



The Dirac equation as a linear tensor equation for one component

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Abstract The Dirac equation is one of the most fundamental equations of modern physics. It is a spinor equation, but some tensor equivalents of the equation were proposed previously. Those equivalents were either nonlinear or involved several components of the Dirac field. On the other hand, the author showed previously that the Dirac equation in electromagnetic field is equivalent to a fourth-order equation for one component of the Dirac spinor. The equivalency is used in this work to derive a linear tensor equivalent of the Dirac equation for just one component. This surprising result can be used in applications of the Dirac equation, for example, in general relativity or for lattice approximation of the Dirac field, and can improve our understanding of the Dirac equation.

1 Introduction

“Of all the equations of physics, perhaps the most ‘magical’ is the Dirac equation. It is the most freely invented, the least conditioned by experiment, the one with the strangest and most startling consequences” [1]. The Dirac equation is a spinor equation, but previously, some authors offered tensor equations equivalent to the Dirac equation. Those equations were either nonlinear [2] or, in the case of the Dirac–Kähler equation [3–5], had a large number of components. The approach of [6] allows to significantly reduce the number of components, but still requires several components.

The surprising result of this work is that it is possible to derive a linear tensor equation for just one component that is generally equivalent to the Dirac equation. This result builds on the previous work [7–9] (see also [10], pp. 24–25, [11]),

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where it was shown that, in a general case, three out of four complex components of the Dirac spinor can be algebraically eliminated from the Dirac equation in an arbitrary electromagnetic field. Therefore, the Dirac equation is generally equivalent to a fourth-order partial differential equation for just one component, which can be made real (at least locally) by a gauge transformation. Some applications of this result are discussed in [12].

One usually emphasizes the advantages of tensor, rather than spinor, equations for applications in general relativity [2] and lattice approximation of Dirac fields [13], but there can be other applications of the fundamental result of this work in the future. As Feynman said in his Nobel lecture, “a good theoretical physicist today might find it useful to have a wide range of physical viewpoints and mathematical expressions of the same theory” [14].

Section 2 summarizes the results of the previous work [7–9] used in the present article.

Section 3 contains the general derivation of the linear tensor equation for one component that is equivalent to the Dirac equation.

Formulations of the linear tensor equation in terms of anti-symmetric second-rank tensors and 3D vectors are provided, respectively, in Sects. 4 and 5.

The supplementary file includes a Mathematica notebook (see it also at [15]). The notebook contains the proofs of several algebraic statements (usually indicated by phrases “One can check that...”) in Sects. 3, 4, and 5, using the chiral representation of gamma-matrices, which is sufficient for our purpose. Proofs of the statements in Sect. 2 can be found in [8, 9].

2 The Dirac equation as a fourth-order linear equation for one component

To make this article reasonably self-contained, let us provide some notation and results from [8, 9] (the more tra-

ditional four-component spinor formalism is used in this work, although the two-component spinor formalism has some important strengths [16]).

We use the Dirac equation in the form:

$$(i\cancel{\partial} - \cancel{A})\psi = \psi, \tag{1}$$

where, e.g., $\cancel{A} = A_\mu \gamma^\mu$ (the Feynman slash notation). A system of units $\hbar = c = m = 1$ is used, and the electric charge e is included in A_μ ($eA_\mu \rightarrow A_\mu$). The metric tensor is

$$g = (g^{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2}$$

Multiplying both sides of (1) by $(i\cancel{\partial} - \cancel{A})$ from the left and using notation

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu], \tag{3}$$

$$F^{\mu\nu} = A^{\nu,\mu} - A^{\mu,\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -H^3 & H^2 \\ E^2 & H^3 & 0 & -H^1 \\ E^3 & -H^2 & H^1 & 0 \end{pmatrix}, \tag{4}$$

$$F = \frac{1}{2}F_{\nu\mu}\sigma^{\nu\mu}, \tag{5}$$

we obtain:

$$(\square' + F)\psi = 0, \tag{6}$$

where the modified d'Alembertian \square' is defined as follows:

$$\begin{aligned} \square' &= \partial^\mu \partial_\mu + 2iA^\mu \partial_\mu + iA_{,\mu}^\mu - A^\mu A_\mu + 1 \\ &= -(i\partial_\mu - A_\mu)(i\partial^\mu - A^\mu) + 1. \end{aligned} \tag{7}$$

We assume that the set of γ -matrices satisfies the standard hermiticity conditions [17]:

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0, \gamma^{5\dagger} = \gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3. \tag{8}$$

Then a charge conjugation matrix C can be chosen in such a way [18, 19] that

$$\begin{aligned} C\gamma^\mu C^{-1} &= -\gamma^{\mu T}, C\gamma^5 C^{-1} = \gamma^{5T}, \\ C\sigma^{\mu\nu} C^{-1} &= -\sigma^{\mu\nu T}, \end{aligned} \tag{9}$$

$$\begin{aligned} C^T &= C^\dagger = -C, CC^\dagger = C^\dagger C = I, \\ C^2 &= -I, \end{aligned} \tag{10}$$

where the superscript T denotes transposition, and I is the unit matrix. The properties of the charge conjugation matrix (9,10) are extensively used in the following.

We choose constant Dirac spinors ξ and η (they do not depend on the spacetime coordinates $x = (x^0, x^1, x^2, x^3)$, so, e. g., $\partial_\mu \xi \equiv 0$) in such a way that they are both eigenvectors of γ^5 with the same eigenvalue $+1$ or -1 , and the normalization condition

$$\bar{\xi}\eta^c = 1 \tag{11}$$

is satisfied, where η^c is the charge conjugated spinor

$$\eta^c = C\bar{\eta}^T. \tag{12}$$

Due to the normalization condition, the spinors ξ and η are linearly independent. If they are eigenvectors of γ^5 with the same eigenvalue ± 1 , the Dirac conjugated spinors $\bar{\xi}$ and $\bar{\eta}$ are linearly independent left eigenvectors of γ^5 with the same eigenvalue ∓ 1 . One can check that spinors $\bar{\xi}F$ and $\bar{\eta}F$ are also left eigenvectors of γ^5 with the same eigenvalue ∓ 1 , so, as the space of left eigenvectors of γ^5 with eigenvalue ∓ 1 is two-dimensional, $\bar{\xi}F$ and $\bar{\eta}F$ are linear combinations of $\bar{\xi}$ and $\bar{\eta}$. One can calculate the coefficients and obtain:

$$\bar{\xi}F = (\bar{\xi}F\eta^c)\bar{\xi} - (\bar{\xi}F\xi^c)\bar{\eta} = f_{\xi\eta}\bar{\xi} - f_{\xi\xi}\bar{\eta}, \tag{13}$$

$$\bar{\eta}F = (\bar{\eta}F\eta^c)\bar{\xi} - (\bar{\eta}F\xi^c)\bar{\eta} = f_{\eta\eta}\bar{\xi} - f_{\eta\xi}\bar{\eta}, \tag{14}$$

where the following notation for components of the electromagnetic field is used:

$$f_{\alpha\beta} = \bar{\alpha}F\beta^c. \tag{15}$$

Here α and β are some Dirac spinors.

Multiplying (6) by $\bar{\xi}$ and $\bar{\eta}$ from the left and using (13) and (14), we obtain:

$$\begin{aligned} \square'(\bar{\xi}\psi) + f_{\xi\eta}(\bar{\xi}\psi) - f_{\xi\xi}(\bar{\eta}\psi) &= 0, \\ \square'(\bar{\eta}\psi) + f_{\eta\eta}(\bar{\xi}\psi) - f_{\eta\xi}(\bar{\eta}\psi) &= 0, \end{aligned} \tag{16}$$

so

$$\bar{\eta}\psi = f_{\xi\xi}^{-1}(\square'(\bar{\xi}\psi) + f_{\xi\eta}(\bar{\xi}\psi)) \tag{17}$$

and

$$((\square' - f_{\xi\eta})f_{\xi\xi}^{-1}(\square' + f_{\xi\eta}) + f_{\eta\eta})(\bar{\xi}\psi) = 0 \tag{18}$$

(one can show that $f_{\xi\eta} = f_{\eta\xi}$). Thus, it is possible to algebraically eliminate three out of four components of the Dirac equation from (6) and derive an equation for the remaining component $\bar{\xi}\psi$.

One can derive the recipes for calculation of the other components of the Dirac spinor. If $\bar{\xi}\psi$ is known, another component, $\bar{\eta}\psi$, can be determined using equation (17). Then ψ can be expressed as a sum of a right-handed and a left-handed spinors ψ_+ and ψ_- , where $\gamma^5\psi_\pm = \pm\psi_\pm$:

$$\psi = \psi_+ + \psi_-, \tag{19}$$

$$\psi_{\pm} = \frac{1}{2}(1 \pm \gamma^5)\psi. \tag{20}$$

Then one can check that

$$\psi_{\mp} = (\bar{\xi}\psi)\eta^c - (\bar{\eta}\psi)\xi^c. \tag{21}$$

When ψ_{\mp} is found in this way, ψ_{\pm} can be found using the Dirac equation (1):

$$\begin{aligned} (i\partial - \mathcal{A})\psi_{\mp} &= \frac{1}{2}(i\partial - \mathcal{A})(1 \mp \gamma^5)\psi \\ &= \frac{1}{2}(1 \pm \gamma^5)(i\partial - \mathcal{A})\psi = \psi_{\pm}, \end{aligned} \tag{22}$$

thus, the Dirac spinor can be fully restored if the component $\bar{\xi}\psi$ is known. Equation (18), together with the recipes for calculation of the other components of the Dirac spinor, is generally equivalent to the Dirac equation. Some “non-transversal” cases where there is no equivalency, for example, if $f_{\xi\xi} \equiv 0$, do not seem to be important from the experimental point of view. For example, $f_{\xi\xi} \equiv 0$ for a free Dirac particle, and it is an important case theoretically, but an arbitrarily weak electromagnetic field, for example, the Coulomb field of a single charged particle in the Universe, restores the equivalency.

The denominator $f_{\xi\xi}$ in (18) may look inaeesthetic. One can get rid of it by multiplying (18) by $(f_{\xi\xi})^3$, but this is a matter of preference.

3 Derivation of the linear tensor equivalent of the Dirac equation

3.1 Antisymmetric second-rank tensors built from pairs of chiral spinors

Let us consider a Dirac spinor χ . It can be presented as a sum of two chiral spinors χ_+ and χ_- that are eigenvectors of γ^5 with eigenvalues of +1 and -1, respectively:

$$\chi = \chi_+ + \chi_-, \gamma^5\chi_+ = \chi_+, \gamma^5\chi_- = -\chi_-. \tag{23}$$

In the chiral representation of γ -matrices [17],

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \\ \gamma^5 &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, C = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}, \end{aligned} \tag{24}$$

where index i runs from 1 to 3, and σ^i are the Pauli matrices. Then χ , χ_+ , and χ_- have the following components:

$$\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{pmatrix}, \chi_+ = \begin{pmatrix} \chi_1 \\ \chi_2 \\ 0 \\ 0 \end{pmatrix}, \chi_- = \begin{pmatrix} 0 \\ 0 \\ \chi_3 \\ \chi_4 \end{pmatrix}. \tag{25}$$

Similarly, let us introduce spinors ζ , ζ_+ , and ζ_- . It is possible to build antisymmetric second-rank tensors from pairs of chiral spinors [20, Section 153]. Let us consider the following matrices θ_+ and θ_- :

$$(\theta_+^{\mu\nu}) = (\bar{\chi}_+\sigma^{\mu\nu}\zeta_+) = \begin{pmatrix} 0 & i(\chi_1^*\zeta_1^* - \chi_2^*\zeta_2^*) & \chi_1^*\zeta_1^* + \chi_2^*\zeta_2^* & -i(\chi_2^*\zeta_1^* + \chi_1^*\zeta_2^*) \\ -i(\chi_1^*\zeta_1^* - \chi_2^*\zeta_2^*) & 0 & \chi_2^*\zeta_1^* + \chi_1^*\zeta_2^* & -i(\chi_1^*\zeta_1^* + \chi_2^*\zeta_2^*) \\ -\chi_1^*\zeta_1^* - \chi_2^*\zeta_2^* & -\chi_2^*\zeta_1^* - \chi_1^*\zeta_2^* & 0 & -\chi_1^*\zeta_1^* + \chi_2^*\zeta_2^* \\ i(\chi_2^*\zeta_1^* + \chi_1^*\zeta_2^*) & i(\chi_1^*\zeta_1^* + \chi_2^*\zeta_2^*) & \chi_1^*\zeta_1^* - \chi_2^*\zeta_2^* & 0 \end{pmatrix} \tag{26}$$

and

$$(\theta_-^{\mu\nu}) = (\bar{\chi}_-\sigma^{\mu\nu}\zeta_-) = \begin{pmatrix} 0 & i(\chi_3^*\zeta_3^* - \chi_4^*\zeta_4^*) & \chi_3^*\zeta_3^* + \chi_4^*\zeta_4^* & -i(\chi_4^*\zeta_3^* + \chi_3^*\zeta_4^*) \\ -i(\chi_3^*\zeta_3^* - \chi_4^*\zeta_4^*) & 0 & -\chi_4^*\zeta_3^* - \chi_3^*\zeta_4^* & i(\chi_3^*\zeta_3^* + \chi_4^*\zeta_4^*) \\ -\chi_3^*\zeta_3^* - \chi_4^*\zeta_4^* & \chi_4^*\zeta_3^* + \chi_3^*\zeta_4^* & 0 & \chi_3^*\zeta_3^* - \chi_4^*\zeta_4^* \\ i(\chi_4^*\zeta_3^* + \chi_3^*\zeta_4^*) & -i(\chi_3^*\zeta_3^* + \chi_4^*\zeta_4^*) & -\chi_3^*\zeta_3^* + \chi_4^*\zeta_4^* & 0 \end{pmatrix}. \tag{27}$$

Proper orthochronous Lorentz transformations acting on Dirac spinors in the chiral representation have the following form [21]:

$$\begin{pmatrix} s \\ (s^\dagger)^{-1} \end{pmatrix}, \tag{28}$$

where

$$s = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix},$$

$$(s^\dagger)^{-1} = \begin{pmatrix} s_{22}^* & -s_{21}^* \\ -s_{12}^* & s_{11}^* \end{pmatrix}, \tag{29}$$

and the determinant $|s| = 1$. The relevant Lorentz transformation acting on tensors is

$$\Lambda = (\Lambda^\mu_\nu) = \frac{1}{2} \begin{pmatrix} s_{11} & s_{12} & s_{21} & s_{22} \\ s_{21} & s_{22} & s_{11} & s_{12} \\ -is_{21} & -is_{22} & is_{11} & is_{12} \\ s_{11} & s_{12} & -s_{21} & -s_{22} \end{pmatrix}$$

$$\times \begin{pmatrix} s_{11}^* & s_{12}^* & -is_{12}^* & s_{11}^* \\ s_{12}^* & s_{11}^* & is_{11}^* & -s_{12}^* \\ s_{21}^* & s_{22}^* & -is_{22}^* & s_{21}^* \\ s_{22}^* & s_{21}^* & is_{21}^* & -s_{22}^* \end{pmatrix}. \tag{30}$$

The derivation of (30) is similar to that in [21]. Our (25) differs from equation (62) of [21], ultimately, because of a different choice of the sign of γ^0 . One can check that $g_{\mu\nu}\Lambda^\mu_\rho\Lambda^\nu_\lambda = g_{\rho\lambda}$ [21], or $\Lambda^T g_l \Lambda = g_l$, where the metric tensor with lower indices $g_l = (g_{\mu\nu})$, and that replacement of $\chi_1, \chi_2, \zeta_1, \zeta_2$ in θ_+ with $\chi'_1, \chi'_2, \zeta'_1, \zeta'_2$ in accordance with formulas

$$\begin{pmatrix} \chi'_1 \\ \chi'_2 \end{pmatrix} = s \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix},$$

$$\begin{pmatrix} \zeta'_1 \\ \zeta'_2 \end{pmatrix} = s \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \tag{31}$$

gives the same result as $\Lambda^\mu_\rho\Lambda^\nu_\lambda\theta_+^{\rho\lambda}$, or $\Lambda\theta_+\Lambda^T$, so θ_+ transforms as an antisymmetric second-rank tensor. Also, replacement of $\chi_3, \chi_4, \zeta_3, \zeta_4$ in θ_- with $\chi'_3, \chi'_4, \zeta'_3, \zeta'_4$ in accordance with formulas

$$\begin{pmatrix} \chi'_3 \\ \chi'_4 \end{pmatrix} = (s^\dagger)^{-1} \begin{pmatrix} \chi_3 \\ \chi_4 \end{pmatrix},$$

$$\begin{pmatrix} \zeta'_3 \\ \zeta'_4 \end{pmatrix} = (s^\dagger)^{-1} \begin{pmatrix} \zeta_3 \\ \zeta_4 \end{pmatrix} \tag{32}$$

gives the same result as $\Lambda^\mu_\rho\Lambda^\nu_\lambda\theta_-^{\rho\lambda}$, or $\Lambda\theta_-\Lambda^T$, so θ_- transforms as an antisymmetric second-rank tensor. Similarly, θ_\pm^* (complex conjugates of θ_\pm) also transform as antisymmetric second-rank tensors.

One can check that

$$(\theta_\pm^{\mu\nu}) = (\mp i \star \theta_\pm^{\mu\nu}), (\theta_\pm^{*\mu\nu}) = (\pm i \star \theta_\pm^{*\mu\nu}), \tag{33}$$

where the Hodge dual of a second-rank antisymmetric tensor is defined as [22]

$$\star F^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}, \tag{34}$$

and $\epsilon^{\alpha\beta\gamma\delta}$ is the totally antisymmetric Levi-Civita tensor ($\epsilon^{0123} = 1$). That means that tensors $\theta_\pm^{\mu\nu}$ are fully defined by 3D vectors θ_\pm^{0i} , and vice versa, similar to how the Weber vector (Riemann–Silberstein vector) $\mathbf{E} + i\mathbf{H}$ [23] fully defines the real antisymmetric second-rank electromagnetic field tensor $F^{\mu\nu}$. Four-dimensional rotations in the 4D Minkowski space for the tensors correspond to rotations through complex angles in a 3D complex space for the 3D vectors [24, §25]. The following could be presented in terms of 3D vectors, but the author will mostly use tensors and add a short section on 3D vectors. No distinction is drawn between tensors and pseudotensors in this work.

Chiral spinors ψ_\pm can be represented by antisymmetric second-rank tensors ([2], [20, Section 154]):

$$(\psi_\pm^{\mu\nu}) = (\overline{\psi}_\pm^c \sigma^{\mu\nu} \psi_\pm) = (\psi_\pm^T C \sigma^{\mu\nu} \psi_\pm). \tag{35}$$

One can see that these are tensors as $(\psi_\pm^{\mu\nu})$ coincide with $(\theta_\mp^{\mu\nu})$ if $\chi = \psi^c$ and $\zeta = \psi^c$ (one can check that $(\psi^c)^c = \psi$ and $\overline{\psi^c} = \psi^T C$). One can check that

$$\psi_\pm^{\mu\nu} \psi_{\pm\mu\nu} = 0. \tag{36}$$

3.2 Constant tensors and some scalar functions to replace the elements of the Dirac equation

So is it possible to replace the spinor elements of (18) by tensor ones while preserving linearity? Note that we want to avoid the following: while chiral spinors can be represented by antisymmetric second-rank tensors satisfying (33, 36), such tensors define spinors only up to a sign, so we need to avoid the ambiguity related to square roots.

It seems natural to start with the tensors corresponding to the spinors ξ^c and η^c , however such an approach creates the above-mentioned ambiguity. Therefore, we start with a tensor

$$u^{\mu\nu} = (\xi^c)^T C \sigma^{\mu\nu} \xi^c = (C \bar{\xi}^T)^T C \sigma^{\mu\nu} \xi^c$$

$$= \bar{\xi} C^T C \sigma^{\mu\nu} \xi^c = \bar{\xi} \sigma^{\mu\nu} \xi^c, \tag{37}$$

corresponding to ξ^c , and a tensor $v^{\mu\nu} = (\xi^c)^T C \sigma^{\mu\nu} \eta^c = \bar{\xi} \sigma^{\mu\nu} \eta^c$. These tensors satisfy (33), where the signs are chosen depending on whether ξ and η are eigenvectors of γ^5 with eigenvalue $+1$ or -1 . Taking into account the normalization condition (11), one can show for the tensor $w^{\mu\nu} = (\eta^c)^T C \sigma^{\mu\nu} \eta^c = \bar{\eta} \sigma^{\mu\nu} \eta^c$ corresponding to the chiral spinor

η^c that

$$u^{\mu\nu} w_{\mu\nu} = -8, v^{\mu\nu} w_{\mu\nu} = 0, w^{\mu\nu} w_{\mu\nu} = 0. \tag{38}$$

We also have

$$u^{\mu\nu} u_{\mu\nu} = 0, u^{\mu\nu} v_{\mu\nu} = 0, v^{\mu\nu} v_{\mu\nu} = 4. \tag{39}$$

There is only one solution for $w^{\mu\nu}$ satisfying (38) and (33) if $u^{\mu\nu}$ and $v^{\mu\nu}$ satisfy (39) and $u^{\mu\nu} \neq 0$. To find it, one can first choose any tensor $k^{\mu\nu}$ (satisfying (33)) such that $k^{\mu\nu} k_{\mu\nu} = 0$ and $u^{\mu\nu} k_{\mu\nu} \neq 0$. Let us prove it.

First, let us consider an antisymmetric second-rank tensor $a^{\mu\nu}$ satisfying (33), so

$$(a^{\mu\nu}) = \begin{pmatrix} 0 & a^{01} & a^{02} & a^{03} \\ -a^{01} & 0 & \pm ia^{03} & \mp ia^{02} \\ -a^{02} & \mp ia^{03} & 0 & \pm ia^{01} \\ -a^{03} & \pm ia^{02} & \mp ia^{01} & 0 \end{pmatrix}. \tag{40}$$

Such tensor is fully defined by the 3D vector $\mathbf{a} = (a^1, a^2, a^3) = (a^{01}, a^{02}, a^{03})$. Let us define the scalar product of (complex) 3D vectors \mathbf{a} and $\mathbf{b} = (b^1, b^2, b^3)$ as $(\mathbf{a} \cdot \mathbf{b}) = a^i b^i = a^1 b^1 + a^2 b^2 + a^3 b^3$ (no conjugation). One can check that $a^{\mu\nu} b_{\mu\nu} = -4(\mathbf{a} \cdot \mathbf{b})$, where $a^{\mu\nu}$ and $b^{\mu\nu}$ are antisymmetric second-rank tensors corresponding to \mathbf{a} and \mathbf{b} and satisfying (33).

The space of antisymmetric tensors satisfying (33) is 3-dimensional, so one can choose such an antisymmetric tensor $t^{\mu\nu}$ satisfying (33) that $u^{\mu\nu}$, $v^{\mu\nu}$, and $t^{\mu\nu}$ are linearly independent.

Let us prove that $u^{\mu\nu} t_{\mu\nu} \neq 0$. Indeed, otherwise $(\mathbf{u} \cdot \mathbf{t}) = 0$, so let us consider the following matrix (similar to the Gram matrix, where, however, a different scalar product is used for complex vectors):

$$M = \begin{pmatrix} (\mathbf{u} \cdot \mathbf{u}) & (\mathbf{u} \cdot \mathbf{v}) & (\mathbf{u} \cdot \mathbf{t}) \\ (\mathbf{v} \cdot \mathbf{u}) & (\mathbf{v} \cdot \mathbf{v}) & (\mathbf{v} \cdot \mathbf{t}) \\ (\mathbf{t} \cdot \mathbf{u}) & (\mathbf{t} \cdot \mathbf{v}) & (\mathbf{t} \cdot \mathbf{t}) \end{pmatrix} = Q^T Q, \tag{41}$$

where

$$Q = \begin{pmatrix} u^1 & v^1 & t^1 \\ u^2 & v^2 & t^2 \\ u^3 & v^3 & t^3 \end{pmatrix}. \tag{42}$$

From (39), we have $(\mathbf{u} \cdot \mathbf{u}) = (\mathbf{u} \cdot \mathbf{v}) = 0$, and we assumed $(\mathbf{u} \cdot \mathbf{t}) = 0$, so we have for the determinants:

$$0 = |M| = |Q|^2. \tag{43}$$

So $|Q| = 0$, although we assumed that $u^{\mu\nu}$, $v^{\mu\nu}$, and $t^{\mu\nu}$, and, therefore, \mathbf{u} , \mathbf{v} , and \mathbf{t} are linearly independent. Therefore,

the assumption $(\mathbf{u} \cdot \mathbf{t}) = 0$ leads to a contradiction. Thus, we proved that $u^{\mu\nu} t_{\mu\nu} \neq 0$.

Let us seek $k^{\mu\nu}$ in the form $(\alpha u^{\mu\nu} + v^{\mu\nu} + t^{\mu\nu})$. Then we have:

$$0 = (\alpha u^{\mu\nu} + v^{\mu\nu} + t^{\mu\nu}) \times (\alpha u_{\mu\nu} + v_{\mu\nu} + t_{\mu\nu}) = v^{\mu\nu} v_{\mu\nu} + t^{\mu\nu} t_{\mu\nu} + 2\alpha u^{\mu\nu} t_{\mu\nu} + 2v^{\mu\nu} t_{\mu\nu}, \tag{44}$$

so we have exactly one solution for α :

$$\alpha = -\frac{v^{\mu\nu} v_{\mu\nu} + t^{\mu\nu} t_{\mu\nu} + 2v^{\mu\nu} t_{\mu\nu}}{2u^{\mu\nu} t_{\mu\nu}}, \tag{45}$$

and for $k^{\mu\nu} = \alpha u^{\mu\nu} + v^{\mu\nu} + t^{\mu\nu}$ we have $k^{\mu\nu} k_{\mu\nu} = 0$ and $u^{\mu\nu} k_{\mu\nu} \neq 0$, as otherwise $u^{\mu\nu} t_{\mu\nu} = 0$.

The tensors $u^{\mu\nu}$, $v^{\mu\nu}$, $t^{\mu\nu}$ are linearly independent, so the tensors $u^{\mu\nu}$, $v^{\mu\nu}$, $k^{\mu\nu}$ are also linearly independent, so one can seek the solution for $w^{\mu\nu}$ in the form

$$w^{\mu\nu} = \alpha_1 u^{\mu\nu} + \alpha_2 v^{\mu\nu} + \alpha_3 k^{\mu\nu}. \tag{46}$$

Contracting (46) with $u_{\mu\nu}$, we obtain

$$\alpha_3 u_{\mu\nu} k^{\mu\nu} = -8. \tag{47}$$

Contracting (46) with $v_{\mu\nu}$, we obtain

$$4\alpha_2 + \alpha_3 v_{\mu\nu} k^{\mu\nu} = 0. \tag{48}$$

Contracting (46) with $w_{\mu\nu} = \alpha_1 u_{\mu\nu} + \alpha_2 v_{\mu\nu} + \alpha_3 k_{\mu\nu}$, we obtain

$$0 = (\alpha_2)^2 v_{\mu\nu} v^{\mu\nu} + 2\alpha_1 \alpha_3 u_{\mu\nu} k^{\mu\nu} + 2\alpha_2 \alpha_3 v_{\mu\nu} k^{\mu\nu}. \tag{49}$$

One can obtain from (47, 48, 49):

$$\alpha_1 = -\frac{(v^{\mu\nu} k_{\mu\nu})^2}{(u^{\mu\nu} k_{\mu\nu})^2}, \alpha_2 = 2 \frac{v^{\mu\nu} k_{\mu\nu}}{u^{\mu\nu} k_{\mu\nu}}, \alpha_3 = -\frac{8}{u^{\mu\nu} k_{\mu\nu}}. \tag{50}$$

The above is a coordinate-free solution. A more explicit solution (where a specific tensor $k^{\mu\nu}$ is chosen) can be written in coordinates.

If ξ and η are eigenvectors of γ^5 with eigenvalue ± 1 ,

$$u = \begin{pmatrix} 0 & u^1 & u^2 & u^3 \\ -u^1 & 0 & \pm iu^3 & \mp iu^2 \\ -u^2 & \mp iu^3 & 0 & \pm iu^1 \\ -u^3 & \pm iu^2 & \mp iu^1 & 0 \end{pmatrix}, v = \begin{pmatrix} 0 & v^1 & v^2 & v^3 \\ -v^1 & 0 & \pm iv^3 & \mp iv^2 \\ -v^2 & \mp iv^3 & 0 & \pm iv^1 \\ -v^3 & \pm iv^2 & \mp iv^1 & 0 \end{pmatrix}, \tag{51}$$

and

$$\begin{aligned} u^{\mu\nu} u_{\mu\nu} &= -4((u^1)^2 + (u^2)^2 + (u^3)^2) = 0, \\ u^{\mu\nu} v_{\mu\nu} &= -4(u^1 v^1 + u^2 v^2 + u^3 v^3) = 0, \\ v^{\mu\nu} v_{\mu\nu} &= -4((v^1)^2 + (v^2)^2 + (v^3)^2) = 4, \end{aligned} \tag{52}$$

tensor $k^{\mu\nu}$ can be chosen in the following form:

$$k = \begin{pmatrix} 0 & u^{1*} & u^{2*} & u^{3*} \\ -u^{1*} & 0 & \pm i u^{3*} & \mp i u^{2*} \\ -u^{2*} & \mp i u^{3*} & 0 & \pm i u^{1*} \\ -u^{3*} & \pm i u^{2*} & \mp i u^{1*} & 0 \end{pmatrix}, \tag{53}$$

then

$$u^{\mu\nu} k_{\mu\nu} = -4(|u^1|^2 + |u^2|^2 + |u^3|^2). \tag{54}$$

This value does not vanish unless $u = 0$. When tensors $u^{\mu\nu}$, $v^{\mu\nu}$, $w^{\mu\nu}$ are known, it is possible to calculate the components of the electromagnetic field from (18):

$$\begin{aligned} f_{\xi\xi} &= \frac{1}{2} F_{\mu\nu} u^{\mu\nu}, \quad f_{\xi\eta} = \frac{1}{2} F_{\mu\nu} v^{\mu\nu}, \\ f_{\eta\eta} &= \frac{1}{2} F_{\mu\nu} w^{\mu\nu}. \end{aligned} \tag{55}$$

To rewrite (18) in terms of tensors completely, one needs to find an appropriate form for $\bar{\xi}\psi$.

If ξ is an eigenvector of γ^5 with eigenvalue +1,

$$\bar{\xi}\psi = \bar{\xi}\psi_- = -\psi_3\xi_1^* - \psi_4\xi_2^*. \tag{56}$$

The tensor corresponding to ψ_- is

$$\psi_-^{\mu\nu} = \psi_-^T C \sigma^{\mu\nu} \psi_-. \tag{57}$$

One can check that

$$\psi_-^{\mu\nu} u_{\mu\nu} = -8(\bar{\xi}\psi)^2. \tag{58}$$

If ξ is an eigenvector of γ^5 with eigenvalue -1,

$$\bar{\xi}\psi = \bar{\xi}\psi_+ = -\psi_1\xi_3^* - \psi_2\xi_4^*. \tag{59}$$

The tensor corresponding to ψ_+ is

$$\psi_+^{\mu\nu} = \psi_+^T C \sigma^{\mu\nu} \psi_+. \tag{60}$$

One can check that

$$\psi_+^{\mu\nu} u_{\mu\nu} = -8(\bar{\xi}\psi)^2. \tag{61}$$

Thus, if ξ is an eigenvector of γ^5 with eigenvalue ± 1 ,

$$\bar{\xi}\psi = \left(-\frac{\psi_{\mp}^{\mu\nu} u_{\mu\nu}}{8} \right)^{\frac{1}{2}}. \tag{62}$$

So $\bar{\xi}\psi$ can be expressed using tensors as a square root of a scalar, but as (18) is a linear equation with respect to $\bar{\xi}\psi$, this does not create any ambiguities related to the sign of the square root (when the sign of the square root is chosen in one point, the choice can be expanded by continuity, at least locally). Thus, (18) can be expressed using tensors only.

3.3 Restoration of the Dirac current

One can also show that the Dirac current can be found using tensors only. Up to a constant factor, the current is

$$j^\mu = \bar{\psi}\gamma^\mu\psi = j_+^\mu + j_-^\mu, \tag{63}$$

where

$$j_\pm^\mu = \bar{\psi}_\pm\gamma^\mu\psi_\pm. \tag{64}$$

One can check that for tensors $\psi_\pm^{\mu\nu}$ (defined by (57) and (60)) the following is true:

$$j_\pm^{\mu\nu} = g_{\sigma\lambda}\psi_\pm^{\sigma\mu}(\psi_\pm^{\lambda\nu})^* = -2j_\pm^\mu j_\pm^\nu. \tag{65}$$

Thus, if we know tensors $\psi_\pm^{\mu\nu}$, we know tensors $j_\pm^\mu j_\pm^\nu$, and, if we know the latter tensors, we can find vectors j_\pm^μ , as

$$j_\pm^\mu = \frac{j_\pm^\mu j_\pm^0}{\sqrt{(j_\pm^0)^2}}. \tag{66}$$

The non-negative value of the square root should be chosen in (66) as we know that $j_\pm^0 \geq 0$. It is not clear whether the option to choose the negative value of the square root and therefore obtain negative charge density for the tensor equivalent of the Dirac equation can be advantageous, e. g., to better describe the antiparticles.

Thus, to prove that one can find out j^μ from one of the scalars $\psi_\mp^{\mu\nu} u_{\mu\nu}$, it is sufficient to prove that one can find both tensors $\psi_\pm^{\mu\nu}$ from the scalar. If ξ and η are eigenvectors of γ^5 with eigenvalue ± 1 , then $\bar{\xi}\psi = \bar{\xi}\psi_\mp$. We have (21), and

$$\begin{aligned} \psi_\mp^{\mu\nu} &= \psi_\mp^T C \sigma^{\mu\nu} \psi_\mp \\ &= ((\bar{\xi}\psi)(\eta^c)^T - (\bar{\eta}\psi)(\xi^c)^T) C \sigma^{\mu\nu} ((\bar{\xi}\psi)\eta^c - (\bar{\eta}\psi)\xi^c) \\ &= (\bar{\xi}\psi)^2 (\bar{\eta}\sigma^{\mu\nu}\eta^c) - (\bar{\eta}\psi)(\bar{\xi}\psi)(\bar{\xi}\sigma^{\mu\nu}\eta^c) \\ &\quad - (\bar{\xi}\psi)(\bar{\eta}\psi)(\bar{\eta}\sigma^{\mu\nu}\xi^c) + (\bar{\eta}\psi)^2 (\bar{\xi}\sigma^{\mu\nu}\xi^c) \\ &= (\bar{\xi}\psi)^2 w^{\mu\nu} - 2(\bar{\eta}\psi)(\bar{\xi}\psi)v^{\mu\nu} + (\bar{\eta}\psi)^2 u^{\mu\nu}, \end{aligned} \tag{67}$$

as

$$\begin{aligned} \bar{\eta}\sigma^{\mu\nu}\xi^c &= (\bar{\eta}\sigma^{\mu\nu}\xi^c)^T = (\xi^c)^T (\sigma^{\mu\nu})^T (\bar{\eta})^T \\ &= (C\bar{\xi}^T)^T (-C\sigma^{\mu\nu}C^{-1})(-C)\eta^c \\ &= \bar{\xi}C^T (C\sigma^{\mu\nu}C)(-C)\eta^c = \bar{\xi}\sigma^{\mu\nu}\eta^c. \end{aligned} \tag{68}$$

Using (17) and (55), we obtain

$$\bar{\eta}\psi = (F_{\mu\nu}u^{\mu\nu})^{-1}(2\Box' + F_{\mu\nu}v^{\mu\nu})(\bar{\xi}\psi). \tag{69}$$

Using (62) and (69), we can express products $(\bar{\xi}\psi)^2$, $(\bar{\eta}\psi)(\bar{\xi}\psi)$, and $(\bar{\eta}\psi)^2$ in (67) via $\psi_{\mp}^{\mu\nu}u_{\mu\nu}$, and the results do not depend on the choice of the value of the square root in (62). Therefore, we can express $\psi_{\mp}^{\mu\nu}$ via the scalar function $\psi_{\mp}^{\mu\nu}u_{\mu\nu}$ (and its derivatives).

Let us now express $\psi_{\pm}^{\mu\nu}$ via the same scalar function. Using (22) and (21), we obtain:

$$\begin{aligned} \psi_{\pm} &= \gamma^{\mu}(i\partial_{\mu} - A_{\mu})\psi_{\mp} \\ &= \gamma^{\mu}(i\partial_{\mu} - A_{\mu})((\bar{\xi}\psi)\eta^c - (\bar{\eta}\psi)\xi^c) \\ &= \gamma^{\mu}(B_{\mu}\xi^c + C_{\mu}\eta^c), \end{aligned} \tag{70}$$

where

$$\begin{aligned} B_{\mu} &= -(i\partial_{\mu} - A_{\mu})(\bar{\eta}\psi), \\ C_{\mu} &= (i\partial_{\mu} - A_{\mu})(\bar{\xi}\psi). \end{aligned} \tag{71}$$

We have

$$\begin{aligned} \psi_{\pm}^T C &= (\gamma^{\mu}(B_{\mu}\xi^c + C_{\mu}\eta^c))^T C \\ &= (B_{\mu}(\xi^c)^T + C_{\mu}(\eta^c)^T)(-C\gamma^{\mu}C^{-1})C \\ &= (B_{\mu}\bar{\xi}C^T + C_{\mu}\bar{\eta}C^T)(-C\gamma^{\mu}) = -(B_{\mu}\bar{\xi} + C_{\mu}\bar{\eta})\gamma^{\mu}. \end{aligned} \tag{72}$$

Then

$$\begin{aligned} \psi_{\pm}^{\nu\sigma} &= \psi_{\pm}^T C\sigma^{\nu\sigma}\psi_{\pm} \\ &= -(B_{\mu}\bar{\xi} + C_{\mu}\bar{\eta})\gamma^{\mu}\sigma^{\nu\sigma}\gamma^{\lambda}(B_{\lambda}\xi^c + C_{\lambda}\eta^c) \\ &= -B_{\mu}B_{\lambda}\bar{\xi}\gamma^{\mu}\sigma^{\nu\sigma}\gamma^{\lambda}\xi^c - C_{\mu}B_{\lambda}\bar{\eta}\gamma^{\mu}\sigma^{\nu\sigma}\gamma^{\lambda}\xi^c \\ &\quad - B_{\mu}C_{\lambda}\bar{\xi}\gamma^{\mu}\sigma^{\nu\sigma}\gamma^{\lambda}\eta^c - C_{\mu}C_{\lambda}\bar{\eta}\gamma^{\mu}\sigma^{\nu\sigma}\gamma^{\lambda}\eta^c. \end{aligned} \tag{73}$$

Let us use the following formula for the product of four γ -matrices [25, p. 17]:

$$\begin{aligned} \gamma^{\mu}\gamma^{\nu}\gamma^{\sigma}\gamma^{\lambda} &= g^{\mu\nu}g^{\sigma\lambda} - g^{\mu\sigma}g^{\nu\lambda} + g^{\mu\lambda}g^{\nu\sigma} \\ &\quad + g^{\mu\nu}\gamma^{\sigma\lambda} - g^{\mu\sigma}\gamma^{\nu\lambda} + g^{\mu\lambda}\gamma^{\nu\sigma} + g^{\nu\sigma}\gamma^{\mu\lambda} \\ &\quad - g^{\nu\lambda}\gamma^{\mu\sigma} + g^{\sigma\lambda}\gamma^{\mu\nu} + \gamma^{\mu\nu\sigma\lambda}, \end{aligned} \tag{74}$$

where the antisymmetrized products of γ -matrices are

$$\begin{aligned} \gamma^{\mu\nu} &= \begin{cases} \gamma^{\mu}\gamma^{\nu} & \text{if } \mu \neq \nu, \\ 0 & \text{otherwise} \end{cases}, \\ \gamma^{\mu\nu\sigma\lambda} &= \begin{cases} \gamma^{\mu}\gamma^{\nu}\gamma^{\sigma}\gamma^{\lambda} & \text{if } \{\mu, \nu, \sigma, \lambda\} \\ & \text{is a permutation of } \{0, 1, 2, 3\}, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{75}$$

so, if $\nu \neq \sigma$,

$$\begin{aligned} \gamma^{\mu}\sigma^{\nu\sigma}\gamma^{\lambda} &= \frac{i}{2}\gamma^{\mu}(\gamma^{\nu}\gamma^{\sigma} - \gamma^{\sigma}\gamma^{\nu})\gamma^{\lambda} = i(g^{\mu\nu}g^{\sigma\lambda} \\ &\quad - g^{\mu\sigma}g^{\nu\lambda} + g^{\mu\nu}\gamma^{\sigma\lambda} - g^{\mu\sigma}\gamma^{\nu\lambda} + g^{\mu\lambda}\gamma^{\nu\sigma} \end{aligned}$$

$$-g^{\nu\lambda}\gamma^{\mu\sigma} + g^{\sigma\lambda}\gamma^{\mu\nu} + \gamma^{\mu\nu\sigma\lambda}). \tag{76}$$

Let us also note that

$$\begin{aligned} \bar{\eta}\gamma^{\mu}\sigma^{\nu\sigma}\gamma^{\lambda}\xi^c &= (\bar{\eta}\gamma^{\mu}\sigma^{\nu\sigma}\gamma^{\lambda}\xi^c)^T = \xi^c T\gamma^{\lambda T}\sigma^{\nu\sigma T}\gamma^{\mu T}\bar{\eta}^T \\ &= -\bar{\xi}C(-C\gamma^{\lambda}C^{-1})(-C\sigma^{\nu\sigma}C^{-1})(-C\gamma^{\mu}C^{-1})(-C\eta^c) \\ &= \bar{\xi}\gamma^{\lambda}\sigma^{\nu\sigma}\gamma^{\mu}\eta^c, \end{aligned} \tag{77}$$

so

$$\begin{aligned} -C_{\mu}B_{\lambda}\bar{\eta}\gamma^{\mu}\sigma^{\nu\sigma}\gamma^{\lambda}\xi^c &= -C_{\mu}B_{\lambda}\bar{\xi}\gamma^{\lambda}\sigma^{\nu\sigma}\gamma^{\mu}\eta^c \\ &= -B_{\mu}C_{\lambda}\bar{\xi}\gamma^{\mu}\sigma^{\nu\sigma}\gamma^{\lambda}\eta^c, \end{aligned} \tag{78}$$

and (73) becomes

$$\begin{aligned} \psi_{\pm}^{\nu\sigma} &= -B_{\mu}B_{\lambda}\bar{\xi}\gamma^{\mu}\sigma^{\nu\sigma}\gamma^{\lambda}\xi^c - 2B_{\mu}C_{\lambda}\bar{\xi}\gamma^{\mu}\sigma^{\nu\sigma}\gamma^{\lambda}\eta^c \\ &\quad - C_{\mu}C_{\lambda}\bar{\eta}\gamma^{\mu}\sigma^{\nu\sigma}\gamma^{\lambda}\eta^c. \end{aligned} \tag{79}$$

Now we need to substitute (76) in (79). Let us show how some of the terms can be calculated. For example (one can check that $\bar{\xi}\xi^c = \bar{\eta}\eta^c = 0$),

$$\begin{aligned} \bar{\xi}g^{\mu\nu}g^{\sigma\lambda}\eta^c &= g^{\mu\nu}g^{\sigma\lambda}, \\ \bar{\xi}g^{\mu\nu}g^{\sigma\lambda}\xi^c &= \bar{\eta}g^{\mu\nu}g^{\sigma\lambda}\eta^c = 0, \end{aligned} \tag{80}$$

$$\begin{aligned} \bar{\xi}g^{\mu\nu}\gamma^{\sigma\lambda}\eta^c &= g^{\mu\nu}\frac{1}{i}v^{\sigma\lambda}, \\ \bar{\xi}g^{\mu\nu}\gamma^{\sigma\lambda}\xi^c &= g^{\mu\nu}\frac{1}{i}u^{\sigma\lambda}, \\ \bar{\eta}g^{\mu\nu}\gamma^{\sigma\lambda}\eta^c &= g^{\mu\nu}\frac{1}{i}w^{\sigma\lambda}, \end{aligned} \tag{81}$$

$$\begin{aligned} \bar{\xi}\gamma^{\mu\nu\sigma\lambda}\eta^c &= \bar{\xi}\epsilon^{\mu\nu\sigma\lambda}\gamma^0\gamma^1\gamma^2\gamma^3\eta^c \\ &= \epsilon^{\mu\nu\sigma\lambda}\bar{\xi}\frac{1}{i}\gamma^5\eta^c = \mp\epsilon^{\mu\nu\sigma\lambda}\frac{1}{i}\bar{\xi}\gamma^{\mu\nu\sigma\lambda}\xi^c \\ &= \bar{\eta}\gamma^{\mu\nu\sigma\lambda}\eta^c = 0. \end{aligned} \tag{82}$$

Thus, we have, if $\nu \neq \sigma$,

$$\begin{aligned} \psi_{\pm}^{\nu\sigma} &= -B_{\mu}B_{\lambda}(g^{\mu\nu}u^{\sigma\lambda} - g^{\mu\sigma}u^{\nu\lambda} + g^{\mu\lambda}u^{\nu\sigma} \\ &\quad - g^{\nu\lambda}u^{\mu\sigma} + g^{\sigma\lambda}u^{\mu\nu}) \\ &\quad - 2B_{\mu}C_{\lambda}(ig^{\mu\nu}g^{\sigma\lambda} - ig^{\mu\sigma}g^{\nu\lambda} + g^{\mu\nu}v^{\sigma\lambda} - g^{\mu\sigma}v^{\nu\lambda} \\ &\quad + g^{\mu\lambda}v^{\nu\sigma} - g^{\nu\lambda}v^{\mu\sigma} + g^{\sigma\lambda}v^{\mu\nu} \mp \epsilon^{\mu\nu\sigma\lambda}) \\ &\quad - C_{\mu}C_{\lambda}(g^{\mu\nu}w^{\sigma\lambda} - g^{\mu\sigma}w^{\nu\lambda} + g^{\mu\lambda}w^{\nu\sigma} \\ &\quad - g^{\nu\lambda}w^{\mu\sigma} + g^{\sigma\lambda}w^{\mu\nu}) \\ &= -B^{\nu}B_{\lambda}u^{\sigma\lambda} + B^{\sigma}B_{\lambda}u^{\nu\lambda} - B^{\lambda}B_{\lambda}u^{\nu\sigma} \\ &\quad + B_{\mu}B^{\nu}u^{\mu\sigma} - B_{\mu}B^{\sigma}u^{\mu\nu} - 2iB^{\nu}C^{\sigma} + 2iB^{\sigma}C^{\nu} \\ &\quad - 2B^{\nu}C_{\lambda}v^{\sigma\lambda} + 2B^{\sigma}C_{\lambda}v^{\nu\lambda} - 2B^{\lambda}C_{\lambda}v^{\nu\sigma} \\ &\quad + 2B_{\mu}C^{\nu}v^{\mu\sigma} - 2B_{\mu}C^{\sigma}v^{\mu\nu} \pm 2B_{\mu}C_{\lambda}\epsilon^{\mu\nu\sigma\lambda} \\ &\quad - C^{\nu}C_{\lambda}w^{\sigma\lambda} + C^{\sigma}C_{\lambda}w^{\nu\lambda} - C^{\lambda}C_{\lambda}w^{\nu\sigma} \\ &\quad + C_{\mu}C^{\nu}w^{\mu\sigma} - C_{\mu}C^{\sigma}w^{\mu\nu} \\ &= -2B^{\nu}B_{\lambda}u^{\sigma\lambda} + 2B^{\sigma}B_{\lambda}u^{\nu\lambda} - B^{\lambda}B_{\lambda}u^{\nu\sigma} \\ &\quad - 2iB^{\nu}C^{\sigma} + 2iB^{\sigma}C^{\nu} \end{aligned}$$

$$\begin{aligned}
 & -2B^\nu C_\lambda v^{\sigma\lambda} + 2B^\sigma C_\lambda v^{\nu\lambda} - 2B^\lambda C_\lambda v^{\nu\sigma} \\
 & + 2B_\mu C^\nu v^{\mu\sigma} - 2B_\mu C^\sigma v^{\mu\nu} \pm 2B_\mu C_\lambda \epsilon^{\mu\nu\sigma\lambda} \\
 & - 2C^\nu C_\lambda w^{\sigma\lambda} + 2C^\sigma C_\lambda w^{\nu\lambda} - C^\lambda C_\lambda w^{\nu\sigma}, \tag{83}
 \end{aligned}$$

as, for example,

$$\begin{aligned}
 & -B^\nu B_\lambda u^{\sigma\lambda} + B_\mu B^\nu u^{\mu\sigma} = -B^\nu B_\lambda u^{\sigma\lambda} \\
 & + B^\nu B_\lambda u^{\lambda\sigma} = -2B^\nu B_\lambda u^{\sigma\lambda}. \tag{84}
 \end{aligned}$$

Thus, we can express $\psi_{\pm}^{\mu\nu}$ (and, finally, the current) via the scalar function $\psi_{\mp}^{\mu\nu} u_{\mu\nu}$ (and its derivatives), and the results do not depend on the choice of the value of the square root in (62), as, for example, products $B^\nu C_\lambda$ do not depend on the choice of the square root.

4 Formulation in terms of antisymmetric second-rank tensors

4.1 Formulation

Let us now formulate the tensor equivalent of the Dirac equation. First, we choose a constant non-zero antisymmetric second-rank tensor $u^{\mu\nu}$ satisfying the following conditions:

$$(u^{\mu\nu}) = (\mp i \star u^{\mu\nu}), u^{\mu\nu} u_{\mu\nu} = 0. \tag{85}$$

Let us then denote the unknown scalar function of the tensor equation φ_u (instead of $\bar{\xi}\psi$). Then let us choose any constant antisymmetric second-rank tensor $v^{\mu\nu}$ satisfying the following conditions:

$$\begin{aligned}
 & (v^{\mu\nu}) = (\mp i \star v^{\mu\nu}), v^{\mu\nu} u_{\mu\nu} = 0, \\
 & u^\mu{}_\sigma v^{\sigma\nu} = -i u^{\mu\nu}. \tag{86}
 \end{aligned}$$

If we use again 3D vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$, corresponding to the antisymmetric second-rank tensors $u^{\mu\nu}, v^{\mu\nu}, w^{\mu\nu}$, such that, for example, $\mathbf{u} = (u^1, u^2, u^3) = (u^{01}, u^{02}, u^{03})$, one can check that (85) implies $(\mathbf{u} \cdot \mathbf{u}) = 0$ and (86) implies $(\mathbf{v} \cdot \mathbf{u}) = 0$. One can also check that the last equality in (86) implies

$$\begin{aligned}
 & \mathbf{u} = \mp \mathbf{u} \times \mathbf{v} \\
 & = \mp (u^2 v^3 - u^3 v^2, -u^1 v^3 + u^3 v^1, u^1 v^2 - u^2 v^1), \tag{87}
 \end{aligned}$$

so

$$\begin{aligned}
 & \mathbf{u} = \pm \mathbf{v} \times \mathbf{u} = \mathbf{v} \times (\mathbf{v} \times \mathbf{u}) \\
 & = (\mathbf{v} \cdot \mathbf{u})\mathbf{v} - (\mathbf{v} \cdot \mathbf{v})\mathbf{u} = -(\mathbf{v} \cdot \mathbf{v})\mathbf{u}, \tag{88}
 \end{aligned}$$

yielding $(\mathbf{v} \cdot \mathbf{v}) = -1$ and $v^{\mu\nu} v_{\mu\nu} = 4$. One can check that the last equality in (86) is generally (if $u^{01} \neq 0$) equivalent to

$$u^{03} v^{02} - u^{02} v^{03} = \pm u^{01}. \tag{89}$$

Then, using the procedure from Sect. 3, we can calculate a constant antisymmetric second-rank tensor $w^{\mu\nu}$ satisfying the following conditions:

$$\begin{aligned}
 & (w^{\mu\nu}) = (\mp i \star w^{\mu\nu}), v^{\mu\nu} w_{\mu\nu} = 0, \\
 & w^{\mu\nu} w_{\mu\nu} = 0, u^{\mu\nu} w_{\mu\nu} = -8. \tag{90}
 \end{aligned}$$

Using (18) and (55), we obtain the following linear tensor equivalent of the Dirac equation:

$$\begin{aligned}
 & (2\Box' - F_{\mu\nu} v^{\mu\nu})(F_{\mu\nu} u^{\mu\nu})^{-1} (2\Box' + F_{\mu\nu} v^{\mu\nu}) \\
 & + F_{\mu\nu} w^{\mu\nu} \varphi_u = 0. \tag{91}
 \end{aligned}$$

The current can be restored using the procedure from Sect. 3. In particular, based on (62), we should use the following:

$$\varphi_u = \left(-\frac{\psi_{\mp}^{\mu\nu} u_{\mu\nu}}{8} \right)^{\frac{1}{2}}. \tag{92}$$

Let us provide examples of the tensors $u^{\mu\nu}, v^{\mu\nu}, w^{\mu\nu}$, which play a role similar to that of γ -matrices in the spinor formulation of the Dirac equation. One of the choices for the tensors to be used with the upper signs in the equations like (85) are:

$$\begin{aligned}
 & (u^{\mu\nu}) = \begin{pmatrix} 0 & i & 1 & 0 \\ -i & 0 & 0 & -i \\ -1 & 0 & 0 & -1 \\ 0 & i & 1 & 0 \end{pmatrix}, (v^{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \\
 & (w^{\mu\nu}) = \begin{pmatrix} 0 & -i & 1 & 0 \\ i & 0 & 0 & -i \\ -1 & 0 & 0 & 1 \\ 0 & i & -1 & 0 \end{pmatrix}. \tag{93}
 \end{aligned}$$

One of the choices for the tensors to be used with the lower signs in the equations like (85) are:

$$\begin{aligned}
 & (u^{\mu\nu}) = \begin{pmatrix} 0 & i & 1 & 0 \\ -i & 0 & 0 & i \\ -1 & 0 & 0 & 1 \\ 0 & -i & -1 & 0 \end{pmatrix}, (v^{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \\
 & (w^{\mu\nu}) = \begin{pmatrix} 0 & -i & 1 & 0 \\ i & 0 & 0 & i \\ -1 & 0 & 0 & -1 \\ 0 & -i & 1 & 0 \end{pmatrix}. \tag{94}
 \end{aligned}$$

4.2 Proof of equivalency

Let us prove that (91) is equivalent to the Dirac equation. For example, if we use the upper signs in (85, 86, 90),

$$\begin{aligned}
 (u^{\mu\nu}) &= \begin{pmatrix} 0 & u^{01} & u^{02} & u^{03} \\ -u^{01} & 0 & iu^{03} & -iu^{02} \\ -u^{02} & -iu^{03} & 0 & iu^{01} \\ -u^{03} & iu^{02} & -iu^{01} & 0 \end{pmatrix} \\
 &= \begin{pmatrix} 0 & u^1 & u^2 & u^3 \\ -u^1 & 0 & iu^3 & -iu^2 \\ -u^2 & -iu^3 & 0 & iu^1 \\ -u^3 & iu^2 & -iu^1 & 0 \end{pmatrix}, \tag{95}
 \end{aligned}$$

Let us choose one of the solutions and find such spinor

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ 0 \\ 0 \end{pmatrix} \tag{101}$$

that

$$\begin{aligned}
 (v^{\mu\nu}) &= \begin{pmatrix} 0 & v^{01} & v^{02} & v^{03} \\ -v^{01} & 0 & iv^{03} & -iv^{02} \\ -v^{02} & -iv^{03} & 0 & iv^{01} \\ -v^{03} & iv^{02} & -iv^{01} & 0 \end{pmatrix} = \begin{pmatrix} 0 & v^1 & v^2 & v^3 \\ -v^1 & 0 & iv^3 & -iv^2 \\ -v^2 & -iv^3 & 0 & iv^1 \\ -v^3 & iv^2 & -iv^1 & 0 \end{pmatrix} \\
 &= (\bar{\xi}\sigma^{\mu\nu}\eta^c) = \begin{pmatrix} 0 & i(\eta_1^*\xi_1^* - \eta_2^*\xi_2^*) & \eta_1^*\xi_1^* + \eta_2^*\xi_2^* & -i(\eta_2^*\xi_1^* + \eta_1^*\xi_2^*) \\ -i(\eta_1^*\xi_1^* - \eta_2^*\xi_2^*) & 0 & \eta_2^*\xi_1^* + \eta_1^*\xi_2^* & -i(\eta_1^*\xi_1^* + \eta_2^*\xi_2^*) \\ -\eta_1^*\xi_1^* - \eta_2^*\xi_2^* & -\eta_2^*\xi_1^* - \eta_1^*\xi_2^* & 0 & -\eta_1^*\xi_1^* + \eta_2^*\xi_2^* \\ i(\eta_2^*\xi_1^* + \eta_1^*\xi_2^*) & i(\eta_1^*\xi_1^* + \eta_2^*\xi_2^*) & \eta_1^*\xi_1^* - \eta_2^*\xi_2^* & 0 \end{pmatrix}. \tag{102}
 \end{aligned}$$

let us find such spinor

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ 0 \\ 0 \end{pmatrix} \tag{96}$$

that

$$(u^{\mu\nu}) = (\bar{\xi}\sigma^{\mu\nu}\xi^c) = \begin{pmatrix} 0 & i((\xi_1^*)^2 - (\xi_2^*)^2) & (\xi_1^*)^2 + (\xi_2^*)^2 & -2i\xi_1^*\xi_2^* \\ -i((\xi_1^*)^2 - (\xi_2^*)^2) & 0 & 2\xi_1^*\xi_2^* & -i((\xi_1^*)^2 + (\xi_2^*)^2) \\ -(\xi_1^*)^2 - (\xi_2^*)^2 & -2\xi_1^*\xi_2^* & 0 & -(\xi_1^*)^2 + (\xi_2^*)^2 \\ 2i\xi_1^*\xi_2^* & i((\xi_1^*)^2 + (\xi_2^*)^2) & (\xi_1^*)^2 - (\xi_2^*)^2 & 0 \end{pmatrix}. \tag{97}$$

This is equivalent to the following system of equations for η_1^* and η_2^* :

$$\begin{aligned}
 v^1 &= i(\eta_1^*\xi_1^* - \eta_2^*\xi_2^*), \quad v^2 = \eta_1^*\xi_1^* + \eta_2^*\xi_2^*, \\
 v^3 &= -i(\eta_2^*\xi_1^* + \eta_1^*\xi_2^*). \tag{103}
 \end{aligned}$$

One can check that, due to the properties of u and v , these equations are compatible and have one solution for the spinor η :

This is equivalent to the following system of equations for ξ_1^* and ξ_2^* :

$$\begin{aligned}
 u^1 &= i((\xi_1^*)^2 - (\xi_2^*)^2), \quad u^2 = (\xi_1^*)^2 + (\xi_2^*)^2, \\
 u^3 &= -2i\xi_1^*\xi_2^*. \tag{98}
 \end{aligned}$$

As $(u^1)^2 + (u^2)^2 + (u^3)^2 = 0$, this system always has solutions and defines the spinor ξ up to a factor ± 1 :

$$\xi_1^* = \left(\frac{-iu^1 + u^2}{2}\right)^{\frac{1}{2}}, \quad \xi_2^* = \frac{u^3}{(-2i\xi_1^*)}, \tag{99}$$

or, if $-iu^1 + u^2 = 0$,

$$\xi_2^* = \left(\frac{i u^1 + u^2}{2}\right)^{\frac{1}{2}}, \quad \xi_1^* = \frac{u^3}{(-2i\xi_2^*)}. \tag{100}$$

$$\eta_1^* = \frac{-iv^1 + v^2}{2\xi_1^*}, \quad \eta_2^* = \frac{iv^1 + v^2}{2\xi_2^*}, \tag{104}$$

or, if $\xi_1^* = 0$,

$$\eta_1^* = \frac{v^3}{-i\xi_2^*}, \quad \eta_2^* = \frac{v^1}{-i\xi_2^*}, \tag{105}$$

or, if $\xi_2^* = 0$,

$$\eta_1^* = \frac{v^1}{i\xi_1^*}, \quad \eta_2^* = \frac{v^3}{-i\xi_1^*}. \tag{106}$$

Then

$$\begin{aligned}
 \bar{\xi} \eta^c &= \eta_2^* \xi_1^* - \eta_1^* \xi_2^* = \frac{iv^1 + v^2}{2\xi_2^*} \xi_1^* - \frac{-iv^1 + v^2}{2\xi_1^*} \xi_2^* \\
 &= \frac{(iv^1 + v^2)(\xi_1^*)^2 - (-iv^1 + v^2)(\xi_2^*)^2}{2\xi_1 \xi_2^*} \\
 &= \frac{(iv^1 + v^2) \frac{-iu^1 + u^2}{2} - (-iv^1 + v^2) \frac{(u^3)^2}{(-2i)^2 \frac{-iu^1 + u^2}{2}}}{iu^3} \\
 &= \frac{(iv^1 + v^2) \frac{-iu^1 + u^2}{2} - (-iv^1 + v^2) \frac{-(u^1)^2 - (u^2)^2}{-2(-iu^1 + u^2)}}{iu^3} = \\
 &= \frac{(iv^1 + v^2)(-iu^1 + u^2) + (-iv^1 + v^2)(-iu^1 - u^2)}{2iu^3} \\
 &= \frac{v^1 u^2 - v^2 u^1}{u^3} = 1, \tag{107}
 \end{aligned}$$

due to (87).

As was shown in Sect. 3, if antisymmetric tensors $u^{\mu\nu}$ and $v^{\mu\nu}$ satisfy (39) and (33), and $u^{\mu\nu} \neq 0$, there is only one antisymmetric tensor $w^{\mu\nu}$ satisfying (38) and (33). On the other hand, the tensor $\bar{\xi} \sigma^{\mu\nu} \eta^c$ satisfies (38) and (33), so $\bar{\xi} \sigma^{\mu\nu} \eta^c = w^{\mu\nu}$. Using this and (5, 15, 97, 102), we obtain from (91):

$$((\square' - f_{\xi\eta}) f_{\xi\xi}^{-1} (\square' + f_{\xi\eta}) + f_{\eta\eta}) \varphi_u = 0, \tag{108}$$

which coincides with the equivalent (18) of the Dirac equation after we identify φ_u with $\bar{\xi} \psi$.

5 Formulation in terms of 3D vectors

Now let us rewrite the Dirac equation in terms of complex 3D vectors. For example, instead of the antisymmetric second-rank tensors $u^{\mu\nu}$, $v^{\mu\nu}$, $w^{\mu\nu}$, we are going to use 3D vectors, such as $\mathbf{u} = (u^1, u^2, u^3)$, where $u^i = u^{0i}$. Lorentz transformations correspond to 3D rotations of such vectors through complex angles.

We require

$$(\mathbf{u} \cdot \mathbf{u}) = 0, (\mathbf{u} \cdot \mathbf{v}) = 0, \mathbf{u} = \mp \mathbf{u} \times \mathbf{v}. \tag{109}$$

One can check that the last equality of (109) generally yields

$$u^{03} v^{02} - u^{02} v^{03} = \pm u^{01} \tag{110}$$

and

$$(\mathbf{v} \cdot \mathbf{v}) = -1. \tag{111}$$

We also require

$$(\mathbf{u} \cdot \mathbf{w}) = 2, (\mathbf{v} \cdot \mathbf{w}) = 0, (\mathbf{w} \cdot \mathbf{w}) = 0. \tag{112}$$

Adapting the calculation of $w^{\mu\nu}$ in Sect. 3, we can obtain, using (46, 50, 51, 52, 53):

$$\begin{aligned}
 \mathbf{w} &= -\frac{(\mathbf{v} \cdot \mathbf{u}^*)^2}{(\mathbf{u} \cdot \mathbf{u}^*)^2} \mathbf{u} + \\
 &2 \frac{(\mathbf{v} \cdot \mathbf{u}^*)}{(\mathbf{u} \cdot \mathbf{u}^*)} \mathbf{v} + \frac{2}{(\mathbf{u} \cdot \mathbf{u}^*)} \mathbf{u}^*, \tag{113}
 \end{aligned}$$

where $\mathbf{u}^* = (u^{1*}, u^{2*}, u^{3*})$. Then, using (4), one can check that, e.g., $F_{\mu\nu} u^{\mu\nu} = 2(\mathbf{u} \cdot (\mathbf{E} \mp i\mathbf{H}))$, where $\mathbf{E} = (E^1, E^2, E^3)$, $\mathbf{H} = (H^1, H^2, H^3)$, introduce vector $\mathbf{F} = \mathbf{E} \mp i\mathbf{H}$, and use (91) to obtain

$$((\square' - (\mathbf{F} \cdot \mathbf{v}))(\mathbf{F} \cdot \mathbf{u})^{-1} (\square' + (\mathbf{F} \cdot \mathbf{v})) + (\mathbf{F} \cdot \mathbf{w})) \varphi_u = 0. \tag{114}$$

Again, the current can be restored using the procedure similar to that in Sect. 3. In particular, based on (62), we should use the following:

$$\varphi_u = \left(\frac{(\boldsymbol{\psi}_{\mp} \cdot \mathbf{u})}{2} \right)^{\frac{1}{2}}, \tag{115}$$

where vector $\boldsymbol{\psi}_{\mp} = (\psi_{\mp}^{01}, \psi_{\mp}^{02}, \psi_{\mp}^{03})$.

Let us provide examples of the vectors \mathbf{u} , \mathbf{v} , \mathbf{w} , which play a role similar to that of γ -matrices in the spinor formulation of the Dirac equation. One of the choices for the vectors are (cf. (93,94)):

$$\mathbf{u} = (i, 1, 0), \mathbf{v} = (0, 0, \mp i), \mathbf{w} = (-i, 1, 0). \tag{116}$$

6 Conclusion

Previously [7–9], the author showed that the Dirac equation is generally equivalent to a fourth-order linear equation for just one component. On the other hand, chiral spinors can be represented by antisymmetric second-rank tensors [2]. Thus, for a Dirac spinor ψ , one can build tensors $\psi_{\pm}^{\mu\nu}$ corresponding to chiral spinors ψ_{\pm} , such that $\psi = \psi_+ + \psi_-$ and $\gamma^5 \psi_{\pm} = \pm \psi_{\pm}$. Then we introduce constant antisymmetric second-rank tensors $u^{\mu\nu}$, $v^{\mu\nu}$ satisfying certain tensor conditions. The choice of $u^{\mu\nu}$ determines the component for which we obtain the linear equivalent of the Dirac equation, and different choices of $v^{\mu\nu}$ lead to equivalent equations. The choice of $u^{\mu\nu}$ and $v^{\mu\nu}$ determines the choice of another constant antisymmetric second-rank tensor $w^{\mu\nu}$ satisfying certain tensor conditions. Eventually, we derive a linear tensor equivalent of the Dirac equation for just one component

$$\varphi_u = \left(-\frac{\psi_{\mp}^{\mu\nu} u_{\mu\nu}}{8} \right)^{\frac{1}{2}}.$$

There is a popular phrase: “A spinor is a square root of a vector” [26, Chapter II, para. 4]. In our case, one can say that a square root of a scalar is a scalar.

Let us make some comments.

- There are two slightly different versions of the formalism depending on whether we start with $\psi_+^{\mu\nu}$ or $\psi_-^{\mu\nu}$.
- When we have a solution of the linear tensor equivalent of the Dirac equation, there is an unequivocal recipe for calculation of the Dirac current.
- The linear tensor equivalent of the Dirac equation is an equation for one component, which component is (in general) complex, but it can be made real by a gauge transformation (at least locally).

The author hopes that the linear tensor equivalent of the Dirac equation for one component will be useful for some applications and for better understanding of the Dirac equation’s “magic”.

Supplementary information

The supplementary file includes a Mathematica notebook. The notebook contains the proofs of several algebraic statements.

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