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Weyl quadratic gravity as a gauge theory and non-metricity vs torsion duality

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Abstract We review (non-supersymmetric) gauge theories of four-dimensional space-time symmetries and their quadratic action. The only true gauge theory of such a symmetry (with a physical gauge boson) that has an exact geometric interpretation, generates Einstein gravity in its spontaneously broken phase and is anomaly-free, is that of Weyl gauge symmetry (of dilatations). Gauging the full conformal group does not generate a true gauge theory of physical (dynamical) associated gauge bosons. Regarding the Weyl gauge symmetry, it is naturally realised in Weyl conformal geometry, where it admits two different but equivalent geometric formulations, of same quadratic action: one non-metric but torsion-free, the other Weyl gauge-covariant and metric (with respect to a new differential operator). To clarify the origin of this intriguing result, a third equivalent formulation of this gauge symmetry is constructed using the standard, modern approach on the tangent space (uplifted to space-time by the vielbein), which is metric but has vectorial torsion. This shows an interesting duality vectorial non-metricity vs vectorial torsion of the corresponding formulations, related by a projective transformation. We comment on the physical meaning of these results.

1 Motivation

The principle of gauge symmetries has been remarkably successful in high energy physics. Here we use it in gauge theories of space-time symmetries such as the Weyl group (Poincaré $\times$ dilatations) and the conformal group, see [1] for a review. In our view a realistic gauge theory with such symmetry should: a) recover Einstein gravity in its (spontaneously) broken phase, b) have a geometric interpretation (as a theory of gravity) and c) be anomaly-free.

Weyl gauge symmetry (of dilatations) is naturally built in Weyl conformal geometry [2,3] (for a review [4]) and thus it does have a geometric formulation. The Weyl gauge boson of dilatations (ω_μ) is dynamical, with a field strength $F_{\mu\nu}$ as the length curvature tensor – a clear geometric origin. This means that the length is not integrable, which means that the geometry is non-metric i.e. there is a non-zero $\tilde{\nabla}_\mu g_{\alpha\beta} = -2\omega_\mu g_{\alpha\beta}$. The Weyl gauge symmetry of the associated (Weyl) *quadratic* action of this gauge theory is spontaneously broken à la Stueckelberg to Einstein gravity [5,6], so ω_μ becomes massive and decouples, hence non-metricity effects are strongly suppressed. Since the Standard Model (SM) with vanishing Higgs mass parameter is scale invariant, it is naturally embedded in Weyl geometry with *no* additional degrees of freedom [7]. This gauge symmetry can be maintained at quantum level which indicates it is anomaly-free [8], as required for a consistent (quantum) gauge theory. Successful inflation is possible [9,10] being just a gauged version of Starobinsky inflation [11]. Good fits for the galaxies rotation curves are also found [12,13] and associated black hole solutions and physics were studied in [14]. All this suggests that Weyl gauge symmetry with its underlying Weyl conformal geometry are the fundamental symmetry and geometry beyond the SM and Einstein gravity.

One can also gauge the full conformal group, in which case one obtains conformal gravity [15,16] (for a review [1]). However, in this case the gauge boson of special conformal transformations f^a_μ is just an auxiliary field absent in the final action. Neither f^a_μ nor ω_μ are then dynamical (i.e. physical), hence this is not a true gauge theory of the conformal group, in the high energy theory sense. Finally, gauging the Poincaré group will generate an action with an infinite series of higher derivative terms, for which we see little motivation.

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Returning to Weyl gauge symmetry, it admits [8] two equivalent geometric formulations in Weyl geometry: one is *non-metric* but *torsion-free*, the other is manifestly *Weyl gauge covariant and metric* with respect to a new differential operator ($\hat{\nabla}$). This intriguing result requires further investigation and this is the main motivation of this work. To this purpose we construct a gauge theory of dilatations in a standard tangent space-time approach uplifted to space-time by the vielbein; this is shown to generate *exactly* the Weyl quadratic action associated to Weyl geometry. This gives a third equivalent formulation, *metric* but *with torsion*, showing a duality (equivalence) vectorial non-metricity vs vectorial torsion. All three formulations are equally good, equivalent descriptions of Weyl quadratic gravity with this gauge symmetry. We comment briefly on some physical aspects of this duality.

2 Weyl gauge symmetry and geometry: equivalent pictures

2.1 Non-metric formulation

Let us discuss Weyl gauge symmetry in its formulation in Weyl geometry.[1] By definition, Weyl geometry is given by equivalence classes $(g_{\alpha\beta}, \omega_\mu)$ of the metric $(g_{\alpha\beta})$ and the Weyl gauge field (ω_μ), which in $d = 4 - 2\epsilon$ dimensions are related by the transformations below, in the absence (a) and presence (b) of scalars (ϕ) and fermions (ψ)

$$\begin{aligned}(a)\, & g'_{\mu\nu} = \Sigma^2\, g_{\mu\nu}, \\ & \omega'_\mu = \omega_\mu - \partial_\mu \ln \Sigma, \qquad \sqrt{g'} = \Sigma^{2d}\sqrt{g}, \\ (b)\, & \phi' = \Sigma^{q_\phi}\phi, \qquad \psi' = \Sigma^{q_\psi}\,\psi,\end{aligned} \tag{1}$$

Without loss of generality, for $g_{\mu\nu}$ we set a Weyl charge $q = 2$, then $q_\phi = -(d-2)/2$ and $q_\psi = -(d-1)/2$ as dictated by their canonical kinetic terms. This defines the *Weyl gauge symmetry* or gauged dilatations symmetry. This should be distinguished from what is generically called "Weyl symmetry" where there is no gauge field. By definition Weyl geometry is *non-metric* i.e. $\tilde{\nabla}_\mu g_{\nu\rho} \neq 0$, with:

$$\begin{aligned}&(\tilde{\nabla}_\lambda + 2\,\omega_\lambda) g_{\mu\nu} = 0, \qquad \text{where} \\ &\tilde{\nabla}_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \tilde{\Gamma}^\rho_{\lambda\mu} g_{\rho\nu} - \tilde{\Gamma}^\rho_{\lambda\nu} g_{\mu\rho}.\end{aligned} \tag{2}$$

The Weyl connection $\tilde{\Gamma}^\rho_{\mu\nu}$ is found from (2). In this non-metric formulation of Weyl geometry one assumes a symmetric connection (i.e. no torsion) $\tilde{\Gamma}^\rho_{\mu\nu} = \tilde{\Gamma}^\rho_{\nu\mu}$, giving a solution

$$\begin{aligned}\tilde{\Gamma}^\rho_{\mu\nu} &= \mathring{\Gamma}^\rho_{\mu\nu} \\ &+ \big[\delta^\rho_\mu \omega_\nu + \delta^\rho_\nu \omega_\mu - g_{\mu\nu}\omega^\rho\big],\end{aligned} \tag{3}$$

with $\mathring{\Gamma}^\lambda_{\mu\nu}$ the Levi-Civita (LC) connection. The Riemann curvature tensor in Weyl geometry associated to this connection is defined as in a Riemannian case, but now in terms of $(\tilde{\Gamma})$:

$$\tilde{R}^\rho{}_{\sigma\mu\nu} = \partial_\mu \tilde{\Gamma}^\rho_{\nu\sigma} - \partial_\nu \tilde{\Gamma}^\rho_{\mu\sigma} + \tilde{\Gamma}^\rho_{\mu\tau}\tilde{\Gamma}^\tau_{\nu\sigma} - \tilde{\Gamma}^\rho_{\nu\tau}\tilde{\Gamma}^\tau_{\mu\sigma}, \tag{4}$$

$\tilde{R}^\rho{}_{\sigma\mu\nu}$ can be expressed in terms of ω_μ, for technical details see Appendix A in [8]. From Eq. (4) one finds the expressions of the Ricci tensor $\tilde{R}_{\mu\nu}$ and scalar $\tilde{R}$ in Weyl geometry

$$\begin{aligned}\tilde{R}_{\mu\nu} &= \tilde{R}^\rho{}_{\mu\rho\nu} = \mathring{R}_{\mu\nu} + \frac{d}{2} F_{\mu\nu} - (d-2)\mathring{\nabla}_{(\mu}\omega_{\nu)} \\ &- g_{\mu\nu}\mathring{\nabla}_\lambda \omega^\lambda + (d-2)(\omega_\mu\omega_\nu - g_{\mu\nu}\omega_\lambda\omega^\lambda),\end{aligned} \tag{5}$$

$$\tilde{R} = g^{\mu\nu}\tilde{R}_{\mu\nu} = \mathring{R} - 2(d-1)\,\mathring{\nabla}_\mu\omega^\mu - (d-1)(d-2)\,\omega_\mu\omega^\mu, \tag{6}$$

$\mathring{R}_{\mu\nu}$, $\mathring{R}$ are the Ricci tensor and scalar in a Riemannian case, respectively, $\mathring{\nabla}$ is the covariant derivative of Riemannian geometry (with LC connection); $\mathring{\nabla}_{(\mu}\omega_{\nu)} \equiv (1/2)(\mathring{\nabla}_\mu\omega_\nu + \mathring{\nabla}_\nu\omega_\mu)$. While $\tilde{R}^\rho{}_{\sigma\mu\nu}$, $\tilde{R}_{\mu\nu}$ are invariant since $\tilde{\Gamma}$ is, $\tilde{R}$ transforms covariantly under (1), like $g^{\mu\nu}$.

The Weyl tensor in Weyl geometry $(\tilde{C}_{\mu\nu\rho\sigma})$ associated to $\tilde{R}_{\mu\nu\rho\sigma}$ is related to the Riemannian $\mathring{C}_{\mu\nu\rho\sigma}$ [8]

$$\tilde{C}^2_{\mu\nu\rho\sigma} = \mathring{C}^2_{\mu\nu\rho\sigma} + (d^2 - 2d + 4)/(d-2)\, F^2_{\mu\nu}. \tag{7}$$

In Weyl geometry there also exists a so-called length curvature tensor $F_{\mu\nu} = \tilde{\nabla}_\mu\omega_\nu - \tilde{\nabla}_\nu\omega_\mu = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu$, which is interpreted as the field strength of ω_μ, where we used that $\tilde{\Gamma}$ is symmetric and $\tilde{\nabla}_\mu\omega_\nu = \partial_\mu\omega_\nu - \tilde{\Gamma}^\rho_{\mu\nu}\omega_\rho$. This ends our geometric definitions.

With this information, the *most general* Lagrangian of Weyl quadratic gravity associated to Weyl geometry in the absence of matter can be written as [3]

$$S = \int d^4x \sqrt{g}\,\Big\{a_0 \tilde{R}^2 + b_0 \tilde{F}^2_{\mu\nu} + c_0 \tilde{C}^2_{\mu\nu\rho\sigma} + d_0 \tilde{G}\Big\}, \tag{8}$$

where a_0, b_0, c_0, d_0 are constants and $\tilde{G}$ is the Chern–Euler–Gauss–Bonnet term (hereafter called Euler term) which is a total derivative (only) for $d = 4$; its expression in d dimensions is found in [8, (Eq. (A-14))]. *No other independent terms are allowed in S by the symmetry!*

Each term in S is separately Weyl gauge invariant, as one can easily check. Since the theory is *non-metric*, in applications one is forced to use the (metric) Riemannian formulation obtained from S by using relations (5), (6), (7) to curvature tensors and scalar of Riemannian geometry. For more technical details see Appendix A in [8].

As discussed extensively in [5–7], the gauge theory of action S has spontaneous breaking à la Stueckelberg to Einstein gravity and a small cosmological constant, after *dynamical* ω_μ becomes massive and decouples after "eating" the

[1] Our conventions [17]: $g_{\mu\nu}$ with $(+,-,-,-)$, $g = |\det g_{\mu\nu}|$. To restore the gauge coupling α of dilatations, rescale $\omega_\mu \to \omega_\mu\alpha$. For $g_{\mu\nu}$ of charge q rescale $\Sigma \to \Sigma^{q/2}$. We work in $d = 4 - 2\epsilon$, as needed at quantum level.

dilaton $\ln\phi$; here ϕ is the scalar field propagated by the (geometric) $\tilde{R}^2$ term in the action. Hence, Einstein gravity is just a "low-energy" effective theory obtained in the broken phase of action (8) and this breaking takes place in the absence of matter. Mass generation (Planck mass, cosmological constant, m_ω) has *geometric origin*, being proportional to $\langle\phi\rangle$, and is also related to a non-vanishing (geometric) length-curvature tensor, $F_{\mu\nu}\neq 0$ [18].

In the presence of the SM, this mechanism receives corrections from the Higgs itself, see Section 2.5 in [7], (also [23]) where the phenomenology of SM embedded in Weyl geometry was studied in detail. Other phenomenological aspects of action S such as successful inflation were discussed in [9,10] together with interesting implications for dark matter [12,13] and black hole physics [14].

2.2 Weyl gauge-covariant formulation

For a gauge theory one would actually like to have manifest Weyl gauge-covariance. The gauge theory formulation in Sect. 2.1 is not entirely satisfactory because it is not manifestly covariant, as one can easily see: the partial derivative ∂_μ in $\tilde{\nabla}_\mu$ when acting on the (geometric) tensors like $\tilde{R}_{\mu\nu}$, etc, or on scalar $\tilde{R}$, is not Weyl gauge-covariant. The explanation is that one should account for the effect of their Weyl charges in the derivative acting on them, etc. A related issue is that the geometry is not metric ($\tilde{\nabla}_\mu g_{\nu\rho}\neq 0$) making calculations difficult and forcing one to go to a Riemannian picture.

The non-metricity and the absence of manifest Weyl gauge covariance in the previous geometric formulation can be addressed and solved simultaneously. Since $(\tilde{\nabla}_\lambda + q\omega_\lambda)g_{\mu\nu}=0$, where $q=2$ is the charge of $g_{\alpha\beta}$, this suggests that for any tensor T, including $g_{\mu\nu}$, of Weyl charge[2] q_T ($T'=\Sigma^{q_T}T$) one should introduce a new differential operator $\hat{\nabla}$ (replacing $\tilde{\nabla}$)

$$\hat{\nabla}_\lambda T \equiv (\tilde{\nabla}_\lambda + q_T\,\omega_\lambda)\,T. \tag{9}$$

This new operator transforms covariantly under (1), as seen by using that $\tilde{\Gamma}$ is invariant: $\hat{\nabla}'_\mu T' = \Sigma^{q_T}\hat{\nabla}_\mu T$. The theory is then metric with respect to the new operator: $\hat{\nabla}_\mu g_{\alpha\beta}=0$.

For reasons that become clear shortly, we also define a new Riemannian and Ricci tensors and Ricci scalar of Weyl geometry [8,19,20]

$$\begin{aligned}
&\hat{R}_{\mu\nu\rho\sigma} = \tilde{R}_{\mu\nu\rho\sigma} - g_{\mu\nu}\hat{F}_{\rho\sigma},\\
&\hat{R}_{\nu\sigma} = \tilde{R}_{\nu\sigma} - \hat{F}_{\nu\sigma},\\
&\hat{R} = \tilde{R}.
\end{aligned} \tag{10}$$

with $\hat{F}_{\mu\nu}=F_{\mu\nu}=\partial_\mu\omega_\nu-\partial_\nu\omega_\mu$. Note also that $\hat{R}_{\mu\nu}-\hat{R}_{\nu\mu}=(d-2)F_{\mu\nu}$, relevant later. With (4), (5) one easily writes these curvatures in terms of their Riemannian counterparts.

One benefit of the new "hat" basis is that the new Weyl tensor $\hat{C}_{\mu\nu\rho\sigma}$ associated to $\hat{R}_{\mu\nu\rho\sigma}$ and Euler terms become [8, (Section 3.1)]

$$\begin{aligned}
&\hat{C}_{\mu\nu\rho\sigma} = \mathring{C}_{\mu\nu\rho\sigma},\\
&\hat{G} = \hat{R}_{\mu\nu\rho\sigma}\hat{R}^{\rho\sigma\mu\nu} - 4\hat{R}_{\mu\nu}\hat{R}^{\nu\mu} + \hat{R}^2.
\end{aligned} \tag{11}$$

The new Weyl tensor is identical to that in Riemannian geometry, while $\hat{G}$ is $\tilde{G}$ of previous section but in the "hat basis" and is a generalisation to Weyl geometry of the Euler term.

A second important benefit is the Weyl gauge covariance under transformation (1)

$$\begin{aligned}
X' = \Sigma^{-4}X,\quad X = \hat{R}^2_{\mu\nu\rho\sigma},\ \hat{R}^2_{\mu\nu},\ \hat{R}^2,\\
\hat{C}^2_{\mu\nu\rho\sigma},\ \hat{G},\ \hat{F}^2_{\mu\nu}.
\end{aligned} \tag{12}$$

$$\begin{aligned}
&\hat{\nabla}'_\mu \hat{R}' = \Sigma^{-2}\hat{\nabla}_\mu\hat{R},\\
&\hat{\nabla}'_\mu\hat{\nabla}'^\mu\hat{R}' = \Sigma^{-4}\hat{\nabla}_\mu\hat{\nabla}^\mu\hat{R},\quad \hat{\nabla}'_\rho\hat{R}'_{\mu\nu} = \hat{\nabla}_\rho\hat{R}_{\mu\nu},\ \text{etc.}
\end{aligned} \tag{13}$$

Unlike its Riemannian version, the Euler term $\hat{G}$ is now Weyl covariant in arbitrary d dimensions (just like $\hat{C}^2_{\mu\nu\rho\sigma}$) which is very important for maintaining this symmetry at quantum level and avoiding the Weyl anomaly [8]. With this information, action (8) becomes

$$S = \int d^4x\sqrt{g}\,\Big\{a_0\hat{R}^2 + b_0\hat{F}^2_{\mu\nu} + c_0\hat{C}^2_{\mu\nu\rho\sigma} + d_0\hat{G}\Big\}. \tag{14}$$

up to a redefinition of b_0. Each term in S is again separately invariant under (1) for $d=4$.

The Weyl covariance of $\hat{R}$ enables us to maintain Weyl gauge symmetry also in $d=4-2\epsilon$ dimensions by a natural "geometric" analytical continuation

$$\begin{aligned}
S = \int d^dx\sqrt{g}\,\Big\{&a_0\hat{R}^2 + b_0\hat{F}^2_{\mu\nu}\\
&+c_0\hat{C}^2_{\mu\nu\rho\sigma} + d_0\hat{G}\Big\}\,\hat{R}^{2(d-4)/4}.
\end{aligned} \tag{15}$$

Quantum calculations can now be done [8] in this metric-like, Weyl gauge covariant picture.[3]

To conclude, with respect to the new $\hat{\nabla}$ operator we simultaneously have a *metric-like* formulation and a Weyl gauge-covariant description of *geometric* operators (curvature tensors/scalar) *and* of their derivatives, as in any gauge theory. Action (14) is equivalent to (8) up to a re-definition of b_0, so it gives the same physics. We thus presented a *manifestly*

[2] The charge q_T is in principle arbitrary. For the objects used in this paper they are given on page 5.

[3] This Weyl invariant regularisation implicitly assumes $\tilde{R}\neq 0$, which is verified a-posteriori [8].

covariant, metric formulation of Weyl geometry as a gauge theory of space-time dilatations. Quantum calculations can now be done directly in this (metric) formulation of Weyl geometry [8] using (15) while keeping a manifest Weyl gauge symmetry in d dimensions for each term in the action; in this way one shows that S of (15) is anomaly-free [8].

2.3 Tangent space formulation has torsion

In the previous sections we presented the Weyl gauge symmetry from its realisation in Weyl geometry, using a geometric approach that lead to two equivalent formulations. This equivalence demands some clarification in the modern gauge theory approach. We do this by constructing the gauge theory of the Weyl group on the tangent space and uplifting it to space-time by the vielbein, see [1,21].

The Weyl group is a subgroup of the conformal group which consists of the Poincaré group × dilatations (it does not include special conformal transformations). The gauge algebra is

$$\begin{aligned}&[P_a, M_{bc}] = \eta_{ab} P_c - \eta_{ac} P_a,\\ &[D, P_a] = P_a, \qquad [P_a, P_b] = 0, \qquad [D, M_{ab}] = 0,\\ &[M_{ab}, M_{cd}] = \eta_{ac} M_{db} - \eta_{bc} M_{da} - \eta_{ad} M_{cb} + \eta_{bd} M_{ca},\end{aligned} \tag{16}$$

where η_{ab} is the Minkowski metric and $a, b, \ldots$ denote tangent space indices. P_a, M_{ab} and D are the generators of translations, Lorentz transformations (rotations) and dilatations, respectively. Their associated gauge fields are the vielbein e_μ^a, spin connection ${\rm w}_\mu{}^{ab}$ and Weyl boson ω_μ, respectively. The corresponding structure constants can be read from the Lie algebra $[T_A, T_B] = f_{AB}{}^C T_C$, where T_A stands for P_a, M_{ab}, D. The gauge curvature $R^A_{\mu\nu}$ of the gauge field B^A_μ is $R^A_{\mu\nu} = 2\partial_{[\mu} B^A_{\nu]} + B^B_\mu B^C_\nu f_{BC}{}^A$, (here $x_{[\mu} y_{\nu]} \equiv (1/2)(x_\mu y_\nu - x_\nu y_\mu)$).

With the structure constants from (16) we find the field strength of local translations, rotations and dilatations

$$R_{\mu\nu}(P^a) = 2D_{[\mu} e^a_{\nu]} + 2\omega_{[\mu} e^a_{\nu]}, \tag{17}$$

$$\begin{aligned}R_{\mu\nu}(M^{ab}) &= \partial_\mu {\rm w}_\nu{}^{ab} - \partial_\nu {\rm w}_\mu{}^{ab} + {\rm w}_\mu{}^a{}_c {\rm w}_\nu{}^{cb}\\ &\quad - {\rm w}_\nu{}^a{}_c {\rm w}_\mu{}^{cb} \equiv R^{ab}{}_{\mu\nu}\,,\end{aligned} \tag{18}$$

$$R_{\mu\nu}(D) = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu \equiv F_{\mu\nu}\,, \tag{19}$$

where

$$D_\mu e^a_\nu = \partial_\mu e^a_\nu + {\rm w}_\mu{}^a{}_b e^b_\nu, \tag{20}$$

is the Lorentz covariant derivative. $F_{\mu\nu}$ denotes the field strength of the Weyl gauge field of dilatation ω_μ, and $R^a{}_{b\mu\nu}$ is the usual two-form curvature tensor defined from the commutator of the tangent space (Lorentz) covariant derivatives

$$R^a{}_{b\mu\nu} := e^\sigma_b [D_\mu, D_\nu] e^a_\sigma. \tag{21}$$

Under a general (infinitesimal) gauge transformation $\delta_\epsilon \equiv \epsilon^A T_A = \xi^a P_a + (1/2)\lambda^{ab} M_{ab} + \lambda_D D$ the gauge field change as $\delta_\epsilon B^A_\mu = -\partial_\mu \epsilon^A + \epsilon^B B^C_\mu f_{BC}{}^A$, while the curvatures transform covariantly $\delta_\epsilon R^A_{\mu\nu} = \epsilon^B R^C_{\mu\nu} f_{BC}{}^A$. For the case at hand, considering only dilatations (i.e. setting to zero all gauge parameters except λ_D) we find

$$\begin{aligned}&\delta_\epsilon e^a_\mu = \lambda_D e^a_\mu, \qquad \delta_\epsilon {\rm w}_\mu{}^{ab} = 0,\\ &\delta_\epsilon \omega_\mu = -\partial_\mu \lambda_D,\end{aligned} \tag{22}$$

and

$$\begin{aligned}&\delta_\epsilon R_{\mu\nu}(P^a) = \lambda_D R_{\mu\nu}(P^a), \quad \delta_\epsilon R_{\mu\nu}(M^{ab}) = 0,\\ &\delta_\epsilon R_{\mu\nu}(D) = 0.\end{aligned} \tag{23}$$

Notice that Eq. (22) is an infinitesimal version of (1) of Weyl geometry, with $\Sigma = \exp(\lambda_D)$.

Let us mention the particular case of gauging the Poincaré symmetry recovered from the above formulae by setting $\omega_\mu = 0$. The diffeomorphism invariance of the theory is then implemented by the constraint $R_{\mu\nu}(P^a) = 0$ which in the Poincaré case gives $D_{[\mu} e^a_{\nu]} = 0$. This is just the first Cartan structure equation without torsion which gives the well-known result for the spin-connection $\mathring{\rm w}_\mu{}^{ab} = 2e^{\nu[a}\partial_{[\mu} e_{\nu]}{}^{b]} - e^{\nu[a} e^{b]\sigma} e_{\mu c} \partial_\nu e_\sigma{}^c$.

Compared to the Poincaré case, $R_{\mu\nu}(P^a)$ in Eq. (17) contains now an extra term due to ω_μ. This term can be interpreted as torsion in the first Cartan structure equation

$$D_{[\mu} e^a_{\nu]} = -2\omega_{[\mu} e^a_{\nu]} \equiv T_{\mu\nu}{}^a \tag{24}$$

As a result, the curvature constraint $R_{\mu\nu}(P^a) = 0$ gives a Weyl spin connection ${\rm w}_\mu{}^{ab}$

$${\rm w}_\mu{}^{ab} = \mathring{\rm w}_\mu{}^{ab} + 2e^{[a}_\mu e^{b]\nu} \omega_\nu. \tag{25}$$

It is important to note that the constraint $R_{\mu\nu}(P^a) = 0$ is invariant under dilatations, see (23). Since the original spin-connection is also invariant, see (22), this guarantees that solution (25) does not transform under dilatations. Furthermore, the curvature two-form $R^a{}_{b\mu\nu}$ is also invariant and hence is the correct geometrical object (together with $F_{\mu\nu}$) for building an invariant action.

The above tangent space formulas can now be "uplifted" to space-time with the vielbein. The affine connection $\Gamma^\rho_{\mu\nu} \equiv e^\rho_a D_\mu e^a_\nu$ corresponding to ${\rm w}_\mu{}^{ab}$ becomes

$$\Gamma^\rho_{\mu\nu} = \mathring{\Gamma}^\rho_{\mu\nu} + \delta^\rho_\mu \omega_\nu - g_{\mu\nu}\omega^\rho\,, \tag{26}$$

and is metric compatible $\nabla_\mu g_{\nu\rho} = 0$ but now we have torsion $T_{\mu\nu}{}^\rho \equiv \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} = 2\delta^\rho_{[\mu}\omega_{\nu]}$. For a later discussion, notice that Γ is related to symmetric $\tilde\Gamma$ of (3) of the non-metric formulation, by a projective transformation[4]

$$\tilde\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} + \delta^\rho_\nu \omega_\mu. \tag{27}$$

[4] See [22] for more on projective transformations in the context of Weyl geometry.

Further, the Riemann tensor associated to Γ is the uplifted version of Eq. (21)

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\tau}\Gamma^{\tau}_{\nu\sigma} - \Gamma^{\rho}_{\nu\tau}\Gamma^{\tau}_{\mu\sigma}, \tag{28}$$

and it is antisymmetric in both the first and last pair of indices; however, it is not symmetric in the exchange of the first pair with the last pair. One finds

$$\begin{aligned} R_{\rho\sigma\mu\nu} &= \mathring{R}_{\rho\sigma\mu\nu} \\ &\quad + \Big[g_{\mu\sigma}\mathring{\nabla}_\nu\omega_\rho - g_{\mu\rho}\mathring{\nabla}_\nu\omega_\sigma + g_{\nu\rho}\mathring{\nabla}_\mu\omega_\sigma - g_{\nu\sigma}\mathring{\nabla}_\mu\omega_\rho \Big] \\ &\quad + \omega^2 (g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}) + \omega_\mu(\omega_\rho g_{\nu\sigma} - \omega_\sigma g_{\nu\rho}) \\ &\quad + \omega_\nu(\omega_\sigma g_{\mu\rho} - \omega_\rho g_{\mu\sigma}), \end{aligned} \tag{29}$$

$$\begin{aligned} R_{\mu\nu} &= \mathring{R}_{\mu\nu} - (d-2)\mathring{\nabla}_\nu\omega_\mu - g_{\mu\nu}\mathring{\nabla}_\alpha\omega^\alpha + (d-2)\omega_\mu\omega_\nu \\ &\quad - (d-2) g_{\mu\nu}\omega^\alpha\omega_\alpha \end{aligned} \tag{30}$$

$$R = \mathring{R} - 2(d-1)\,\mathring{\nabla}_\mu\omega^\mu - (d-1)(d-2)\,\omega_\mu\omega^\mu\,. \tag{31}$$

Remarkably, the expressions for $R_{\rho\sigma\mu\nu}$, $R_{\mu\nu}$ and R are identical to those in the Weyl covariant formulation of Eq. (10) with replacements (4), (5), and obtained in the "hat" basis which is metric with respect to $\hat{\nabla}_\mu$. Below we clarify the origin of this equivalence.

In a true gauge theory we need fully covariant derivative operators. Therefore we introduce the derivative $\hat{D}_\mu$ by its action on a tangent space vector V^a of (arbitrary) Weyl weight q_V

$$\hat{D}_\mu V^a = \partial_\mu V^a + q_V\,\omega_\mu\, V^a + \mathrm{w}_\mu{}^a{}_b\, V^b\,. \tag{32}$$

Since $\hat{D}_\mu$ coincides with the standard tangent space derivative D_μ (defined by a spin connection $\mathrm{w}_\mu{}^{ab}$) when acting on tensors with zero Weyl weight it is straightforward to see that $\hat{D}_\mu$ is compatible with the metric η_{ab} as $\hat{D}_\mu\eta_{ab} = D_\mu\eta_{ab} = 0$.

Translating this derivative $\hat{D}_\mu$ to space time by

$$\hat{\nabla}_\mu V^\nu = e^\nu_a \hat{D}_\mu V^a\,, \tag{33}$$

We find precisely $\hat{\nabla}_\mu$ defined in the previous section.

Consider now a Weyl invariant vector on the tangent space V^a. We can write

$$\hat{\nabla}_\mu V^\nu = e^\nu_a \hat{D}_\mu V^a = e^\nu_a D_\mu V^a = \nabla_\mu V^\nu\,. \tag{34}$$

This implies

$$[\hat{\nabla}_\mu, \hat{\nabla}_\nu]V^\rho = [\nabla_\mu, \nabla_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma - T_{\mu\nu}{}^\sigma\nabla_\sigma V^\rho\,, \tag{35}$$

which shows that the Riemann tensor $\hat{R}^\rho{}_{\sigma\mu\nu}$ associated to the metric gauge covariant derivative $\hat{\nabla}_\mu$ is geometrically expressed in terms of a connection with torsion (see Eqs. (26) and (28)). In conclusion we have shown that we have the identity

$$\hat{R}^\rho{}_{\sigma\mu\nu} = R^\rho{}_{\sigma\mu\nu}\,, \tag{36}$$

and similar relations for its contractions, as already checked, see text after Eq. (31). This also confirms that the tensors $R^\rho{}_{\sigma\mu\nu}$ and $R_{\mu\nu}$ are invariant while R transforms covariantly under the gauged dilatation transformation, as already seen in Sect. 2.2.

In conclusion, the Weyl gauge-covariant picture of Sect. 2.2 gives rise to the same curvature tensors/scalar as in the formulation of this section that is metric, with torsion.

We can now write the action for the gauge theory of the Weyl group. It is natural to consider the most general invariant action quadratic in the curvatures, as in any gauge theory, with indices contracted with the metric $g_{\mu\nu}$ or the completely antisymmetric ϵ-density $\epsilon_{\mu_1\ldots\mu_d}$ (or their tangent space counterparts). To derive the general action, one uses the Weyl charges of various fields under gauged dilatations, which are:

Field	e^a_μ	e^μ_a	$g_{\mu\nu}$	$g^{\mu\nu}$	$\mathrm{w}_\mu{}^{ab}$	$\sqrt{g}$	$R^a{}_{b\mu\nu}$	$R_{\mu\nu}$	R	$F_{\mu\nu}$	ϕ	ψ
q	1	- 1	2	- 2	0	d	0	0	- 2	0	$-\frac{d-2}{2}$	$-\frac{d-1}{2}$

By analysing the symmetries of the possible terms, one shows that there are four independent terms in the action, R^2, $R_{(\mu\nu)}R^{(\mu\nu)}$, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ and $F_{\mu\nu}F^{\mu\nu}$ or their combinations. In $d=4$ one can also build topological invariants by using the ϵ-density. We consider the Euler term[5] term G, which for a connection with torsion is given by

$$G = R^2 - 4R_{\mu\nu}R^{\nu\mu} + R_{\mu\nu\rho\sigma}R^{\rho\sigma\mu\nu}. \tag{37}$$

Notice the position of the contracted indices which is essential in making G a topological invariant for a connection with torsion in four dimensions. In d dimensions, G is no longer a topological invariant but it is Weyl gauge-covariant like its counterpart in Sect. 2.2 to which is actually identical. A convenient choice of independent quadratic terms each invariant under gauged dilatations gives the following action (with constants a_0, ..., d_0):

$$S = \int d^4x\sqrt{g}\,\Big[a_0\,R^2 + b_0\,F^2_{\mu\nu} + c_0 C^2_{\rho\sigma\mu\nu} + d_0\,G\Big]\,. \tag{38}$$

This action is identical (up to a redefinition of couplings a_0,..., d_0) to that discussed in the two "geometric" formulations of the previous sections. In d dimensions this action

[5] For a four-dimensional manifold M (compact, orientable, without border) the Euler characteristic can be computed from a general *metric* connection with the formula $\chi(M) = \int_M e(R) = 1/(2\pi)^2\int_M \mathrm{Pf}(R) = 1/(2\pi)^2\int_M 1/(2!\,2^2)\epsilon_{abcd}R^{ab}\wedge R^{cd} = 1/(32\pi^2)\int d^4x\,\sqrt{g}\,(R^2 - 4R_{\mu\nu}R^{\nu\mu} + R_{\mu\nu\rho\sigma}R^{\rho\sigma\mu\nu})$.

can be continued analytically as in Eq. (15). In conclusion, gauging the Poincaré × dilatations symmetry gives rise to the same theory as in the non-metric or in the Weyl gauge-covariant formulations, so we have (in d dimensions) three equivalent formulations of this symmetry.[6]

3 Torsion vs non-metricity duality

So far we found three different formulations of theories with Weyl gauge symmetry: one in terms of a non-metric connection $\tilde{\Gamma}$, one in terms of a (metric) connection with torsion Γ and one fully covariant formulation in terms of the operators $\hat{\nabla}_\mu$ and $\hat{D}_\mu$. In this section we want to analyse more closely the relation between these formulations.

The non-metric connection ($\tilde{\Gamma}$) of (3) is invariant under transformations Eq. (1), symmetric in (μ, ν) and thus torsionless, but it does not preserve the metric: $\tilde{\nabla}_\mu g_{\nu\rho} = -2\omega_\mu g_{\nu\rho}$. The spin connection associated to it can be computed from the usual formula

$$\begin{aligned} \tilde{\mathrm{w}}_\mu{}^a{}_b &= -e_b^\nu \tilde{\nabla}_\mu e_\nu^a \\ &= \mathring{\mathrm{w}}_\mu{}^a{}_b + e_\mu^a e_b^\nu \omega_\nu - e_{b\mu} e^{a\nu}\omega_\nu + \delta_b^a \omega_\mu. \end{aligned} \tag{39}$$

The last term in the rhs, symmetric in (a, b), spoils the tangent space metricity because

$$\tilde{D}_\mu \eta_{ab} = -2\omega_\mu \eta_{ab}. \tag{40}$$

Unlike $\mathrm{w}_\mu{}^{ab}$ which was Weyl gauge invariant, $\tilde{\mathrm{w}}_\mu{}^{ab}$ transforms like a gauge field (its trace is proportional to ω_μ). This gives the non-metricity one-form with components $Q_{\mu ab} = -2\omega_\mu \eta_{ab}$. Similarly, in the tangent space formulation we had the torsion two-form given by (24). Both formulations comprise additional degrees of freedom compared to Riemannian geometry. In both cases the extra degrees of freedom are vectors (in d dimensions) which are identified with the Weyl gauge boson ω_μ. Therefore we have a special relation between vectorial non-metricity and vectorial torsion on which we shall comment later. One can associate a curvature tensor to $\tilde{\mathrm{w}}$ via the commutator

$$\begin{aligned} \tilde{R}^a{}_{b\mu\nu} := e_b^\sigma [\tilde{D}_\mu, \tilde{D}_\nu] e_\sigma^a &= \partial_\mu \tilde{\mathrm{w}}_\nu{}^a{}_b - \partial_\nu \tilde{\mathrm{w}}_\mu{}^a{}_b \\ &+ \tilde{\mathrm{w}}_\mu{}^a{}_c \tilde{\mathrm{w}}_\nu{}^c{}_b - \tilde{\mathrm{w}}_\nu{}^a{}_c \tilde{\mathrm{w}}_\mu{}^c{}_b \,, \end{aligned} \tag{41}$$

This corresponds to the usual curvature tensor in the non-metric formulation of Weyl gravity which gives Eq. (4). Note that the only symmetry of this tensor is the antisymmetry in the last two indices. With this, the curvature tensor, Ricci tensor and scalar in the tangent space and non-metric formulations are then related

$$R_{\rho\sigma\mu\nu} = \tilde{R}_{\rho\sigma\mu\nu} - g_{\rho\sigma} F_{\mu\nu}, \qquad R_{\mu\nu} = \tilde{R}_{\mu\nu} - F_{\mu\nu}, \qquad R = \tilde{R}\,, \tag{42}$$

as already noticed in Eqs. (10), (36) in the Weyl covariant picture.

We now have a clear description of the transition between the tangent space formulation with torsion and the (torsion-free) non-metric formulation: the affine and spin connections of these formulations are related by a projective transformation

$$\begin{aligned} \tilde{\mathrm{w}}_\mu{}^{ab} &= \mathrm{w}_\mu{}^{ab} + \eta^{ab}\omega_\mu \,, \\ \tilde{\Gamma}^\rho_{\mu\nu} &= \Gamma^\rho_{\mu\nu} + \delta^\rho_\nu \omega_\mu \,, \end{aligned} \tag{43}$$

where the last terms in the rhs of these equations account for the non-metricity of the lhs connections. With the new (spin) connection $\tilde{\mathrm{w}}$, Eq. (24) becomes $\tilde{D}_{[\mu} e^a_{\nu]} = 0$, and thus has zero torsion. Hence the same equation admits two interpretations, one in terms of torsion and the other in terms of non-metricity. We thus have a "dual picture" and interpretation of vectorial torsion vs vectorial non-metricity (see [24,25] for a related study).

The vielbein postulate can also be written in different ways, depending on which affine and spin connections one is using. Indeed, we have the following equivalent equations

$$\nabla_\mu e^a_\nu + \mathrm{w}_\mu{}^a{}_b e^b_\nu = 0\,, \tag{44}$$

$$\tilde{D}_\mu e^a_\nu - e^a_\rho \tilde{\Gamma}^\rho_{\mu\nu} = 0\,, \tag{45}$$

$$(\tilde{\nabla}_\mu + \omega_\mu)\, e^a_\nu + \mathrm{w}_\mu{}^a{}_b e^b_\nu = 0\,. \tag{46}$$

Equation (44) reflects the choice of working with the metric affine connection $\Gamma^\rho_{\mu\nu}$ of Eq. (26) with torsion, and the invariant (and metric) spin connection[7] of Eq. (25), as in Sect. 2.3.

Equation (45) implies that one is choosing an invariant non-metric affine connection (3) paired with a non-invariant (and non-metric) spin connection (39) which, however, covariantises the corresponding tangent space derivative $\tilde{D}_\mu$ when acting on the vielbein (since $\tilde{D}_\mu e^a_\nu = \hat{D}_\mu e^a_\nu$). Therefore, Eqs. (44) and (45) pair (non-)metricity in the space-time with (non-) metricity on the tangent space, respectively. This was used in Sect. 2.1.

A mixed choice is also possible. Indeed, in Eq. (46), because $\tilde{\nabla}_\mu e^a_\nu$ is not covariant with respect to dilatations, one adds a further covariantisation $(\tilde{\nabla}_\mu + \omega_\mu) e^a_\nu = \hat{\nabla}_\mu e^a_\nu$. This is the choice that corresponds to the Weyl covariant picture in Sect. 2.2, with both the affine and spin connections invariant and seems suitable for physical applications. This case

[6] There is a special limit of action (38) when ω_μ is "pure gauge", so $F_{\mu\nu} = 0$; ω_μ can then be integrated out, to leave an action with Weyl symmetry only (no ω_μ field), see [18,23] for an extensive discussion.

[7] The spin connection is invariant because in this case $\nabla_\mu e^a_\nu = \hat{\nabla}_\mu e^a_\nu$ and hence the first term is covariant.

pairs a non-metric connection in space-time with a metric spin connection on the tangent space.

There is an additional interesting aspect of the duality we found (covariant) non-metric versus torsion formulations. It is well-known that connections with torsion preserve the norm of vectors under parallel transport. In agreement with our equivalence of formulations, and contrary to a long-held (wrong) view, the (torsion-free) non-metric formulation of Weyl geometry also preserves the norm of the vectors under their parallel transport along a curve. This result applies provided that 1) vectors are Weyl invariant in the tangent space (i.e. vanishing charge in tangent space $q_v = 0$), and 2) their parallel transport preserves the Weyl gauge covariance, as demanded in a gauge theory, something missed by the long-held view. This result is shown in eqs. (B-8) to (B-13) in [18]. This is consistent with the above equivalence of the formulations of Weyl geometry as a gauge theory of gravity.

More generally, for vectors of arbitrary tangent-space charge ($q_v \neq 0$), parallel transport is again physically meaningful only if Weyl gauge covariance is maintained, so the gauge covariant derivative (i.e. $\hat{\nabla}$) is used; since this operator is metric compatible, the norm changes only by the charge of the tangent space vector. To detail, consider the infinitesimal covariant parallel transport of a vector v i.e. $dx^\mu \hat{\nabla}_\mu v^\nu = 0$, then by the metricity of the gauge covariant derivative ($\hat{\nabla}_\mu g_{\alpha\beta} = 0$), one has that the norm is covariantly constant $dx^\mu \hat{\nabla}_\mu |v|^2 = 0$. This implies the following variation $d\,|v|^2 = -2 q_v \omega_\mu |v|^2 dx^\mu$ with q_v the tangent space charge; if $q_v = 0$ we re-obtain the norm is invariant. This result is identical in the metric formulation with torsion (using $\hat{\nabla}$) or non-metric covariant formulation [18, (Eq. (B-12))].

In conclusion, for a description of Weyl gauge symmetry all three formulations are equally good. Weyl gauge symmetry does not prefer one connection or the other, although, from a high energy theory viewpoint, the Weyl-covariant formulation may be preferable. The above equivalence of the three formulations of the quadratic gravity, as a gauge theory associated to Weyl geometry, is specific for the vectorial non-metricity of Weyl geometry (and vectorial torsion), but the situation changes in more general cases [29]. This is easily understood, because torsion and non-metricity have in general a different physical meaning. This distinction is more intuitive in solid state physics, see section 4.4 in [18]. Consider a 3D crystalline structure: defects of dimension $d = 0$ known as point defects (missing atoms, extra atoms, etc) that destroy the local notion of length are naturally associated with non-metricity. Torsion is associated with defects of dimension $d = 1$ known as dislocations of the lattice. Hence, there is a clear difference between torsion and non-metricity. Then why is there no such difference apparent in our study above?

To understand this, note that we only considered vectorial non-metricity and vectorial torsion, that lead to the dual, equivalent interpretations. This is because both torsion and non-metricity have a vector component under $so(4)$ algebra decomposition, which is "tested" here. But torsion and non-metricity tensors have additional degrees of freedom beyond this vector component that do distinguish between these two tensors both mathematically and physically. In other words, the equivalent dual interpretation discussed here will fail beyond the vectorial non-metricity/torsion and then the physical aspects of non-metricity and torsion are indeed different in a general case [29].

So far we discussed only gauged dilatations. The general result by Coleman-Mandula [26] allows us to have the conformal group as the maximal space-time symmetry. In addition to the Weyl group, the conformal group includes special conformal transformations. Using these transformations we can always set to zero the gauge field ω_μ of dilatations[8] [1]. Moreover, at quadratic order in curvatures, no kinetic term for the gauge field of special conformal transformations can be written, so the corresponding gauge field is not dynamical (physical), either [15,16]. Thus, in this case we cannot talk about a true gauge theory (in the same way the electroweak theory without kinetic terms for the gauge bosons $W^\pm$, Z cannot be regarded as a gauge theory of weak interactions). Therefore, only gauged dilatations give a true (and anomaly-free) gauge theory of a four-dimensional space-time symmetry of the action. In this case, Weyl geometry seems the natural underlying geometry that realises this symmetry, even in the absence of matter. It may actually be the unique geometry to do so in a realistic way, given the equivalent dual formulation we found, as discussed in [29].

So far our analysis did not discuss the effect of adding matter fields. It is easy to see that our results remain valid when the SM is embedded in Weyl geometry. First, the SM gauge sector is invariant under (1) while the fermions Dirac action is identical to that in Riemannian geometry and is invariant under (1) [7]; this is because fermions do not couple classically to ω_μ [27,28]. Of the SM action only the Higgs sector couples to $\tilde{R}$ (as in $\tilde{R} H^\dagger H$) and also to ω_μ through its kinetic term [7]; however, these couplings are not changed by transformations (10), (42) considered here, hence our results do not change in the presence of SM. More details will be presented elsewhere [29].

[8] It is for this reason that one can construct Poincaré gravity/supergravity as gauged fixed theories with conformal/superconformal symmetry.

4 Conclusions

We reviewed (non-supersymmetric) gauge theories of $d = 4$ space-time symmetries and studied their quadratic action. In our view, such gauge theory should: a) have, as a theory of gravity, an exact geometric interpretation and origin for their degrees of freedom, b) recover Einstein gravity in their (spontaneously) broken phase, and c) this symmetry should be anomaly-free, as any (quantum) gauge symmetry. Theories based on Weyl gauge group (Poincaré $\times$ dilatations) meet these criteria. However, gauging the full conformal group does not generate a true gauge theory since the associated gauge bosons (of special conformal symmetry and dilatation) are not physical (dynamical). In other words, conformal gravity is a gauge theory of conformal group as much as, say, the electroweak theory without kinetic terms for $W^{\pm}$, Z gauge bosons is a gauge theory of weak interactions.

The gauge theory of the Weyl group gives rise to Weyl quadratic gravity and this is naturally realised in Weyl conformal geometry where this gauge symmetry is built in. This quadratic gravity (gauge) theory has two equivalent geometric formulations, that have the same action and thus same physics: a familiar formulation with vectorial non-metricity but no torsion, and a formulation that is manifestly Weyl-covariant and metric with respect to a new differential operator ($\hat{\nabla}$). The theory recovers Einstein gravity in its (spontaneously) broken phase. In the absence of the SM all degrees of freedom have geometric origin, and the gauge symmetry is manifestly maintained in d dimensions which indicates it is anomaly-free, as it was recently shown elsewhere.

To clarify the origin of the above equivalence, we compared these two equivalent geometric formulations of Weyl gauge symmetry to the standard, modern approach of constructing a gauge theory (of dilatations) by using the tangent space-time formulation "uplifted" to space-time by the vielbein. This lead to a gauge theory of dilatations that has an identical associated quadratic gravity action and that is metric but has vectorial torsion. This third formulation is "dual" (equivalent) to the non-metric formulation in Weyl geometry, to which it is related by a simple projective transformation. This duality vectorial non-metricity vs vectorial torsion was explained in detail. This equivalence fails beyond the vectorial non-metricity and vectorial torsion, due to the different, additional number of degrees of freedom of these tensors in the general case (that even break the Weyl gauge symmetry of the action). The above three equivalent realisations of Weyl gauge symmetry: non-metric, Weyl-covariant and metric with torsion remain equivalent when the SM is added. The above results suggest that the gauged dilatation may be a fundamental symmetry beyond both the SM and Einstein gravity and deserves further investigation.

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