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$f(R, \Box R)$-gravity and equivalency with the modified GUP Scalar field models

Andronikos Paliathanasis[1,2,a]

[1] Institute of Systems Science, Durban University of Technology, Durban 4000, South Africa
[2] Departamento de Matemáticas, Universidad Católica del Norte, Avda. Angamos 0610, 1280 Casilla, Antofagasta, Chile

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Abstract Inspired by the generalization of scalar field gravitational models with a minimum length we study the equivalent theory in modified theories of gravity. The quadratic generalized uncertainty principle (GUP) gives rise to a deformed Heisenberg algebra in the application, resulting in the emergence of additional degrees of freedom described by higher-order derivatives. The new degrees of freedom can be attributed to the introduction of a new scalar field, transforming the resulting theory into a two-scalar field theory. Thus, in order to describe all the degrees of freedom we investigate special forms of the sixth-order modify $f(R, \Box R)$ −theory of gravity, where the gravitational Lagrangian has similar properties to that of the GUP scalar field theory. Finally, the cosmological applications are discussed, and we show that the de Sitter universe can be recovered without introducing a cosmological constant.

1 Introduction

Dark energy is an exotic matter source introduced in Einstein's field equations of General Relativity in order to explain the cosmic acceleration as it is observed by the cosmological data [1–5]. Currently, there are no observable phenomena directly linked to the nature and characteristics of dark energy. As a result, the physical nature and the origin of dark energy it is up for debate in the scientific community. Introducing the cosmological constant in the Einstein–Hilbert action integral represents one of the simplest approaches to address the issue of dark energy. The ΛCDM cosmology is an analytic solution of the field equations of General Relativity for a spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) geometry with a nonzero cosmological constant term and a pressureless dust fluid. ΛCDM has achieved considerable success in describing a large span of astronomical and cosmological data. Nevertheless, the cosmological constant can not explain the complete evolution of the cosmic history [6,7], we refer the reader to the recent discussion [8].

In order to overcome the problems of the cosmological constant and to explain the late-time cosmic acceleration, in recent years, cosmologists have proposed various models which can categorized to two large families of theories, the dark energy theories and the modified theories of gravity. For the dark energy models, an energy-momentum tensor that attributes the new degrees of freedom is introduced in field equations of General Relativity. The dynamics driven by the newly introduced degrees of freedom provide an explanation for various cosmological phenomena. Some of the most common dark energy models are the quintessence scalar field [9–12], phantom scalar field [13,14], chameleon mechanism [15], scalar-tensor models [16–18], Galileons [19,20]; multiscalar field models [21–23], Chaplygin gas-like fluids [24–28], k-essence [29,30], tachyons [31–33]. On the other hand, in modified theories of gravity the Einstein–Hilbert action integral is modified with the introduction of geometric invariants [34–36]. As a result, the gravitational field equations are modified such that new geometrodynamical components to be introduced and provide an effective geometric matter source to explain the acceleration of the universe [37]. The "zoology" of modified theories of gravity can be categorized based on the geometric invariant used to modify the gravitational Action Integral and the order of derivatives involved. Within the realm of modified theories of gravity, a particular family of interest comprises the so-called $f(X)$-theories, where the gravitational Action Integral is a function f of the geometric invariant X. The latter can be the Ricci scalar leads to $f(R)$-gravity [38], the torsion scalar T of the Weitzenböck connection in teleparallelism [39], the non-metricity scalar Q in symmetric teleparallel theory [40], the Gauss–Bonnet

[a] e-mail: anpaliat@phys.uoa.gr (corresponding author)

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term [41], for other proposed theories see for instance [42–47] and references therein.

$f(R)$-gravity [48,49] has been widely studied in the literature with many interesting results in cosmological studies [50–52] as also important results in the description of astrophysical objects [53–55]. The quadratic theory of gravity $f(R) = R + qR^2$ [56–58], has been used successfully [59] for the description of another inflationary epoch of our universe. Specifically, the quadratic $f(R)$-gravity has been used as a mechanism for acceleration in the early stages of the universe [60]. The quadratic term R^2 follows from the analytic expressions for the quantum-gravitational effects in the one-loop approximation. Indeed the origin of the R^2 term is the vacuum polarization of the physical space [61,62].

The existence of a minimum length, i.e. maximum energy in nature, is supported by various approaches to quantum gravity, like string theory, doubly special relativity and the black hole physics. The incorporation of a minimum length necessitates a modification of Heisenberg's uncertainty principle, giving rise to the generalized uncertainty principle (GUP) [63]. By modifying the uncertainty principle governing quantum observables, we arrive at a revised definition of the Heisenberg algebra, which in turn leads to adjustments in the Poisson brackets in the classical limits [64–66]. The modified Poisson brackets revise the equations of motion such that new degrees of freedom to be introduced. There is a plethora of studies of GUP in gravitational physics, see for instance [67–69]. As far as the cosmological constant is concerned, it has been found that it is related to the GUP and the minimum length [70,71]. Another extension of the uncertainty principle in the extended uncertainty principle (EUP) [72,73], however in this work we are focus in GUP. We refer the reader to the recent review [74].

The quintessence model, employed to describe dark energy in [75] incorporates modifications to the scalar field Lagrangian through the application of the quadratic GUP, that is, in the equation of motion for the scalar field, i.e. the Klein–Gordon equation, new higher-order derivatives have been introduced, as a result of the deformed Heisenberg algebra. These new terms are related to the existence of the minimum length. The higher-order derivatives can be described by a second-scalar field and this modified quintessence model is equivalent to a multi-scalar field cosmological model with interaction between the two scalar fields. It was found that the GUP components affect the cosmological evolution of a FLRW geometry not only in the early stages of the universe but also in the late-universe, while the effects for the existence of the deformed Heisenberg algebra are observable in the cosmological perturbations [76]. The case where a matter source is included in the field equations coupled to the scalar field was the subject of study in [77]. Furthermore, in [78] the deformed Heisenberg algebra for the quadratic GUP was considered to study the effects of the minimum length in the case of a gravitational theory, which satisfies Mach's principle. Specifically, the case of the Bran–Dicke scalar field was considered modified by the quadratic GUP. It was found that the modified theory is a multi-scalar field theory with different dynamics and evolution from the unmodified theory. An important result is that the nature of the asymptotic solutions does not depend on the value of the Brans–Dicke parameter, which is different from the case of the unmodified theory. Last but not least, the effects of the GUP modification in the Brans–Dicke theory are observable not only in the early stages of the universe but also in the late-time.

There exists a unique connection between some modified theories of gravity and scalar field models. Indeed, the new degrees of freedom provided by the geometric scalars in modified theories of gravity can be attributed to scalar fields. For instance $f(R)$-gravity [48] is equivalent with the so-called O'Hanlon theory [79] which belongs to the family of scalar-tensor theories [80] and specifically to the Brans–Dicke gravity [81]. Recall that $f(R)$-gravity is fourth-order theory for a nonlinear function f, while it reduces to a second-order theory in the limit of general relativity when f is a linear function. Thus, with the introduction of a Lagrange multiplier the field equations can be written in the equivalent order of second-order derivatives but in the same time the number of the dependent variables increases. In general, Lagrange multipliers are applied for the introduction of constraints for the cosmological model. There are various studies in the literature for the application of Lagrange multipliers in gravitational theories, see for instance [82–84].

In this study, we investigate if there exists a modified theory of gravity where we can recover the quadratic GUP corrections that follow from the deformation of the Heisenberg algebra in scalar field theories. With such analysis, we will be able to find the geometric equivalent of GUP in modified theories of gravity. The GUP scalar field models studied before, in the Einstein and Jordan frames [75,78], are second-order multi-scalar field models, specifically the matter source is attributed to two-scalar fields. Hence, in order to be able to recover a two scalar field theory in modified theories of gravity we shall consider a sixth-order theory. $F(R, \Box R)$-theory has been introduced before [85,86] as a sixth-order gravity and extension of $f(R)$ theory, where $\Box$ is the Laplace operator. The introduction of higher-order derivatives in the gravitational Action Integral is in agreement with quantum gravity [85]. The effects of the $\Box R$ terms in $F(R, \Box R)$ −theory has been widely studied before in the description of inflation [87–89]. A detailed analysis of the cosmological dynamics in $F(R, \Box R)$ performed recently in [90] where it was found that higher-order terms can dominate the evolution of the universe. For more applications of $F(R, \Box R)$-theory in gravitational physics we refer the reader to [91–95] and references therein. The structure of the paper is as follows.

In Sect. 2 we present the basic properties and definitions of GUP and we focus on the case of quadratic GUP. We define the deformed Heisenberg algebra, and we derive the modified Klein–Gordon equation for a spin-0 particle. The latter modified Lagrangian is used in Sect. 3 in order to introduce the effects of the deformed Heisenberg algebra in scalar field cosmological models. The modified quintessence and modified Brans–Dicke models are presented. In Sect. 4 we consider the $F(R, \Box R)$-gravity which is a theory of gravity of sixth-order. We introduce Lagrange multipliers in order to increase the number of the dependent variables with the introduction of scalar fields, and at the same time reduce the theory into a second-order gravitational model. We found that a separable function $F(R, \Box R)$ is equivalent to two-scalar field theory with similar properties to that of GUP scalar field models. In Sect. 5 we focus in the case of $F(R, \Box R) = R + K(\Box R)$ gravity, where the term $K(\Box R)$ introduces similar corrections terms in the field equations as that of the minimum length of GUP. Indeed, the $K(\Box R)$ we can say that follows from the deformation algebra of the Einstein–Hilbert action integral. For a spatially flat FLRW background geometry in Sect. 5 it was found that the de Sitter universe is a unique attractor for the cosmological solution without necessary introduce a cosmological constant term. Thus, the existence of a minim length in the early universe leads to an accelerated universe. Finally, in Sect. 6 we draw our conclusions.

2 Generalized uncertainty principle

The existence of a minimum length leads to the modification of the Heisenberg's uncertainty principle as

$$\Delta X_i \Delta P_j \geqslant \frac{\hbar}{2}[\delta_{ij}(1+\beta P^2) + 2\beta P_i P_j], \tag{1}$$

where parameter β is the deformed parameter defined as $\beta = \beta_0/M_{Pl}^2 c^2$, where M_{Pl} is the Planck mass and $M_{Pl}c^2$ is the Planck energy, or equivalently $\beta = \beta_0 \ell_{Pl}^2/\hbar^2$ where ℓ_{Pl} ($\approx 10^{-35}\ m$) is the Planck length. Parameter β_0 usually is selected to be positive and equal to one, however in order to have observable quantum effects, the parameter β_0 can have different values [96].

The usual choice of the parameter β_0 is $\beta_0 = 1$, however, such a choice could lead do not observable quantum effects; however in [96], it has been shown that the dimensionless parameter β_0 could has upper bound such as $\beta_0 >> 1$. However, there are studies where shown that the deformation parameter β_0 can be negative [97–100].

Therefore, the modified uncertainty principle (1) leads to the deformed Heisenberg algebra [101,102]

$$[X_i, P_j] = i\hbar\left[\delta_{\alpha\beta}\left(1+\beta_0\frac{\ell_{Pl}^2}{2\hbar^2}P^2\right) + \beta_0\frac{\ell_{Pl}^2}{\hbar^2}P_\alpha P_\beta\right]. \tag{2}$$

Hence, the coordinate representation of the momentum operator, which satisfies the commutation relation (2), can be defined as $P_i = p_i(1+\beta p^2)$, where we have selected to keep underformed the position underformed, that is, $X_i = x_i$. Representation (x, p) is the canonical representation satisfying $[x_i, p_j] = i\hbar\delta_{ij}$.

In the relativistic limit the commutation relation (2) reads [103]

$$[X_\mu, P_\nu] = -i\hbar\left[\left(1+\beta_0\frac{\ell_{Pl}^2}{2\hbar^2}(\eta^{\mu\nu}P_\mu P_\nu)\right)\eta_{\mu\nu} + \beta_0\frac{\ell_{Pl}^2}{\hbar^2}P_\mu P_\nu\right], \tag{3}$$

where $\eta_{\mu\nu}$ is the flat metric. Therefore, the corresponding deformed operators for (3) are $P_\mu = p_\mu(1-\beta(\eta^{\alpha\gamma}p_\alpha p_\gamma))$, $X_\nu = x_\nu$.

In the relativistic limit the equation of motion for a spin-0 particle with rest mass zero is

$$\left[\eta^{\mu\nu}P_\mu P_\nu - (mc)^2\right]\Psi = 0, \tag{4}$$

that is,

$$\Box\Psi - 2\beta\hbar^2\Box(\Box\Psi) + \left(\frac{mc}{\hbar}\right)^2\Psi + O\left(\beta^2\right) = 0. \tag{5}$$

where the Laplace operator for the metric $\eta_{\mu\nu}$ is marked with the symbol $\Box$. For a generic metric tensor $g_{\mu\nu}$ the Laplace operator is defined as $\Delta = \Box$, where $\Delta = \frac{1}{\sqrt{-g}}\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\partial_\mu\right)$. In Eq. (5) term $2\beta\hbar^2\Box(\Box\Psi)$ is the quantum correction term which follows from the existence of the minimum length. Because the quantum correction terms introduce the fourth-order derivatives in the Klein–Gordon equation, the $\beta^2 \to 0$, Eq. (5) is a singular perturbative system which means that there is a solution where the term $2\beta\hbar^2\Box(\Box\Psi)$ dominates and drives the dynamics. For the discussion of inner and outer solutions of singular perturbative differential equations we refer the reader in [104].

The modified Klein–Gordon Eq. (5) is a fourth-order partial differential equation. It can be written in the equivalent form of two second-order differential equations by introducing the new scalar field $\Phi = \Box\Psi$. Hence, Eq. (5) in the limit $\beta^2 \to 0$ becomes,

$$\Box\Psi - \Phi = 0, \tag{6}$$

$$2\beta\hbar^2\Box\Phi + (mc)^2\Psi + \Phi = 0. \tag{7}$$

Let us now derive the Lagrangian for the two scalar field model with equations of motion (6) and (7).

We consider the action integral of the Eq. (5)

$$S_{KG}^{\text{mod}} = \int dx^4 \left(\frac{1}{2}\eta^{\mu\nu}\mathcal{D}_\mu \Psi \mathcal{D}_\nu \Psi - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2 \Psi^2 \right), \quad (8)$$

where now $\mathcal{D}_\mu$ is the deformed operator defined as

$$\mathcal{D}_\mu = \nabla_\mu + \beta\hbar^2 \nabla_\mu \Box. \quad (9)$$

Hence, expression (8) becomes

$$S_{KG}^{\text{mod}} = \int dx^4 \left(\frac{1}{2}\eta^{\mu\nu} \left(\nabla_\mu \Psi \nabla_\nu \Psi + 2\beta\hbar^2 \left(\nabla_\nu \nabla_\mu \Box\right) \Psi \right) - \frac{1}{2}\left(\frac{mc}{\hbar}\Psi\right)^2 \right), \quad (10)$$

We include the Lagrange multiplier λ into the dynamical system and the second scalar field Φ with constraint $\Phi = \Box\Psi$, therefore the latter Action Integral after integration by parts is written as follows

$$S_{KG}^{\text{mod}} = \int dx^4 \sqrt{-g} \left(\frac{1}{2}\eta^{\mu\nu} \nabla_\mu \Psi \nabla_\nu \Psi + 2\beta\hbar^2 \eta^{\mu\nu} \nabla_\mu \Psi \nabla_\nu \Phi + \beta\hbar^2 \Phi^2 - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2 \Psi^2 \right). \quad (11)$$

3 Scalar field theories modified by GUP

We review previous results on the application of the quadratic GUP in scalar field theories.

Consider the scalar–tensor action integral [80]

$$S = \int dx^4 \sqrt{-g} \left[\frac{1}{2} F(\phi)^2 R - L_\phi \right], \quad (12)$$

where $F(\phi)$ is the coupling function and L_ϕ is the Lagrangian for the scalar field, that is

$$L_\phi \left(x^\mu, \phi, \nabla_\mu \phi\right) = \frac{\omega}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + V(\phi) \quad (13)$$

where ω is a parameter which defines the nature of the scalar field. Lagrangian function (13) is that of the underformed Heisenberg algebra. Hence, after the application of the quadratic GUP, we end with Lagrangian function (11) which reads

$$L_\phi^{GUP} \left(x^\mu, \phi, \psi, \nabla_\mu \phi, \nabla_\mu \psi\right) = \frac{\omega}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + V(\phi) + \beta \left(2 g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \psi + \psi^2\right) \quad (14)$$

where ψ is the second scalar field which follows from the higher-order derivatives.

We replace in (12), (14) and we end with the Action Integral

$$S = \int dx^4 \sqrt{-g} \left[\frac{1}{2} F(\phi) R - \frac{\omega}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) - \beta \left(2 g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \psi + \psi^2\right) \right]. \quad (15)$$

3.1 Quintessence and phantom cosmologies

In the case of a spatially flat FLRW geometry

$$ds^2 = -dt^2 + a^2(t) \left(dx^2 + dy^2 + dz^2\right), \quad (16)$$

and for a minimally coupled scalar field, that is $F(\phi) = 1$, the cosmological field equations are

$$3H^2 - \frac{1}{2}\dot{\phi}^2 - 2\beta\dot{\phi}\dot{\psi} - V(\phi) + \beta\psi^2 = 0, \quad (17)$$

$$2\dot{H} + 3H^2 + \frac{1}{2}\dot{\phi}^2 + 2\beta\dot{\phi}\dot{\psi} - \left(V(\phi) - \beta\psi^2\right) = 0, \quad (18)$$

$$\ddot{\phi} + 3H\dot{\phi} - \psi = 0, \quad (19)$$

$$\beta\left(\ddot{\psi} + 3H\dot{\psi}\right) + \frac{1}{2}\left(\psi + V_{,\phi}\right) = 0. \quad (20)$$

where $H = \frac{\dot{a}}{a}$ is the Hubble function.

For $\omega = 1$, the scalar field is a quintessence while for $\omega = -1$ scalar field ψ is a phantom field. However, as it was found before in [75], the equation of state parameter for the effective parameter can cross the phantom divide line and for the case of a quintessence field, that is, because the second scalar field ψ can dominates such that $w_{eff} < -1$. Last but not least, from the analysis of the dynamics in [76] it was found that the de Sitter universe is always a late-time attractor independent from the nature of the scalar field potential.

Furthermore, the field Eqs. (17)–(20) can derive from the variation of the point-like Lagrangian

$$L_Q = 3a\dot{a}^2 - \frac{1}{2}a^3\dot{\phi}^2 - 2\beta a^3\dot{\phi}\dot{\psi} + a^3 V(\phi) - \beta a^3\psi^2. \quad (21)$$

3.2 Brans–Dicke cosmology

Brans–Dicke model is recovered when $F(\phi) = \phi^2$, where $\omega = \bar{\omega}_{BD}$ is now the Brans–Dicke parameter. In this case the cosmological field equations in the case of a FLRW spacetime

$$6H^2 + 12H\left(\frac{\dot{\phi}}{\phi}\right) + \frac{\bar{\omega}_{BD}}{2}\left(\dot{\phi}^2 + 2\beta\hbar^2\left(\frac{\dot{\phi}}{\phi}\right)\left(\frac{\dot{\psi}}{\phi}\right) - \frac{\psi^2}{\phi^2}\right) + \frac{V(\phi)}{\phi^2} = 0, \quad (22)$$

$$2\dot{H}+3H^2+4\left(\frac{\dot{\phi}}{\phi}\right)H+\left(\frac{\bar{\omega}_{BD}}{4}-2\right)\left(\frac{\dot{\phi}}{\phi}\right)^2+\frac{V(\phi)}{2\phi^2}$$

$$+\beta\frac{\bar{\omega}_{BD}}{4\phi^2}\left(\psi^2-2\dot{\psi}\dot{\phi}\right)=0, \quad (23)$$

$$\bar{\omega}_{BD}\left(\beta\ddot{\psi}+\ddot{\phi}\right)+3\bar{\omega}_{BD}H\left(\beta\dot{\psi}+\dot{\phi}\right)$$

$$+V_{,\phi}+12\phi\left(\dot{H}+2H^2\right)=0. \quad (24)$$

$$\ddot{\phi}+3H\dot{\phi}+\psi=0, \quad (25)$$

Similarly as before the Lagrangian function which reproduces the field equations reads

$$L\left(a,\dot{a},\phi,\dot{\phi},\psi,\dot{\psi}\right)=-6a\phi^2\dot{a}^2-12\phi a^2\dot{a}\dot{\phi}$$

$$-\frac{\bar{\omega}_{BD}}{2}a^3\left(\dot{\phi}^2+2\beta\hbar^2\dot{\phi}\dot{\psi}-\psi^2\right)+a^3V(\phi). \quad (26)$$

In [78] it was found that the modified field equations provide a different cosmological history. The GUP introduces significant changes to the dynamics and asymptotic solutions of the field equations, resulting in newfound degrees of freedom. These modifications diverge from those of the unmodified model. Moreover, the physical characteristics of the asymptotic solutions are contingent upon the exponent of the potential function rather than the value of the Brans–Dicke parameter, which is the case in the unmodified model.

4 $F(R,\Box R)$-gravity

In this section we incoporate Lagrange multipliers in order to study the scalar-tensor description of $F(R,\Box R)$-gravity. In our study, we focus into the scenario where scalar field models with GUP are derived from $F(R,\Box R)$-gravity.

4.1 $f(R)$-gravity

The Ricciscalar R is defined by second-order derivatives of the metric tensor; hence a Lagrangian which is an arbitrary function of R leads to fourth-order equations of motions. Indeed, $f(R)$-gravity is a fourth-order theory and equivalent with a special case of the scalar-tensor theory.

Consider a Riemannian manifold with metric $g_{\mu\nu}$ and Ricciscalar R. The Action Integral of the $f(R)$-gravity is as follows

$$S=\int dx^4\sqrt{-g}f(R) \quad (27)$$

when $f(R)=R-2\Lambda$ the limit of general relativity is recovered.

We make use of the Lagrange multiplier λ, thus the action integral (27) reads

$$S=\int dx^4\sqrt{-g}\left(f(\chi)+\lambda(\chi-R)\right). \quad (28)$$

The equation of motion for λ is $\frac{\partial S}{\partial\chi}=0$, that is, $\lambda=-f_{,\chi}$.

By replacing λ in (28) we find

$$S=\int dx^4\left[f_{,\chi}R+\left(f-\chi f_{,\chi}\right)\right], \quad (29)$$

or equivalently

$$S=\int dx^4\left[\varphi R+V(\varphi)\right],\ \varphi=f_{,\chi}, \quad (30)$$

and potential function $V(\varphi)=\left(f-\chi f_{,\chi}\right)$.

4.2 $F(R,\Box R)$-gravity

We consider the sixth-order theory of gravity with action integral

$$S=\int dx^4\sqrt{-g}F(R,\Box R) \quad (31)$$

where $\Box R=g^{\mu\nu}R_{;\mu\nu}$.

We consider the new variables χ,ζ and we introduce the Lagrange multipliers $\lambda_1,\ \lambda_2$. Therefore, the action integral (31) is expressed as follows

$$S=\int dx^4\sqrt{-g}\left[F(\chi,\zeta)+\lambda_1(\chi-R)+\lambda_2(\zeta-\Box R)\right]. \quad (32)$$

Variation with respect to the Lagrange multipliers provide the equations of motions $\lambda_1+F_{,\chi}=0$ and $\lambda_2+F_{,\zeta}=0$.

We replace in the gravitational action integral (32) and we find

$$S=\int dx^4\sqrt{-g}\left[\left(F_{,\chi}+F_{,\zeta}\Box\right)R+\left(F-\chi F_{,\chi}-\zeta F_{,\zeta}\right)\right] \quad (33)$$

Integration by parts of the second term of (33) gives

$$\int dx^4\sqrt{-g}\left(F_{,\zeta}\Box R\right)=-\int dx^4\sqrt{-g}\left(F_{,\zeta\zeta}g^{\mu\nu}\chi_{;\mu}\zeta_{;\nu}\right).$$

Consequently we end with the gravitational action integral

$$S=\int dx^4\sqrt{-g}\Big[F_{,\chi}R-F_{,\zeta\zeta}g^{\mu\nu}\nabla_\mu\chi\nabla_\nu\zeta$$

$$+\left(F-\chi F_{,\chi}-\zeta F_{,\zeta}\right)\Big] \quad (34)$$

As we expected we have two scalar fields, the $\{\chi,\zeta\}$, because $F(R,\Box R)$-gravity is a sixth-order gravity. The field χ is a non-minimally coupled field and the field ζ is an extra field which can be seen as perturbation effects. What is of special interest is the non-diagonal term $F_{,\zeta\zeta} g^{\mu\nu}\chi_{;\mu}\zeta_{;\nu}$ which has similarities with the non-diagonal term in (15).

4.2.1 $F(R,\Box R) = f(R) + K(\Box R)$

Assume now the case where $F(R,\Box R)$ is a separable function, i.e. $F_{,\chi\zeta} = 0$. Hence, by replacing $F(R,\Box R) = f(R) + K(\Box R)$ in (34) we derive

$$S = \int dx^4\sqrt{-g}\left[M(\chi)R - g^{\mu\nu}\chi_{;\mu}\psi_{;\nu} + V(\chi) + \hat{V}(\psi)\right] \tag{35}$$

where $\psi = K_{,\zeta}$, $M(\chi) = f_{,\chi}(\chi)$, $V(\chi) = f(\chi) - \chi f_{,\chi}(\chi)$ and $\hat{V}(\psi) = K(\zeta) - \zeta K_{,\zeta}(\zeta)$.

We observe that the action integral (35) is of the form of the GUP scalar-tensor theory (15) for zero value of parameter ω, that is, $\omega = 0$. Moreover, for $K(\Box R) = \frac{1}{2\hat{\beta}}(\Box R)^2$, we find $\hat{V}(\psi) = -\frac{\hat{\beta}}{2}\psi^2$, which means that that the separable $F(R,\Box R) = f(R) + \frac{1}{2\hat{\beta}}(\Box R)^2$ introduce new geometrodynamical terms in the field equations which can be attributed to the corrections provided by the quadratic GUP in scalar-tensor theory. In our case, the quantum corrections of $f(R)$-gravity are on the limit of O'Hanlon theory where $\omega = 0$. Parameter $\hat{\beta}$ is the analogue of the deformed parameter β of GUP.

In the special case where $f(R)$ is a linear function and for the $F(R,\Box R) = R - 2\Lambda + \frac{1}{2\alpha}(\Box R)^2$ theory, from (35) it follows

$$S = \int dx^4\sqrt{-g}\left[R - 2\Lambda - \hat{\beta}g^{\mu\nu}\chi_{;\mu}\psi_{;\nu} - \frac{\hat{\beta}}{2}\psi^2\right]. \tag{36}$$

5 Cosmological solutions

We proceed with the analysis of cosmological evolution for the physical parameters for the gravitational action integral (35) with background geometry the FLRW spacetime (16).

For the spatially flat FLRW geometry, the field equations can be derived from the variation of the point-like Lagrangian function

$$L\left(a,\dot{a},\chi,\dot{\chi},\psi,\dot{\psi}\right) = 6aM(\chi)\dot{a}^2 + 6a^2M_{,\phi}(\chi)\dot{a}\dot{\chi} + a^3\dot{\chi}\dot{\psi} + a^3\left(V(\chi) + \hat{V}(\psi)\right), \tag{37}$$

in which the first modified Friedmann equation is the constraint equation

$$6M(\chi)H^2 + 6M_{,\chi}(\chi)H\dot{\chi} + \dot{\chi}\dot{\psi} - \left(V(\chi) + \hat{V}(\psi)\right) = 0, \tag{38}$$

which can be seen as the conservation laws of energy for the three-dimensional dynamical system described by the point-like Lagrangian (37). The rest of the field equations are the Euler–Lagrange equations $\left(\frac{d}{dt}\frac{\partial}{\partial\dot{\mathbf{y}}} - \frac{\partial}{\partial\mathbf{y}}\right)L(\mathbf{y},\dot{\mathbf{y}}) = 0$, where now $\mathbf{y} = (a,\chi,\psi)$.

Consequently, the second-order differential equations are

$$2M\left(2\dot{H} + 3H^2\right) + 2M_{,\chi}(2H\dot{\chi} + \ddot{\chi}) + 2\dot{\chi}^2M_{,\chi\chi} - \dot{\chi}\dot{\psi} - \left(V(\chi) + \hat{V}(\psi)\right) = 0, \tag{39}$$

$$\ddot{\chi} + 3H\dot{\chi} - \hat{V}(\psi)_{,\psi} = 0, \tag{40}$$

$$\ddot{\psi} + 3H\dot{\psi} + 6M_{,\chi}\left(\dot{H} + 2H^2\right) - V_{,\chi} = 0. \tag{41}$$

Let us focus in the case where $M(\chi) = 1$ and $V(\chi) = 0$. Then the latter gravitational field equations read

$$3H^2 + \frac{1}{2}\left(\dot{\chi}\dot{\psi} - \hat{V}(\psi)\right) = 0, \tag{42}$$

$$2\dot{H} + 3H^2 - \frac{1}{2}\left(\dot{\chi}\dot{\psi} + \hat{V}(\psi)\right) = 0, \tag{43}$$

$$\ddot{\chi} + 3H\dot{\chi} - \hat{V}(\psi)_{,\psi} = 0, \tag{44}$$

$$\ddot{\psi} + 3H\dot{\psi} + 6\left(\dot{H} + 2H^2\right) = 0. \tag{45}$$

Therefore, the effective energy density and pressure components are defined as

$$\rho_{eff} = \frac{1}{2}\left(-\dot{\chi}\dot{\psi} + \hat{V}(\psi)\right), \tag{46}$$

$$p_{eff} = -\frac{1}{2}\left(\dot{\chi}\dot{\psi} + \hat{V}(\psi)\right), \tag{47}$$

where the effective equation of state parameter is written

$$w_{eff} = \frac{\left(\dot{\chi}\dot{\psi} + \hat{V}(\psi)\right)}{\left(\dot{\chi}\dot{\psi} - \hat{V}(\psi)\right)}. \tag{48}$$

Indeed, when the potential function $\hat{V}(\psi)$ dominates the asymptotic solution is that of the de Sitter universe with $w_{eff} = -1$.

If we define the new scalar fields $\chi = \phi + \varpi$, $\dot{\psi} = \phi - \varpi$, then Eq. (42) reads

$$3H^2 + \frac{1}{2}\left(\left(\dot{\phi}^2 - \dot{\varpi}^2\right) - \hat{V}(\phi - \varpi)\right) = 0, \tag{49}$$

which means that the model is equivalent to the quintom cosmological model with a mixed potential term, which is different from the analysis presented in [105]. For a complete analysis of the dynamics in quintom cosmology we refer the reader in [106].

Below we present the phase-space analysis for the field equations (42)–(45) by using dimensionless variables in the H-normalization approach [10].

5.1 Phase-space analysis

We define the new dimensionless variables

$$x = \frac{\dot{\chi}}{H}\ ,\ z = \frac{\dot{\psi}}{6H}\ ,\ w = \frac{\hat{V}(\psi)}{6H^2}\ ,\ \lambda = \frac{\hat{V}_{,\psi}}{\hat{V}},\ \tau = \ln a \quad (50)$$

and we rewrite the field equations (42)–(45) in the form of the subsequent algebraic-differential system

$$\frac{dx}{d\tau} = \frac{3}{2}\left(w\left(4\lambda - x\right) - x\left(1 + x + z\right)\right), \quad (51)$$

$$\frac{dz}{d\tau} = -\frac{3}{2}z\left(1 + w + xz\right), \quad (52)$$

$$\frac{dw}{d\tau} = 3w\left(1 - w - \left(x - 2\lambda\right)z\right), \quad (53)$$

$$\frac{d\lambda}{d\tau} = 6\lambda^2 z\left(\Gamma\left(\lambda\right) - 1\right)\ ,\ \Gamma\left(\lambda\right) = \frac{V_{\psi\psi}V}{\left(V_{,\psi}\right)^2}, \quad (54)$$

with constraint

$$1 + xz - w = 0. \quad (55)$$

By utilizing the constraint equation, we can effectively reduce the dimension of the aforementioned dynamical system by one.

Hence, we arrive at the three-dimensional system

$$\frac{dx}{d\tau} = 3\left(2\lambda - x\right)\left(1 + xz\right), \quad (56)$$

$$\frac{dz}{d\tau} = -3z\left(1 + xz\right), \quad (57)$$

$$\frac{d\lambda}{d\tau} = 6\lambda^2 z\left(\Gamma\left(\lambda\right) - 1\right). \quad (58)$$

At each stationary point $P = (x(P), z(P), \lambda(P))$of the dynamical system (56)-(58) describes an asymptotic solution where the effective fluid source has the equation of state parameter

$$w_{eff} = -1 - 2x(P)z(P). \quad (59)$$

5.2 Exponential potential

In order to reduce further the dimension of the dynamical system we consider the simple case where $\hat{V}(\psi)$ is the exponential potential, that is, $\hat{V}(\psi) = V_0 e^{\lambda_0\psi}$. For this potential we calculate $\Gamma(\lambda) = 1$, and $\lambda = \lambda_0$ is always a constant.

Hence, the stationary points of the dynamical system (56), (57) are

$$P_1 = \left(x_1, -\frac{1}{x_1}\right),\ P_2 = (2\lambda, 0).$$

For the family of points P_1 we derive $w_{eff}(P_1) = 1$, this implies that the points represent a family of stiff fluid solutions. Furthermore, the asymptotic solution at point P_2 describes the accelerated de Sitter universe, because $w_{eff}(P_2) = -1$.

Let us now proceed with the investigation of the stability properties of the points. For the linearization system (56), (57) around the stationary points P_1 we determine the eigenvalues $e_1(P_1) = 6\left(1 - \frac{\lambda}{x_1}\right)$, $e_2(P_1) = 0$. For $1 - \frac{\lambda}{x_1} > 0$ points P_1 are sources, however for $1 - \frac{\lambda}{x_1} < 0$ we will infer about the stability of the points from the phase-space portraits. For point P_2 the eigenvalues of the linearized system are $e_1(P_2) = -3$ and $e_2(P_2) = -3$, which means that the de Sitter universe is a future attractor for the dynamical system.

In Fig. 1 we present the phase-space portrait for the two-dimensional dynamical system (56), (57). We observe that the family of points P_1 describe always unstable solutions and they are the boundaries where the trajectories move to the infinity.

5.2.1 *Analysis at infinity*

We define the Poincare variables

$$x = \frac{X}{\sqrt{1 - X^2 - Z^2}}\ ,\ z = \frac{Z}{\sqrt{1 - X^2 - Z^2}}$$

and the new independent variable $dT = \sqrt{1 - X^2 - Z^2}d\tau$.

In the Poincare variables the field equations read

$$\frac{dX}{dT} = -3\left(X^2 + Z^2 - 1 - XZ\right)\left(2\lambda - X\left(2\lambda X + \sqrt{1 - X^2 - Z^2}\right)\right), \quad (60)$$

$$\frac{dZ}{dT} = 3Z\left(X^2 + Z^2 - 1 - XZ\right)\left(2\lambda X + \sqrt{1 - X^2 - Z^2}\right), \quad (61)$$

and the $w_{eff}(x, z)$ becomes

$$w_{eff}(X, Z) = -1 - \frac{2XZ}{1 - X^2 - Z^2}. \quad (62)$$

Hence, in order the solutions at the infinity to be physical accepted it should be $XZ \geq 0$. For $XZ > 0$ at infinity the solutions describe Big Rip singularities with $w_{eff} \to -\infty$, while in the limit $XZ = 0$, the de Sitter universe is recovered.

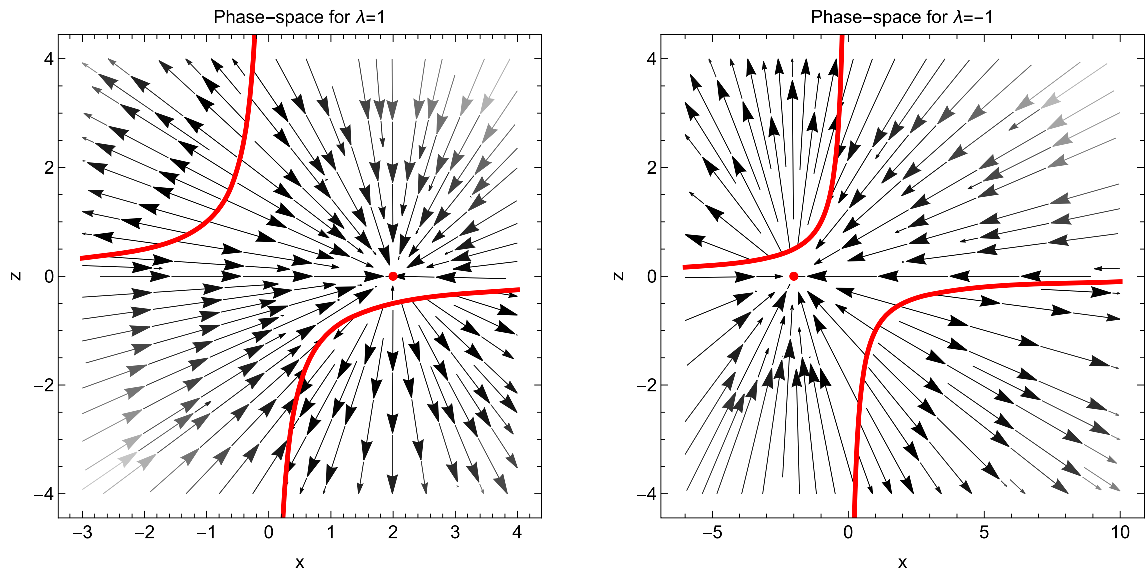

Fig. 1 Phase-space portrait for the two dynamical system (56), (57) for $\lambda = 1$ (left fig.) and $\lambda = -1$ (right fig.). With red lines are the family of points P_1 and with red dot is marked the de Sitter attractor P_2. We observe that points P_1 describe always unstable solutions and are the limits where the trajectories move to the infinity regime

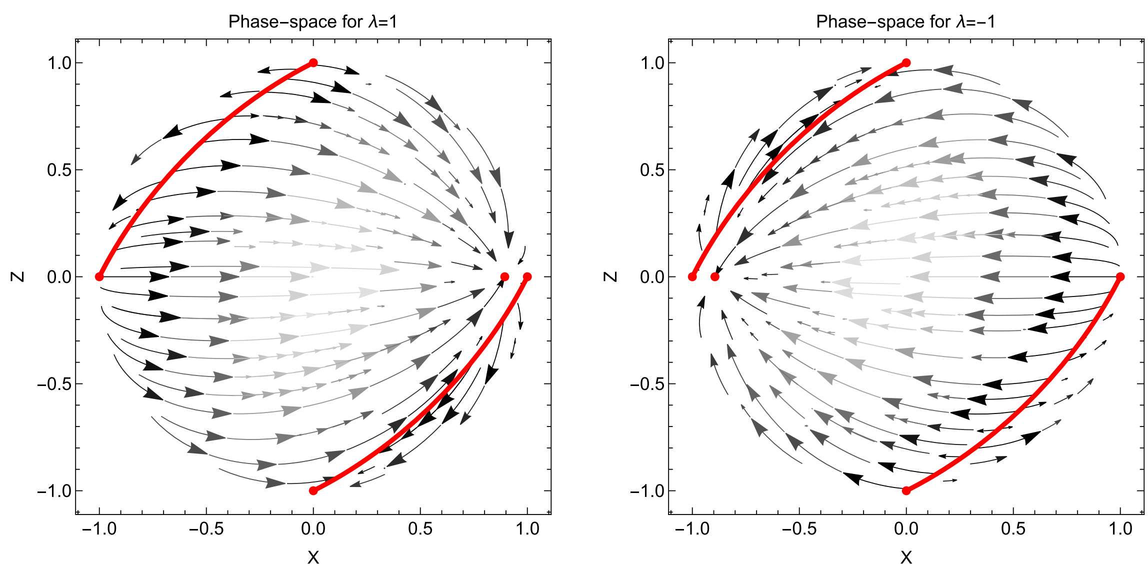

Fig. 2 Phase-space portrait for the two dynamical system (56), (57) for $\lambda = 1$ (left fig.) and $\lambda = -1$ (right fig.) in the Poincare variables (X, Z). With red lines are the family of points P_1 and with red dot are marked the de Sitter attractor P_2 and the stationary points at the infinity $Q_1^{\pm}$ and $Q_2^{\pm}$. The regions outside the red lines (points P_1) lead to unphysical solutions with $w_{eff} > 1$, thus the accepted initial conditions of the dynamical system are those inside the red lines

The stationary points at the infinity are calculated $Q_1^{\pm} = (\pm 1, 0)$ and $Q_2^{\pm} = (0, \pm 1)$, which means that $w_{eff}\left(Q_1^{\pm}\right) = -1$ and $w_{eff}\left(Q_1^{\pm}\right) = -1$. The linearized system at points $Q_1^{\pm}$, $Q_2^{\pm}$ are $e_1\left(Q_1^{\pm}\right) = 0$, $e_2\left(Q_1^{\pm}\right) = 0$ and $e_1\left(Q_2^{\pm}\right) = \pm 6\lambda_2$, $e_2\left(Q_2^{\pm}\right) = 0$ respectively. From the phase-space portrait of Fig. 2, we remark that the stationary points at infinity describe always unstable solutions and the unique attractor of the dynamical system is the de Sitter solution described by the stationary point P_2. The regions outside the red lines (points P_1) lead to unphysical solutions with $w_{eff}\,(X, Z) > 1$, thus the accepted initial conditions of the dynamical system are those inside the red lines.

6 Conclusions

Scalar field gravitational theories which are modified by the presence of a minimum length in the Uncertainty Principle are equivalent with two-scalar field theories of gravity. In this work we investigate the equivalent modified theory of gravity which can be described by the latter GUP scalar field theory. We assumed $F(R, \Box R)$-gravitational model which is a sixth-order theory. We found that the separable model $F(R, \Box R) = f(R) + K(\Box R)$ can be in comparison with the GUP scalar field theory. Specifically, for the $K(\Box R) = \frac{1}{2\hat{\beta}}(\Box R)^2$ model, there are introduced geometrodynamical terms in the field equations which are in the same form with that introduced by the deformed Heisenberg algebra of the quadratic GUP. Indeed, parameter $\hat{\beta}$ is linearly related with the deformation parameter β of GUP.

We focused in the case of $F(R, \Box R) = R + K(\Box R)$ where now the $K(\Box R)$-terms can attribute the quantum corrections in the limit of General Relativity. For the latter theory and for a spatially flat FLRW background geometry we investigated the cosmological dynamics. From the analysis of the phase-space it was found that there exist always a future attractor which corresponds to the de Sitter solution without to introduce a cosmological constant term. We conclude that the quantum corrections related by the $K(\Box R)$ can drive the dynamics such that inflation to occurred.

In a future work we plan to investigate further the effects of the introduction of higher-order terms $K(\Box R)$ in $f(R)$f theory for the cosmological evolution and investigate the case of EUP in gravitational models.

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