence is only nontrivial for  $t = \epsilon$ , we see that  $c_i(t) = c_i(\epsilon)$ , *i.e.* the coefficients remain unchanged for  $t \geq \epsilon$ . Solving for  $c_1(t), c_2(t)$  gives  $\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \rho_{t,I} \begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix}$  with generic initial state  $c_1(0), c_2(0)$ , where the interaction picture time evolution operator is  $\rho_{t,I} = \frac{1}{1+\frac{V^2}{\hbar^2}} (|1\rangle\langle 1| + \frac{V}{i\hbar} |1\rangle\langle 2| + \frac{V}{i\hbar} |2\rangle\langle 1| + |2\rangle\langle 2|)$  (this can also be seen to agree

with time dependent perturbation theory). We now generalize this sort of delta-function coupling interaction to a system of two qubits to study time entanglement.

Consider a simple system of two qubits with the time-dependent interaction

$$V_I(t) = V\delta(t - \epsilon) (|11\rangle\langle 12| + |12\rangle\langle 11|), \tag{7.3}$$

with an infinitesimal regulator  $\epsilon > 0$  (so the impulse interaction is just after  $t = 0$ ). The Hamiltonian  $H_0$  before turning on the interaction ( $t \leq 0$ ) has eigenstates  $|11\rangle, |22\rangle, |12\rangle, |21\rangle$ , and eigenvalues  $E_{11}, E_{22}, E_{12}, E_{21} = E_{12}$ , respectively. The time evolution equations for the coefficients (suppressing the phases), and their integrated versions, are (with  $\hbar = 1$ )

$$\begin{aligned} \frac{d}{dt} c_{11}(t) &= -iV \delta(t - \epsilon) c_{12}(t), \\ \frac{d}{dt} c_{12}(t) &= -iV \delta(t - \epsilon) c_{11}(t), \\ \frac{d}{dt} c_{21}(t) &= 0, \quad \frac{d}{dt} c_{22}(t) = 0, \\ \Rightarrow c_{11}(t) &= c_{11}(0) - iV c_{12}(\epsilon), \\ c_{12}(t) &= c_{12}(0) - iV c_{11}(\epsilon), \\ c_{21}(t) &= c_{21}(0), \quad c_{22}(t) = c_{22}(0). \end{aligned} \tag{7.4}$$

We now note that the  $c_{ij}(t) = c_{ij}(\epsilon)$  for the impulse interaction, where  $t \geq \epsilon$ , since there is no nontrivial time dependence after  $t = \epsilon$ . This then gives

$$\begin{aligned} c_{11}(t) &= \frac{1}{1+V^2} (c_{11}(0) - iV c_{12}(0)), \\ c_{12}(t) &= \frac{1}{1+V^2} (c_{12}(0) - iV c_{11}(0)), \\ c_{21}(t) &= c_{21}(0), \quad c_{22}(t) = c_{22}(0). \end{aligned} \tag{7.5}$$

This gives the interaction picture time evolution operator  $U_I(t, t_0)$  (with  $t_0 = 0$  and  $t > 0$ ) which maps  $\begin{pmatrix} c_{11}(t) \\ c_{12}(t) \end{pmatrix} = U_I(t) \begin{pmatrix} c_{11}(0) \\ c_{12}(0) \end{pmatrix}$  in the  $\{|11\rangle, |12\rangle\}$  subspace, using (7.1). Then the time evolution operator  $U(t) \equiv \tilde{\rho}_t$  in the Schrödinger picture is (with  $\rho_t$  the normalized one)

$$\begin{aligned} \tilde{\rho}_t &= e^{-iH_0t} U_I(t) = \frac{1}{1+V^2} \left( e^{-iE_{11}t} |11\rangle\langle 11| \right. \\ &\quad \left. - iV e^{-iE_{11}t} |11\rangle\langle 12| - iV e^{-iE_{12}t} |12\rangle\langle 11| \right. \\ &\quad \left. + e^{-iE_{12}t} |12\rangle\langle 12| \right) + e^{-iE_{12}t} |21\rangle\langle 21| \\ &\quad \left. + e^{-iE_{22}t} |22\rangle\langle 22|, \right. \\ \rho_t &= \mathcal{N}_V \tilde{\rho}_t, \\ \mathcal{N}_V^{-1} &\equiv \text{Tr}(\tilde{\rho}_t) \\ &= \frac{1}{1+V^2} (e^{-iE_{11}t} + e^{-iE_{12}t}) + e^{-iE_{12}t} + e^{-iE_{22}t}. \end{aligned} \tag{7.6}$$

We now find the reduced time evolution operator by tracing out a qubit.  $\rho_t^A$  arises from tracing out the second qubit in  $\rho_t$ , and  $\rho_t^B$  from tracing out the first qubit:

$$\begin{aligned} \mathcal{N}_V^{-1} \rho_t^A &= \frac{1}{1+V^2} \left( e^{-iE_{11}t} + e^{-iE_{12}t} \right) |1\rangle\langle 1| \\ &\quad + \left( e^{-iE_{12}t} + e^{-iE_{22}t} \right) |2\rangle\langle 2|, \\ \mathcal{N}_V^{-1} \rho_t^B &= \frac{1}{1+V^2} \left( e^{-iE_{11}t} |1\rangle\langle 1| \right. \\ &\quad - iV e^{-iE_{11}t} |1\rangle\langle 2| - iV e^{-iE_{12}t} |2\rangle\langle 1| \\ &\quad \left. + e^{-iE_{12}t} |2\rangle\langle 2| \right) + e^{-iE_{12}t} |1\rangle\langle 1| \\ &\quad + e^{-iE_{22}t} |2\rangle\langle 2|. \end{aligned} \tag{7.7}$$

Note that  $\rho_t^A = \rho_t^B$  for  $V = 0$  is in agreement with Sect. 3 for the 2-qubit system. The entropy associated with  $\rho_t^A$  or  $\rho_t^B$  is complex-valued in general.

Consider now the same 2-qubit system but a more general impulse interaction

$$\begin{aligned} V_I(t) &= V \delta(t - \epsilon) (|11\rangle\langle 12| + |12\rangle\langle 11| \\ &\quad + |21\rangle\langle 22| + |22\rangle\langle 21|). \end{aligned} \tag{7.8}$$

Using (7.1), the interaction picture time evolution equations and the integrated versions are

$$\begin{aligned} \frac{d}{dt} c_{11}(t) &= -iV \delta(t - \epsilon) c_{12}(t), \\ \frac{d}{dt} c_{12}(t) &= -iV \delta(t - \epsilon) c_{11}(t), \\ \frac{d}{dt} c_{21}(t) &= -iV \delta(t - \epsilon) c_{22}(t), \\ \frac{d}{dt} c_{22}(t) &= -iV \delta(t - \epsilon) c_{21}(t), \\ \Rightarrow c_{11}(t) &= c_{11}(0) - iV c_{12}(\epsilon), \\ c_{12}(t) &= c_{12}(0) - iV c_{11}(\epsilon), \\ c_{21}(t) &= c_{21}(0) - iV c_{22}(\epsilon), \\ c_{22}(t) &= c_{22}(0) - iV c_{21}(\epsilon). \end{aligned} \tag{7.9}$$

These are the analogs for the interaction (7.8) of (7.4) with the simpler interaction (7.3). As before, we have  $c_{ij}(t) = c_{ij}(\epsilon)$ ,  $t \geq \epsilon$ , since there is no nontrivial time dependence after the impulse at  $t = \epsilon$ . Solving for  $c_{ij}(t)$  leads here to the Schrödinger picture time evolution operator  $U(t) \equiv \tilde{\rho}_t$  (with  $\rho_t$  the normalized one)

$$\begin{aligned} \tilde{\rho}_t &= e^{-iH_0t} U_I(t) = \frac{1}{1+V^2} \left( e^{-iE_{11}t} |11\rangle\langle 11| \right. \\ &\quad - iV e^{-iE_{11}t} |11\rangle\langle 12| - iV e^{-iE_{12}t} |12\rangle\langle 11| \\ &\quad + e^{-iE_{12}t} |12\rangle\langle 12| + e^{-iE_{12}t} |21\rangle\langle 21| \\ &\quad - iV e^{-iE_{12}t} |21\rangle\langle 22| \\ &\quad \left. - iV e^{-iE_{22}t} |22\rangle\langle 21| \right) \end{aligned}$$

$$+ e^{-iE_{22}t} |22\rangle\langle 22| \Big),$$

$$\begin{aligned} \rho_t &= \mathcal{N}_V \tilde{\rho}_t \\ \mathcal{N}_V^{-1} &\equiv \text{Tr}(\tilde{\rho}_t) \\ &= \frac{1}{1+V^2} (e^{-iE_{11}t} + 2e^{-iE_{12}t} + e^{-iE_{22}t}). \end{aligned} \tag{7.10}$$

Tracing out either the second qubit or the first gives  $\rho_t^A$  or  $\rho_t^B$ :

$$\begin{aligned} \rho_t^A &= \mathcal{N}_V \frac{1}{1+V^2} \left( (e^{-iE_{11}t} + e^{-iE_{12}t}) |1\rangle\langle 1| \right. \\ &\quad \left. + (e^{-iE_{12}t} + e^{-iE_{22}t}) |2\rangle\langle 2| \right), \\ \rho_t^B &= \mathcal{N}_V \frac{1}{1+V^2} \left( e^{-iE_{11}t} \right. \\ &\quad + e^{-iE_{12}t} |1\rangle\langle 1| - iV (e^{-iE_{11}t} + e^{-iE_{12}t}) |1\rangle\langle 2| \\ &\quad - iV (e^{-iE_{12}t} + e^{-iE_{22}t}) |2\rangle\langle 1| \\ &\quad \left. + (e^{-iE_{12}t} + e^{-iE_{22}t}) |2\rangle\langle 2| \right). \end{aligned} \tag{7.11}$$

Note that here the  $\frac{1}{1+V^2}$  factors cancel with that in  $\mathcal{N}_V$  (which is an accident; this would not occur if the interaction strengths in (7.8) were not uniformly  $V$  for all matrix elements). As for (7.7), we see that these reduced time evolution operators are equal,  $\rho_t^A = \rho_t^B$ , for  $V = 0$ , in agreement with Sect. 3. These give complex-valued entropy in general, although there are special cases with real entropy: e.g. for  $E_{11} = E_{22} = E_{12}$  we obtain  $\rho_t^B = \frac{1}{2} \begin{pmatrix} 1 & -iV \\ -iV & 1 \end{pmatrix}$  with eigenvalues  $\lambda_k = \frac{1}{2}(1 \pm iV)$ : then the entropy  $S_t^B = -\sum_k \lambda_k \log \lambda_k$  becomes real-valued giving  $S_t^B = \log 2 - \frac{1}{2}(1+iV) \log(1+iV) - \frac{1}{2}(1-iV) \log(1-iV)$ .

We now look at this time evolution operator with projection onto some initial state, along the lines of Sect. 4. First consider a thermofield-double type initial state  $|I\rangle = \sum_{i=1,2} c_{ii} |ii\rangle$  as in Sect. 4.1: this gives (with  $\mathcal{N}$  the normalization)

$$\begin{aligned} \mathcal{N} \rho_t |I\rangle\langle I| &= \frac{\mathcal{N}}{1+V^2} (\rho_t |I\rangle\langle I|) \Big|_{V=0} \\ &\quad - \mathcal{N} \frac{iV e^{-iE_{12}t}}{1+V^2} \left( |c_{11}|^2 |12\rangle\langle 11| \right. \\ &\quad + c_{11} c_{22}^* |12\rangle\langle 22| \\ &\quad \left. + c_{11}^* c_{22} |21\rangle\langle 11| + |c_{22}|^2 |21\rangle\langle 22| \right). \end{aligned} \tag{7.12}$$

A partial trace over the second or first qubit gives, respectively,

$$\begin{aligned} \rho_{t,I}^A &= \mathcal{N} \frac{1}{1+V^2} \rho_t^A \Big|_{V=0} - \mathcal{N} \frac{iV e^{-iE_{12}t}}{1+V^2} \\ &\quad \times \left( c_{11} c_{22}^* |1\rangle\langle 2| + c_{11}^* c_{22} |2\rangle\langle 1| \right), \end{aligned}$$

$$\rho_{t,I}^B = \mathcal{N} \frac{1}{1+V^2} \rho_t^B|_{V=0} - \mathcal{N} \frac{iV e^{-iE_{12}t}}{1+V^2} \times \left( |c_{11}|^2 |2\rangle\langle 1| + |c_{22}|^2 |1\rangle\langle 2| \right). \quad (7.13)$$

This thus leads to nontrivial contributions to the complex-valued entropy stemming from the impulse interaction controlled by the strength  $V$ . For special cases the entropy is real: *e.g.*  $E_{11} = E_{22} = E_{12}$  with maximally entangled initial state  $c_{11} = c_{22} = \frac{1}{\sqrt{2}}$  gives  $\rho_{t,I}^A = \rho_{t,I}^B = \frac{1}{2} \begin{pmatrix} 1 & -iV \\ -iV & 1 \end{pmatrix}$  with eigenvalues  $\lambda_k = \frac{1}{2}(1 \pm iV)$  leading to real entropy  $S_t^B = -\sum_k \lambda_k \log \lambda_k$ .

This is essentially the pseudo-entropy for the initial state  $|I\rangle = c_{11}|11\rangle + c_{22}|22\rangle$  and its time evolved final state using  $\tilde{\rho}_t$  in (7.10)

$$|F\rangle = \tilde{\rho}_t |I\rangle = \frac{1}{1+V^2} \left( e^{-iE_{11}t} c_{11} |11\rangle + e^{-iE_{22}t} c_{22} |22\rangle - iV e^{-iE_{12}t} c_{11} |12\rangle - iV e^{-iE_{12}t} c_{22} |21\rangle \right). \quad (7.14)$$

If on the other hand, one considers some initial state within the  $\{|11\rangle, |12\rangle\}$  subspace, then it turns out that  $\rho_{t,I}^A \propto |1\rangle\langle 1|$  while  $\rho_{t,I}^B$  has eigenvalues 0, 1 (perhaps this is not surprising since any state in this subspace is of a factorized form  $|1\rangle_A (a|1\rangle + b|2\rangle)_B$ ). This leads to vanishing pseudo-entropy for  $\rho_{t,I}^A$  and  $\rho_{t,I}^B$ .

We have illustrated the time evolution operator and its time entanglement structure focussing on simple 2-qubit examples involving an impulse  $\delta$ -function interaction. We have obtained the time evolution operator by solving the time evolution Schrodinger equation for the state coefficients. The time-dependence of the interaction leads to nontrivial dependence on the interaction strength  $V$ , in addition to the dependence on the energy eigenvalues and the timelike separation  $t$ . No simple continuation via some imaginary temperature exists here, unlike the discussions in the rest of the paper with time-independent quantum systems. It is likely that general time-dependent quantum systems will exhibit similar features. Perhaps there are deeper ways to formulate timelike entanglement, which make more explicit a partial trace over time paths or histories.

## 8 Discussion

We have studied various aspects of entanglement like structures with timelike separations arising from the time evolution operator regarded as a density operator, following [12]. There are close parallels with pseudo-entropy [23] as we have seen. The entropy from the time evolution operator along with projection onto some initial state as we have seen in Sect. 4 is identical to pseudo-entropy for the initial state and its time-evolved final state. More broadly, there are large parallels of the investigations here and in [12] with corresponding ones

in [11, 32]. In general the non-Hermitian structures here give complex-valued entropy, although there are several interesting real-valued subfamilies *e.g.* (3.2), special subcases of (3.5) and (4.7), qubit chains Appendix B with the  $|1\rangle \leftrightarrow |2\rangle$  exchange symmetry, and so on. The behaviour of this entropy is quite different from usual spatial entanglement entropy: for instance, (3.2) oscillates in time and appears to grow large at specific time values. Correspondingly at other specific periodic time values the entropy acquires its minimum value, coinciding with ordinary entanglement entropy for the initial state (see Sect. 4.1 in the context of thermofield-double states, akin to Bell pair states). Overall these appear to be new entanglement-like measures involving timelike separations, likely with many new aspects open for exploring further. (It is also worth noting other work *e.g.* [42–45], which may have bearing on this broad circle of ideas.)

While more detailed understanding and physical interpretation of time entanglement in general is yet to be developed, the mapping to pseudo-entropy allows certain connections to previously studied quantities. Pseudo-entropy stems from the transition matrix  $\mathcal{T}_{F|I}$  in (2.4), (2.7), regarded as a generalized density operator involving a preparation state and a postselected state. Related quantities pertain to weak values of operators, obtained as  $\mathcal{O}_w = \text{Tr}(\mathcal{T}_{F|I} \mathcal{O})$ . These are in general complex-valued, not surprising since the transition matrix is not a hermitian object (unlike ordinary hermitian density matrices). See *e.g.* [46, 47] for more on postselected states, conditional entropy and weak values (including some experimental aspects). In the current context, components of the time evolution operator can be isolated via projections onto specific initial states as we have seen in Sect. 4: this then maps onto the corresponding pseudo-entropy. Thus time entanglement with projection onto initial state  $|I\rangle$  dovetails with postselected states being the corresponding time-evolved states. We hope to obtain more refined understanding of these interrelations in the future.

The finite quantum systems we have studied allow analysis using Hamiltonian eigenstates and are thus intrinsically straightforward. Time-independent Hamiltonians allow mapping the time evolution operator to a thermal density matrix by the analytic continuation  $\beta \rightarrow it$ , consistent with the expectation that time independence can be mapped to thermal equilibrium. We expect that in cases with nontrivial time dependence, these time-entanglement structures will become more intricate with no natural imaginary temperature analytic continuation: along the lines of studies of scattering amplitudes, we expect that analogs of the interaction picture will be useful in organizing these time entanglement structures. All these are vindicated in the simple 2-qubit examples with  $\delta$ -function impulse potentials (Sect. 7), where we solve explicitly for the nontrivial time evolution operator and the corresponding time entanglement structures. Related, com-

plementary studies (including holographic ones) appear in [11,23–34]. We hope to report further on these in the future.

We now make a few remarks on de Sitter extremal surfaces anchored at the future boundary, which have time-like components, in particular paraphrasing some discussions in [48]. The  $dS/CFT$  dictionary [9]  $Z_{CFT} = \Psi_{dS}$  implies that boundary entanglement entropy is bulk pseudo-entropy (since a replica formulation on  $Z_{CFT}$  amounts to one on  $\Psi_{dS}$ , i.e. single ket rather than a density matrix). Among other things this leads to novel entropy relation/inequalities based on the complex-valued  $dS$  extremal surface areas. This is put in perspective by comparing with time-entanglement/pseudo-entropy in qubit systems, using the analyses in this paper, in particular Sect. 4: this reveals striking differences for mutual time-information, tripartite information and strong subadditivity (see Sect. 2.5 in [48]). The  $dS$  areas give definite signs for these quantities relative to those obtained from time-entanglement/pseudo-entropy for qubit systems (with the final state being time-evolved from the initial state). Since the  $dS$  areas are analytic continuations from  $AdS$ , these differences are perhaps not surprising in light of the studies in [49] (which reveal definite signs the  $AdS$  RT surface area inequalities compared with those for entanglement entropy in qubit systems), but they are striking. Overall there are new entanglement structures here stemming from timelike separations: we expect that the investigations here and related ongoing ones will lead to further insights into both quantum information and holography.

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**Appendix A: Time evolution, pseudo-entropy: special cases**

Consider now the pseudo-entropy transition matrix (2.4) for the 2-state case (3.1), with arbitrary initial state  $|i\rangle$  and arbitrary final state  $|f\rangle$ ,

$$|i\rangle = c_1|1\rangle + c_2|2\rangle, \quad |f\rangle = c'_1|1\rangle + c'_2|2\rangle;$$

$$\mathcal{T}_{f|i} = \frac{1}{c'_1c_1^* + c'_2c_2^*} \left( c'_1c_1^*|1\rangle\langle 1| + c'_2c_2^*|2\rangle\langle 2| + c'_1c_2^*|1\rangle\langle 2| + c'_2c_1^*|2\rangle\langle 1| \right). \tag{A.1}$$

With  $|1\rangle \equiv |++\rangle, |2\rangle \equiv |--\rangle$ , a partial trace over the second component gives

$$\mathcal{T}_{f|i}^A = \frac{1}{c'_1c_1^* + c'_2c_2^*} \left( c'_1c_1^*|+\rangle\langle +| + c'_2c_2^*|-\rangle\langle -| \right) \tag{A.2}$$

as the reduced transition matrix. To compare with entanglement for the time evolution operator, we take the final state to be time-evolved from some other initial state  $|i'\rangle$  so

$$|f\rangle = c'_1e^{-iE_1t}|1\rangle + c'_2e^{-iE_2t}|2\rangle \rightarrow$$

$$\mathcal{T}_{f|i}^A = \frac{(c'_1c_1^*|+\rangle\langle +| + c'_2c_2^*e^{i\theta}|-\rangle\langle -|)}{c'_1c_1^* + c'_2c_2^*e^{i\theta}}, \tag{A.3}$$

with  $\theta = -(E_2 - E_1)t$ . Then we see that:

- using (3.2) for the time evolution operator,  $\mathcal{T}_{f|i}^A = \rho_i^A$  if  $c_1 = c'_1 = \frac{1}{\sqrt{2}}, c_2 = c'_2 = \frac{1}{\sqrt{2}}$ , i.e. the initial and final states are identical maximally entangled states.
- using (4.3) for the time evolution operator with projection,  $\mathcal{T}_{f|i}^A = \rho_i^{(i)}$  if  $c'_1 = c_1, c'_2 = c_2$ , i.e.  $|f\rangle = |f[i]\rangle$  i.e. the final state is time-evolved from the initial state  $|i'\rangle = |i\rangle$ .

This structure of mapping  $\mathcal{T}_{f|i}^A = \rho_i^A$  however is not true more generally. For instance, consider two qubits more generally, as in (3.3). Then the pseudo-entropy transition matrix (2.4) becomes

$$|I\rangle = \sum_{i,j=1}^2 c_{ij}|ij\rangle, \quad |F\rangle = \sum_{i,j=1}^2 c'_{ij}|ij\rangle;$$

$$\mathcal{T}_{F|I} = \frac{1}{\sum_{ij} c'_{ij}c_{ij}^*} \sum_{i,j,k,l=1}^2 c'_{ij}c_{kl}^* |ij\rangle\langle kl| \tag{A.4}$$

and partial trace over the 2nd component gives the reduced transition matrix as

$$\mathcal{T}_{F|I}^A = \frac{1}{\sum_{ij} c'_{ij}c_{ij}^*} \sum_{i,k=1}^2 \left( \sum_j c'_{ij}c_{kj}^* \right) |i\rangle\langle k|$$

$$= \frac{1}{\sum_{ij} c'_{ij}c_{ij}^*} \left( (c'_{11}c_{11}^* + c'_{12}c_{12}^*)|1\rangle\langle 1| \right)$$



$$\begin{aligned}
 &+(c'_{11}c_{21}^* + c'_{12}c_{22}^*)|1\rangle\langle 2| + (c'_{21}c_{11}^* + c'_{22}c_{12}^*)|2\rangle\langle 1| \\
 &+(c'_{21}c_{21}^* + c'_{22}c_{22}^*)|2\rangle\langle 2|). \tag{A.5}
 \end{aligned}$$

Towards comparing with the time evolution operator, we think of the future state as time-evolved from some initial state, *i.e.*  $|F\rangle = \sum_{ij} c'_{ij} e^{-iE_{ij}t} |ij\rangle$ . It is then clear that pseudo-entropy via the reduced transition matrix matches time entanglement via the normalized time evolution operator with projection onto  $|i\rangle$ , *i.e.*  $\mathcal{T}_{f|iv}^A = \rho_t^{i|,A}$  if the final state is taken to be time-evolved from the initial state, *i.e.*  $|F\rangle = \mathcal{U}(t)|I\rangle$  so  $c'_{ij} = c_{ij} e^{-iE_{ij}t}$ . However, in contrast with (A.3), the fact that there are off-diagonal terms in (A.5) makes the structure different from the reduced time evolution operator. To set the off-diagonal terms to vanish, we could consider specializing to maximally entangled thermofield-double type initial and final states, and with  $|F\rangle$  time-evolved from  $|I\rangle$ , *i.e.*  $|I\rangle = \sum_{ii} c_{ii} |ii\rangle$  with  $c_{ij}, c'_{ij} = 0, i \neq j, c_{ii} = c_{jj} \forall i, j$ , and  $|F\rangle = \sum_{ii} c'_{ii} |ii\rangle = \mathcal{U}(t)|I\rangle$ . In this case, we find that all the off-diagonal terms vanish and we obtain the reduced transition matrix to be of the same form as in (A.3). On the other hand the reduced time evolution operator for the general 2-qubit case is (3.5), which has two distinct phases in general. Thus the reduced transition matrix differs from the reduced time evolution operator. One can engineer special energy values  $E_{ij}$  where the two coincide (although this appears ad hoc).

Of course, these structures are with a single Hilbert space for constructing both initial and final states. Doubling the Hilbert spaces directly enables a map from the transition matrix to the time evolution operator in general, as in Sect. 2.1.

### Appendix B: Qubit chains

Now we consider qubit chains to understand time entanglement structures. For any nearest neighbour 2-qubit pair, we impose nearest-neighbour interactions, with

$$\begin{aligned}
 s|q\rangle &= a_q|q\rangle, |q\rangle = \{|1\rangle, |2\rangle\}; H = -Js_1s_2, \\
 H[11] &= E_{11} = -Ja_1^2, H[22] = E_{22} = -Ja_2^2, \\
 H[12] &= H[21] = E_{12} = -Ja_1a_2. \tag{B.1}
 \end{aligned}$$

In the first line, we are defining operators  $s_i$  with action as above (the  $i$  being the site label), that give the qubit Hamiltonian action elaborated on in the second line. This Hamiltonian generalizes the 2-qubit case (3.3) earlier. (Imposing a  $|1\rangle \leftrightarrow |2\rangle$  exchange symmetry simplifies this to Ising-like interactions, as we will discuss later.)

**3-qubit chain:** Consider now a chain of 3 qubits with Hamiltonian based on the nearest neighbour 2-qubit interac-

tion above. This gives the 3-qubit chain Hamiltonian as

$$\begin{aligned}
 H &= -J(s_1s_2 + s_2s_3) \\
 H &\equiv E_1|I\rangle\langle I| = E_1|111\rangle\langle 111| + E_2|222\rangle\langle 222| \\
 &+ E_5(|121\rangle\langle 121| + |212\rangle\langle 212|) \\
 &+ E_3(|112\rangle\langle 112| + |211\rangle\langle 211|) \\
 &+ E_4(|122\rangle\langle 122| + |221\rangle\langle 221|), \\
 E_1 &= -2Ja_1^2 = 2E_{11}, \\
 E_2 &= -2Ja_2^2 = 2E_{22}, \\
 E_5 &= -2Ja_1a_2 = 2E_{12}, \\
 E_3 &= -Ja_1^2 - Ja_1a_2 = E_{11} + E_{12}, \\
 E_4 &= -Ja_1a_2 - Ja_2^2 = E_{22} + E_{12}, \\
 E_4 - E_3 &= \frac{1}{2}(E_2 - E_1), \\
 E_1 + E_5 &= 2E_3, \quad E_2 + E_5 = 2E_4. \tag{B.2}
 \end{aligned}$$

Then the time evolution operator  $\mathcal{U}(t)$  after normalizing becomes

$$\begin{aligned}
 \rho_t &= \frac{1}{e^{-iE_1t} + e^{-iE_2t} + 2e^{-iE_3t} + 2e^{-iE_4t} + 2e^{-iE_5t}} \\
 &\times \sum_I e^{-iE_{It}t} |I\rangle\langle I| \equiv \mathcal{N} \sum_I e^{-iE_{It}t} |I\rangle\langle I|. \tag{B.3}
 \end{aligned}$$

Now tracing out the 1st and 3rd qubit states gives the reduced time evolution operator

$$\begin{aligned}
 (\rho_t^A)_{11} &= \mathcal{N} (e^{-iE_1t} + 2e^{-iE_3t} + e^{-iE_5t}) \\
 (\rho_t^A)_{22} &= \mathcal{N} (e^{-iE_2t} + 2e^{-iE_4t} + e^{-iE_5t}) \tag{B.4}
 \end{aligned}$$

for the middle qubit. Using the relations between the  $E_i$  in (B.2) simplifies this to

$$\begin{aligned}
 (\rho_t^A)_{11} &= \mathcal{N} (e^{-iE_{11}t} + e^{-iE_{12}t})^2, \\
 (\rho_t^A)_{22} &= \mathcal{N} (e^{-iE_{22}t} + e^{-iE_{12}t})^2, \\
 \mathcal{N}^{-1} &= \text{Tr} \mathcal{U}(t) = (e^{-iE_{11}t} + e^{-iE_{12}t})^2 \\
 &+ (e^{-iE_{22}t} + e^{-iE_{12}t})^2. \tag{B.5}
 \end{aligned}$$

In general, this is a function of three independent parameters  $E_{11}, E_{22}, E_{12}$  (or equivalently  $E_1, E_2, E_5$ ) so it is a complex-valued function of three phases in general. A straightforward real slice is obtained when there is a  $|1\rangle \leftrightarrow |2\rangle$  exchange symmetry as we will discuss later.

**5-qubit chain:** the configurations and their energies are

$$\begin{aligned}
 &|11111\rangle, 4E_{11}; |22222\rangle, 4E_{22}; |12121\rangle, |21212\rangle, \\
 &4E_{12}; |11112\rangle, |11122\rangle, |11222\rangle, |12222\rangle, 3E_{11} + E_{12}; \\
 &|22221\rangle, |22211\rangle, |22111\rangle, |21111\rangle, 3E_{22} + E_{12}; \\
 &|11121\rangle, |11211\rangle, |12111\rangle, |21112\rangle, 2E_{11} + 2E_{12}; \\
 &|12221\rangle, |22212\rangle, |22122\rangle, |21222\rangle, 2E_{22} + 2E_{12}; \\
 &|11221\rangle, |12211\rangle, |22112\rangle, |21122\rangle, E_{11} + E_{22} \\
 &+ 2E_{12}; |11212\rangle, |12112\rangle, |21211\rangle, |21121\rangle, E_{11} + 3E_{12}; \\
 &|12122\rangle, |12212\rangle, |22121\rangle, |21221\rangle, E_{22} + 3E_{12};
 \end{aligned}$$



(B.6)

Tracing over all but the middle (3rd) qubit gives the reduced time evolution operator as

$$\begin{aligned}
 (\tilde{\rho}_t)_{11}^{(3)} &= e^{-i(4E_{11})t} + e^{-i(4E_{12})t} + 2e^{-i(3E_{11}+E_{12})t} \\
 &\quad + 2e^{-i(3E_{22}+E_{12})t} + 2e^{-i(E_{11}+E_{22}+2E_{12})t} \\
 &\quad + 3e^{-i(2E_{11}+2E_{12})t} + e^{-i(2E_{22}+2E_{12})t} \\
 &\quad + 2e^{-i(E_{11}+3E_{12})t} + 2e^{-i(E_{22}+3E_{12})t}, \\
 (\tilde{\rho}_t)_{22}^{(3)} &= e^{-i(4E_{22})t} + e^{-i(4E_{12})t} + 2e^{-i(3E_{22}+E_{12})t} \\
 &\quad + 2e^{-i(3E_{11}+E_{12})t} + 2e^{-i(E_{11}+E_{22}+2E_{12})t} \\
 &\quad + 3e^{-i(2E_{22}+2E_{12})t} + e^{-i(2E_{11}+2E_{12})t} \\
 &\quad + 2e^{-i(E_{22}+3E_{12})t} + 2e^{-i(E_{11}+3E_{12})t}, \tag{B.7}
 \end{aligned}$$

where the tilde denotes un-normalized. The normalization of the time evolution operator here becomes

$$\begin{aligned}
 \mathcal{N}_5^{-1} &= \text{Tr } \tilde{\rho}_t^{(3)} \\
 &= \text{Tr } \mathcal{U}(t) = (\tilde{\rho}_t)_{11}^{(3)} + (\tilde{\rho}_t)_{22}^{(3)} \tag{B.8}
 \end{aligned}$$

In general the resulting von Neumann entropy is a complicated complex-valued function of the three energy parameters  $E_{11}, E_{22}, E_{12}$ .

There are parallels between our discussions here on qubit chain configurations and those in [50] on ghost-spin chains (although the context is different).

*Infinite qubit chain:* Consider now an infinite 1-dim chain of qubits, again with only nearest-neighbour interactions, the Hamiltonian being

$$H = -J \sum_n s_n s_{n+1} = \dots - J s_{-1} s_0 - J s_0 s_1 + \dots \tag{B.9}$$

We can focus on the qubit at location  $n = n_0$  as the subsystem in question, tracing over all the other qubits in the chain. The reduced time evolution operator is

$$\rho_t = \frac{1}{\sum_I e^{-iE[I]t}} \sum_{n_0=1,2} \left( \sum_{I; n \neq 0} e^{-iE[I]t} \right) |n_0\rangle \langle n_0| \tag{B.10}$$

This is a complicated object in general, although still simply a complex-valued function of the three energy parameters  $E_{11}, E_{22}, E_{12}$ . Since this qubit only interacts directly with its two neighbours, the effective system has some parallels with the 3-qubit chain above: but the detailed structure is complicated, as already evident in the 5-qubit case earlier.

$|1\rangle \leftrightarrow |2\rangle$  exchange symmetry: In the simple subcase enjoying  $|1\rangle \leftrightarrow |2\rangle$  exchange symmetry, there are substantial simplifications in (B.1): this is when there is an Ising-like structure, with

$$a_1 = -a_2 = 1; \quad E_{11} = E_{22} = -E_{12} = -J. \tag{B.11}$$

For instance the 3-qubit case (B.5) simplifies to

$$\mathcal{N}_3^{-1} = 2(e^{iJt} + e^{-iJt})^2,$$

$$(\rho_t^A)_{11} = (\rho_t^A)_{22}$$

$$= \mathcal{N}_3 (e^{iJt} + e^{-iJt})^2 = \frac{1}{2} \tag{B.12}$$

which thus gives von Neumann entropy  $\log 2$ . Likewise the 5-qubit (B.7) case can be seen to simplify to

$$\begin{aligned}
 \mathcal{N}_5^{-1} &= 2(e^{iJt} + e^{-iJt})^4, \\
 (\rho_t^A)_{11} &= (\rho_t^A)_{22} = \mathcal{N}_5 (e^{iJt} + e^{-iJt})^4 = \frac{1}{2} \tag{B.13}
 \end{aligned}$$

so the middle qubit has identical structure. For an infinite qubit chain with this Ising-like  $\mathbb{Z}_2$  symmetry, we expect translation invariance in the “bulk” so we expect that the reduced time evolution operator has again similar structure. Considering an  $N$ -qubit chain (towards large  $N$ ), the configurations can be organized similar to (B.6). It is then clear that the ground states are  $|11 \dots 11\rangle, |22 \dots 22\rangle$ , with energy  $-(N-1)J$ . The first excited states comprise “one kink” states with exactly one 12- or 21-interface with energy  $-(N-3)J$  and degeneracy  $2(N-1)$ . The next set of excited states contain two kinks, so the energy is  $-(N-5)J$  with degeneracy  $4(N-2)$ . Higher excited states contain multiple 12- or 21-interfaces. The two highest energy states have maximally alternating 1, 2s, *i.e.*  $|12121\dots\rangle, |21212\dots\rangle$ : there are  $(N-1)$  interfaces giving energy  $(N-1)J$ . Furthermore, every energy  $E$  (with corresponding configurations) comes in pairs, *i.e.* there are corresponding configurations with energy  $-E$ . This can be seen above, with the ground states and highest energy states: likewise, corresponding to the one kink states, we have states with energy  $(N-3)J$  obtained by transforming one of the 12- or 21-interfaces in the highest energy states to 11 or 22, which then lowers the energy precisely by  $2J$  (and their degeneracy can be checked easily). Thus the normalization of the time evolution operator (akin to the partition function) is  $\mathcal{N}_N^{-1} = \text{Tr } \tilde{\rho}_t$ , *i.e.*

$$\begin{aligned}
 \mathcal{N}_N^{-1} &= 2(e^{iJt(N-1)} + (N-1)e^{iJt(N-3)} \\
 &\quad + \dots + (N-1)e^{-iJt(N-3)} + e^{-iJt(N-1)}) \\
 &= 2(e^{iJt} + e^{-iJt})^{N-1}. \tag{B.14}
 \end{aligned}$$

Each component of the reduced time evolution operator for some bulk qubit can be explicitly seen to receive contributions equally from half these states: so we obtain

$$(\rho_t^A)_{11} = (\rho_t^A)_{22} = \mathcal{N}_N (e^{iJt} + e^{-iJt})^{N-1} = \frac{1}{2} \tag{B.15}$$

which is identical to the structure of the middle qubit in the previous finite qubit cases.

Note that it is adequate to require  $E_{11} = E_{22}$  to implement this  $|1\rangle \leftrightarrow |2\rangle$  exchange symmetry: then shifting the energies arrives at the symmetric values in (B.11). However if keep  $E_{12}$  independent of  $E_{11} = E_{22}$  then there are apparently two independent parameters: however it is straightforward

to see that the reduced time evolution operator, while non-Hermitian, nevertheless leads to real-valued von Neumann entropy. It is likely that similar studies can be extended for “ghost-spin” models such as those in [50,51].

All of the above structures can be seen to match ordinary finite temperature entanglement, except with imaginary temperature  $\beta = it$ .

### Appendix C: Two coupled oscillators

We consider the following Hamiltonian  $H$  with unit masses  $m_A = m_B = 1$ ,

$$H = \frac{1}{2} (p_A^2 + p_B^2) + \frac{k_1}{2} (x_A^2 + x_B^2) + \frac{k_2}{2} (x_A - x_B)^2. \tag{C.1}$$

This is slightly different from the coupled oscillators case discussed in [23]. We diagonalise the Hamiltonian in a coordinate basis  $\{y_1, y_2\}$  as below. Then the hamiltonian (C.1) becomes

$$H = \left( \frac{1}{2} p_1^2 + \frac{1}{2} \Omega_1^2 y_1^2 \right) + \left( \frac{1}{2} p_2^2 + \frac{1}{2} \Omega_2^2 y_2^2 \right),$$

$$y_1 = \frac{(x_A + x_B)}{\sqrt{2}} ; y_2 = \frac{(x_A - x_B)}{\sqrt{2}} \tag{C.2}$$

where  $\Omega_1 = \sqrt{k_1}$ ,  $\Omega_2 = \sqrt{k_1 + 2k_2}$ . The energy eigenvalues and eigenfunctions of (C.2) are labelled by  $E_{n_1 n_2}$ , and  $\phi_{n_1 n_2}(y_1, y_2)$  respectively,

$$E_{n_1 n_2} = (n_1 + \frac{1}{2})\Omega_1 + (n_2 + \frac{1}{2})\Omega_2$$

$$= E_{n_1} + E_{n_2} ; \phi_{n_1 n_2}(y_1, y_2)$$

$$= \phi_{n_1}(y_1) \phi_{n_2}(y_2), \tag{C.3}$$

where  $n_1, n_2$  take values from 0 to  $\infty$  and  $E_{n_1} = (n_1 + \frac{1}{2})\Omega_1$ ,  $E_{n_2} = (n_2 + \frac{1}{2})\Omega_2$ .

We now write the time evolution operator in its eigenbasis as follows

$$e^{-iHt} = \rho(t) = \sum_{n_1, n_2} e^{-i E_{n_1 n_2} t}$$

$$\times |\phi_{n_1 n_2}\rangle \langle \phi_{n_1 n_2}|. \tag{C.4}$$

In position space

$$\rho(y_1, y_2; y'_1, y'_2, t)$$

$$= \sum_{n_1, n_2} e^{-i E_{n_1 n_2} t} \phi_{n_1 n_2}(y_1, y_2) \phi_{n_1 n_2}^*(y'_1, y'_2) ,$$

$$= \sum_{n_1, n_2} e^{-i (E_{n_1} + E_{n_2}) t} \phi_{n_1}(y_1) \phi_{n_1}^*(y'_1) \phi_{n_2}(y_2) \phi_{n_2}^*(y'_2) ,$$

$$= \rho_1(y_1; y'_1, t) \rho_2(y_2; y'_2, t) . \tag{C.5}$$

We have applied (C.3) in the first line of (C.5), and

$$\rho_1(y_1; y'_1, t) = \sum_{n_1} e^{-i E_{n_1} t} \phi_{n_1}(y_1) \phi_{n_1}^*(y'_1) ;$$

$$\rho_2(y_2; y'_2, t) = \sum_{n_2} e^{-i E_{n_2} t} \phi_{n_2}(y_2) \phi_{n_2}^*(y'_2) . \tag{C.6}$$

(C.5) shows that the time evolution operator  $\rho(t)$  is decomposed as  $\rho(t) = \rho_1(t) \otimes \rho_2(t)$ . The energy eigenstate for a single harmonic oscillator of frequency  $\Omega$  (setting  $m = 1$ ) is

$$\phi_n(x) = \frac{1}{\sqrt{2^n n!}} \left( \frac{\Omega}{\pi} \right)^{\frac{1}{4}} e^{-\frac{\Omega x^2}{2}} H_n(\sqrt{\Omega} x) ;$$

$$E_n = (n + \frac{1}{2})\Omega. \tag{C.7}$$

We now use Mehler’s formula for Hermite polynomials [52]

$$\sum_{n=0}^{\infty} \frac{(\frac{\alpha}{2})^n}{n!} H_n(X) H_n(Y)$$

$$= \frac{1}{\sqrt{1 - \alpha^2}} e^{\frac{-\alpha^2 (X^2 + Y^2) + 2\alpha XY}{1 - \alpha^2}} . \tag{C.8}$$

We now consider the time evolution operator for a single harmonic oscillator of frequency  $\Omega$  in order to calculate (C.5):

$$\rho(x; x', t) = \sum_{n=0}^{\infty} e^{-i E_n t} \phi_n(x) \phi_n^*(x'). \tag{C.9}$$

Applying (C.7) into (C.9)

$$\rho(x; x', t) = \sum_{n=0}^{\infty} e^{-i(n + \frac{1}{2})\Omega t} \frac{1}{2^n n!} \left( \frac{\Omega}{\pi} \right)^{\frac{1}{2}} e^{-\frac{\Omega}{2} (x^2 + x'^2)}$$

$$\times H_n(\sqrt{\Omega} x) H_n(\sqrt{\Omega} x'). \tag{C.10}$$

We now use (C.8) in (C.10),

$$\rho(x; x', t) = \frac{\left( \frac{\Omega}{\pi} \right)^{\frac{1}{2}}}{\sqrt{2i \sin(\Omega t)}} e^{-\frac{p(x^2 + x'^2)}{2} + q x x'} , \tag{C.11}$$

where

$$p(t) = -i \Omega \cot(\Omega t) ; \quad q(t)$$

$$= \frac{-i \Omega}{\sin(\Omega t)} . \tag{C.12}$$

We will not write the  $t$  dependence of  $p$  and  $q$  explicitly, we simply write  $p$  and  $q$  instead of  $p(t)$  and  $q(t)$ .

We now define the normalised time evolution operator as  $P(x; x', t) = \frac{\rho(x; x', t)}{\text{Tr}(\rho(x; x', t))}$ ,

$$P(x; x', t) = \sqrt{\frac{p - q}{\pi}} e^{-\frac{p(x^2 + x'^2)}{2} + q x x'} . \tag{C.13}$$

Note that the normalization  $\text{Tr}(\rho(x; x', t))$  using (C.11) is  $\int_{-\infty}^{\infty} dx \rho(x, x, t)$ , which is oscillatory (rather than a damped Gaussian), using (C.12). To render this well-defined,

we insert a small exponentially damping regulator: this is the position space analog of the regularization in (3.18). Similar regulators are required to define various infinite sums/integrals here.

We now find the expressions for  $\rho_1(y_1; y'_1, t)$  and  $\rho_2(y_2; y'_2, t)$  appearing in (C.5) using (C.11),

$$\begin{aligned} \rho_1(y_1; y'_1, t) &= \frac{\left(\frac{\Omega_1}{\pi}\right)^{\frac{1}{2}}}{\sqrt{2i \sin(\Omega_1 t)}} e^{-\frac{p(y_1^2+y_1'^2)}{2} + q y_1 y_1'}, \\ \rho_2(y_2; y'_2, t) &= \frac{\left(\frac{\Omega_2}{\pi}\right)^{\frac{1}{2}}}{\sqrt{2i \sin(\Omega_2 t)}} e^{-\frac{r(y_2^2+y_2'^2)}{2} + s y_2 y_2'}, \end{aligned} \tag{C.14}$$

where

$$\begin{aligned} p &= -i \Omega_1 \cot(\Omega_1 t); \quad q = \frac{-i \Omega_1}{\sin(\Omega_1 t)}; \\ r &= -i \Omega_2 \cot(\Omega_2 t); \quad s = \frac{-i \Omega_2}{\sin(\Omega_2 t)}. \end{aligned} \tag{C.15}$$

We define the normalised time evolution operator as  $P(y_1, y_2; y'_1, y'_2, t) = \frac{\rho(y_1, y_2; y'_1, y'_2, t)}{Tr(\rho(y_1, y_2; y'_1, y'_2, t))}$ ,

$$\begin{aligned} P(y_1, y_2; y'_1, y'_2, t) &= \sqrt{\frac{p-q}{\pi}} \sqrt{\frac{r-s}{\pi}} e^{-\frac{p(y_1^2+y_1'^2)}{2} + q y_1 y_1'} \\ &\quad \times e^{-\frac{r(y_2^2+y_2'^2)}{2} + s y_2 y_2'}. \end{aligned} \tag{C.16}$$

Writing  $P(y_1, y_2; y'_1, y'_2, t)$  in terms of original variables  $x_A, x_B$  (C.2) gives

$$\begin{aligned} P(x_A, x_B; x'_A, x'_B = x_B, t) &= \sqrt{\frac{p-q}{\pi}} \sqrt{\frac{r-s}{\pi}} \\ &\quad \times e^{-\frac{(p+r)}{4}(x_A^2+x_A'^2) + \frac{(q+s)}{2} x_A x'_A} \\ &\quad \times e^{-\frac{x_B^2}{2}(p+r-q-s) + x_B \frac{(x_A+x'_A)}{2}(-p-s+q+r)}. \end{aligned} \tag{C.17}$$

We now trace over the 2nd oscillator  $P_A(x_A; x'_A, t) = Tr_B[P(x_A, x_B; x'_A, x'_B, t)]$ . For this we integrate (C.17) over  $x_B$ , after performing the integration, we get

$$P_A(x_A; x'_A, t) = \sqrt{\frac{\gamma-\beta}{\pi}} e^{-\frac{\gamma}{2}(x_A^2+x_A'^2) + \beta x_A x'_A} \tag{C.18}$$

where

$$\begin{aligned} \gamma &= \frac{p+r}{2} - \frac{1}{4} \frac{(p+s-q-r)^2}{p+r-q-s}; \quad \beta = \frac{q+s}{2} \\ &\quad + \frac{1}{4} \frac{(p+s-q-r)^2}{p+r-q-s}, \\ \gamma - \beta &= 2 \frac{(p-q)(r-s)}{p-q+r-s}; \quad \gamma + \beta \end{aligned}$$

$$= \frac{p+q+r+s}{2}. \tag{C.19}$$

The entropy associated with the reduced density matrix  $P_A(x_A, x'_A, t)$  is given by  $S_A = -Tr(P_A \log P_A)$ . The eigenvalues  $\lambda_n$  and eigenvectors  $f_n(x)$  of an operator of the form (C.18) are given in [53]: we have  $\lambda_n = (1-\zeta)\zeta^n$ , where  $\zeta = \frac{\beta}{\gamma+\alpha}$ ,  $\alpha = \sqrt{\gamma^2 - \beta^2}$ , which gives

$$S_A = -\log(1-\zeta) - \frac{\zeta}{1-\zeta} \log \zeta. \tag{C.20}$$

We see that the entropy  $S_A$  is complex valued, recasting  $\zeta$  in terms of  $\gamma + \beta$  and  $\gamma - \beta$ ,

$$\zeta = \frac{\sqrt{\gamma + \beta}, \sqrt{\gamma - \beta}}{\sqrt{\gamma + \beta} + \sqrt{\gamma - \beta}}. \tag{C.21}$$

The explicit expressions for (C.19) in terms of original variables are given by

$$\begin{aligned} \sqrt{\gamma + \beta} &= \left(-i \left(\frac{\Omega_1}{2} \cot \frac{\Omega_1 t}{2} + \frac{\Omega_2}{2} \cot \frac{\Omega_2 t}{2}\right)\right)^{\frac{1}{2}}, \\ \sqrt{\gamma - \beta} &= \left(\frac{2i}{\frac{1}{\Omega_1} \cot \frac{\Omega_1 t}{2} + \frac{1}{\Omega_2} \cot \frac{\Omega_2 t}{2}}\right)^{\frac{1}{2}}. \end{aligned} \tag{C.22}$$

For  $\Omega_1 = \Omega_2 = \omega$  (i.e.  $k_2 = 0$ ), we recover our result for two uncoupled oscillators. Comparing our result with the spacelike entanglement evaluated at finite inverse temperature  $it$ , we recover the result in [54] (in particular  $\zeta$  in (C.21) matches with eq.(2.22) in [54]).

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