Eur. Phys. J. C (2024) 84:330
https://doi.org/10.1140/epjc/s10052-024-12610-2

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

# Observational tests of asymptotically flat $\mathcal{R}^2$ spacetimes

**Tao Zhu**[1,2,a], **Hoang Ky Nguyen**[3,b], **Mustapha Azreg-Aïnou**[4,c], **Mubasher Jamil**[5,d]

[1] Institute for Theoretical Physics and Cosmology, Zhejiang University of Technology, Hangzhou 310023, China
[2] United Center for Gravitational Wave Physics (UCGWP), Zhejiang University of Technology, Hangzhou 310023, China
[3] Department of Physics, Babeş-Bolyai University, Cluj-Napoca 400084, Romania
[4] Başkent University, Engineering Faculty, Bağlica Campus, 06790 Ankara, Turkey
[5] School of Natural Sciences, National University of Sciences and Technology, Islamabad 44000, Pakistan

Received: 16 October 2023 / Accepted: 25 February 2024 / Published online: 29 March 2024


**Abstract** A novel class of Buchdahl-inspired metrics with closed-form expressions was recently obtained based on Buchdahl's seminal work on searching for static, spherically symmetric metrics in $\mathcal{R}^2$ gravity in vacuo. Buchdahl-inspired spacetimes provide an interesting framework for testing predictions of $\mathcal{R}^2$ gravity models against observations. To test these Buchdahl-inspired spacetimes, we consider observational constraints imposed on the deviation parameter, which characterizes the deviation of the asymptotically flat Buchdahl-inspired metric from the Schwarzschild spacetime. We utilize several recent solar system experiments and observations of the S2 star in the galactic center and the black hole shadow. By calculating the effects of Buchdahl-inspired spacetimes on astronomical observations both within and outside of the solar system, including the deflection angle of light by the Sun, gravitational time delay, perihelion advance, shadow, and geodetic precession, we determine observational constraints on the corresponding deviation parameters by comparing theoretical predictions with the most recent observations. Among these constraints, we find that the tightest one comes from the Cassini mission's measurement of gravitational time delay.

## 1 Introduction

Einstein's theory of general relativity (GR) has been the most successful theory describing the dynamics of massive objects under gravitational effects such as the motion of binary stars and planetary motion near their host stars, predicting novel gravitational objects such as black holes, compact stars, and gravitational waves. It remains the only theory of gravity that passes all solar system and astronomical tests. In the last few years, two earliest predictions of GR have been tested and verified via black holes, namely the gravitational deflection of light passing by a black hole, resulting in the formation of a black hole shadow, and the existence of gravitational waves. Recently, the shadows of black holes at the center of the M87 and Milky Way galaxies have been observed and analyzed in detail [1–4]. Astronomers have also detected numerous gravitational wave signals generated by the merger of binary black holes of different masses [5,6]. Despite these successes, some fundamental problems remain in the foundations of GR, including its renormalization and establishing its unison with quantum mechanics, thereby formulating a set of physical laws valid for both length scales, the very small and the very large.

The accelerated cosmic expansion observed in 1998 has spurred efforts to modify GR to account for the enigmatic "dark energy" component. Among the various modified theories of gravitation, the family of $f(\mathcal{R})$ introduced by Buchdahl in the early 1970s has become an active arena of investigation in the past 25 years [7–10]. Within this ghost-free class of theories, pure $\mathcal{R}^2$ gravity stands out for its scale-invariant nature. Attempts to incorporate the matter sector, namely the Glashow–Weinberg–Salam model of particle physics, into pure quadratic gravity to form a renormalizable quantum gravity framework have been made in the form of adimensional gravity, or "agravity" [11,12].

The pure $\mathcal{R}^2$ action

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\mathcal{R}^2 + S_M\left(g_{\mu\nu}, \psi\right) \tag{1}$$

[a] e-mail: zhut05@zjut.edu.cn
[b] e-mail: hoang.nguyen@ubbcluj.ro
[c] e-mail: azreg@baskent.edu.tr
[d] e-mail: mjamil@sns.nust.edu.pk (corresponding author)

can be recast using an auxiliary scalar field $\Phi$ as [8]

$$S = \frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}\left[\Phi\mathcal{R} - \frac{1}{2}\Phi^2\right] + S_M\left(g_{\mu\nu}, \psi\right) \tag{2}$$

resulting in the field equations

$$G_{\mu\nu} = \frac{\kappa}{\Phi}T_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\Phi + \frac{1}{\Phi}\left(\nabla_\mu\nabla_\nu\Phi - g_{\mu\nu}\Box\Phi\right) \tag{3}$$

$$3\Box\Phi = \kappa T \tag{4}$$

in which $\Phi$ is equal to the Ricci scalar $\mathcal{R}$. In vacuum, i.e. $T_{\mu\nu} = 0$, the terms $\frac{1}{4}g_{\mu\nu}\Phi$ and $\frac{1}{\Phi}\left(\nabla_\mu\nabla_\nu\Phi - g_{\mu\nu}\Box\Phi\right)$ act as additional "matter" sources to the Einstein–Hilbert field equation, producing corrections beyond the vacuum solutions of GR. To date, there is yet only empirical evidence of a scalar degree of freedom beside the established tensor ingredients. However, the dynamical nature of $\Phi$, governed by the "harmonic" equation (4), suggests potential manifestations for $\Phi$ near mass sources in the strong field regime. As demonstrated by one of the authors [13], the pure $\mathcal{R}^2$ field equation admits a rich host of vacuum solutions with the *non-constant* Ricci scalar. The Buchdahl-inspired solutions found therein are asymptotically de Sitter (or anti-de Sitter) and entail a new (Buchdahl) parameter $k$, accounting for the term $\frac{1}{\Phi}\left(\nabla_\mu\nabla_\nu\Phi - g_{\mu\nu}\Box\Phi\right)$ and enabling spatial variations in $\Phi$. Notably, considerations of a scalar degree of freedom have appeared in other contexts, such as the scalarization in neutron stars and black holes [14].

One important feature of pure $\mathcal{R}^2$ gravity is its limit to GR. In particular, it concerns the existence of a Newtonian behavior which is an essential requirement for any viable theory of gravitation. The limit is delicate [15]. This is because, whereas the vacuo of GR is Ricci flat, i.e. $\mathcal{R}_{\mu\nu} = 0$, for pure $\mathcal{R}^2$ gravity, the vacuum background far from the mass sources is de Sitter (or anti-de Sitter) with a Ricci scalar $\mathcal{R} = 4\Lambda$. In [15] it was found that, owing to the de Sitter background, the spin-2 tensor graviton excitations are massless instead of massive (which would have been the case if the background were locally flat). The massless modes should effectively carry a long-range interaction rather than a Yukawa short-range interaction. Based on this insight, one of the present authors [16] has recently established the emergence of a gravitational potential with the correct Newtonian tail on a de Sitter background for the pure $\mathcal{R}^2$ theory. That is to say, pure $\mathcal{R}^2$ gravity possesses a proper Newtonian limit, *despite* the absence of the Einstein–Hilbert term in its action (1). This finding strengthens the viability of pure $\mathcal{R}^2$ as a candidate theory of gravitation.

An immediate implication is finding novel static and spherically symmetric spacetime or black hole solutions in $\mathcal{R}^2$ theory which was pioneered by Buchdahl [17]. His work culminated in the formulation of a second-order ordinary differential equation for finding metric coefficients, which remained unsolved until recently, when one of the coauthors (Nguyen) succeeded in obtaining vacuum solutions which are asymptotically de Sitter [13] or asymptotically flat as a special case [18] (with the axisymmetric extensions of the latter solution having been proposed in [19]). Our current study is concerned with this special Buchdahl-inspired solution. The metric involves a free parameter, the Buchdahl parameter $k$, which can be interpreted as a scalar hair, and setting $k = 0$ yields a Schwarzschild spacetime/black hole as a limiting case. In [18], it was shown that the respective static, spherically symmetric black hole solution has different areas of event horizon depending on the chosen value of $k$. Specifically, the horizon area can be 0, $4\pi r_s^2$, $16\pi r_s^2$, and divergent for the following values $k \in (-\infty, r_s) \cup (0, \infty)$, $k = 0$, $k = -r_s$ and $k \in (-r_s, 0)$, respectively. Here, $r_s$ is the radius of the black hole horizon. A further investigation of this new solution is needed to understand its phenomenology.

We should clarify that the special Buchdahl-inspired solution is also Ricci scalar flat, namely, $\Phi \to 0$ everywhere in the vacuum exterior to a mass source. Whereas the term $-\frac{1}{4}g_{\mu\nu}\Phi$ in Eq. (3) is negligible in this limit, contributions from the term $\frac{1}{\Phi}\left(\nabla_\mu\nabla_\nu\Phi - g_{\mu\nu}\Box\Phi\right)$ persist and are encoded by the (dimensionless) Buchdahl parameter $\tilde{k}$, defined to be $k$ normalized by $r_s$ [20]. As a higher-derivative characteristic, the value of $\tilde{k}$ is *system-dependent*. It does not have a universal value but can vary from one system to another, depending on the composition of the matter source. In normal conditions, such as in the solar system, $\tilde{k}$ could be insignificant. In extreme conditions, such as around compact stars, intuitively, $\tilde{k}$ may acquire large values.

In this article, we are motivated to analyze the experimental and observational implications of the special Buchdahl-inspired metric. We would like to see how much the new effects beyond the Schwarzschild case affect the dynamics of particles in geodesic motion in the special Buchdahl-inspired metric. To be more general, we consider a more general form of the Buchdahl-inspired metric by treating several new parameters for describing the solution independently. By relying on the classical relativistic methods, we calculate perihelion shift, gravitational time delay, and geodesic precession of orbits, and test these results with the solar system experiments. We also attempt to consider the observational implications of the Buchdahl-inspired spacetime using the observational data of the S2 star orbit about the Milky Way central black hole and to investigate the shadow of rotating solutions.

Our article is structured as follows. In Sect. 2, we provide a brief review of the general and special Buchdahl-inspired metrics in different sets of coordinate systems. In Sect. 3, we investigate the geodesics of both massless and massive objects in the general Buchdahl-inspired spacetime and derive in detail the effects of the spacetime on observations in the solar system experiments, black hole shadow of M87, and the orbit of the S2 star at the galactic center. The observational bounds on the deviation parameter, which characterizes the deviation of the asymptotically flat Buchdahl-inspired metric from the Schwarzschild spacetime, are obtained by comparing the theoretical predictions with observational data. Then, in Sect. 4, we study a spinning object in the general Buchdahl-inspired spacetime and derive the geodetic procession of its spin vector, from which we obtain the constraints on the corresponding deviation parameter using the Gravity Probe B and lunar laser ranging data. A brief summary of our main results and some discussion are presented in Sect. 5.

## 2 Summary of Buchdahl-inspired vacuum solutions

In a pioneering Nuovo Cimento work in 1962 [17], Buchdahl developed—but prematurely abandoned—a program to find vacuum configurations for pure $\mathcal{R}^2$ gravity. Subsequent advancements made by one of us, documented in Refs. [13,18], completed his program and derives an exhaustive class of metric in compact form, to be summarized in this section.

The field equation in vacuum is

$$\mathcal{R}\left(\mathcal{R}_{\mu\nu}-\frac{1}{4}g_{\mu\nu}\mathcal{R}\right)+g_{\mu\nu}\Box\mathcal{R}-\nabla_\mu\nabla_\nu\mathcal{R}=0, \tag{5}$$

which contains fourth derivatives of the metric components $g_{\mu\nu}$ in $\Box\mathcal{R}$ and $\nabla_\mu\nabla_\nu\mathcal{R}$. The solution in general thus involves two additional parameters. As we shall see, they are the scalar curvature, $4\Lambda$, at spatial infinity, and a new (Buchdahl) parameter $k$ which is of the dimension of length. The case of $\Lambda\neq 0$ is an asymptotically (anti-)de Sitter vacuum solution, whereas the case of $\Lambda=0$ is an asymptotically flat vacuum solution.

The body of our orbital motion study in this paper is on the asymptotically flat vacuum solution outside of a static and spherical symmetric mass source. Nevertheless, for completeness, we shall expose the available representations of the metrics in this section.

### 2.1 Asymptotically de Sitter vacuum solution in standard coordinates

In [13], a general Buchdahl-inspired metric was determined to be in a compact form (with $\mathrm{d}\Omega^2:=\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2$)

$$\mathrm{d}s^2=e^{k\int\frac{\mathrm{d}r}{q(r)r}}\left\{-\frac{p(r)q(r)}{r}\mathrm{d}t^2+\frac{p(r)r}{q(r)}\mathrm{d}r^2+r^2\mathrm{d}\Omega^2\right\}. \tag{6}$$

The variables $p$ and $q$ obey first-order "evolution" rules

$$\frac{\mathrm{d}p(r)}{\mathrm{d}r}=\frac{3k^2}{4r}\frac{p(r)}{q^2(r)}, \tag{7}$$

$$\frac{\mathrm{d}q(r)}{\mathrm{d}r}=\left(1-\Lambda r^2\right)p(r), \tag{8}$$

whereas the Ricci scalar is

$$\mathcal{R}(r)=4\Lambda\,e^{-k\int\frac{\mathrm{d}r}{r\,q(r)}}. \tag{9}$$

The metric involves $\Lambda$, representing the scalar curvature at spatial infinity, and $k$, the Buchdahl parameter. When $k=0$, metric (6) duly recovers the de Sitter metric [13].

### 2.2 Asymptotically de Sitter vacuum solution in "canonical" coordinates

The metric expressed in (6) contains a conformal factor which is inversely proportional to the Ricci scalar. In [19], we considered making $r$ a function of $R$ such that the proper part of the metric satisfies $g_{tt}g_{RR}=-1$. The resulting metric is given by

$$\mathrm{d}s^2=e^{k\int\frac{\mathrm{d}R}{\Psi(R)r^2(R)}}\left\{-\Psi(R)\mathrm{d}t^2+\frac{\mathrm{d}R^2}{\Psi(R)}+r^2(R)\mathrm{d}\Omega^2\right\}, \tag{10}$$

with

$$\Psi(R):=\frac{p(R)\,q(R)}{r(R)}. \tag{11}$$

The "evolution" rules now involve three functions $p(R)$, $q(R)$, and $r(R)$

$$\frac{\mathrm{d}r(R)}{\mathrm{d}R}=\frac{1}{p(R)}, \tag{12}$$

$$\frac{\mathrm{d}p(R)}{\mathrm{d}R}=\frac{3k^2}{4r(R)q^2(R)}, \tag{13}$$

$$\frac{\mathrm{d}q(R)}{\mathrm{d}R}=1-\Lambda\,r^2(R), \tag{14}$$

with the Ricci scalar being given by

$$\mathcal{R}(r)=4\Lambda\,e^{-k\int\frac{\mathrm{d}R}{\Psi(R)r^2(R)}}. \tag{15}$$

### 2.3 Asymptotically flat vacuum solution in standard coordinates

In Ref. [18], we further found an exact closed analytical solution corresponding to the case of $\Lambda=0$, which was called the special Buchdahl-inspired metric. This metric is Ricci scalar flat, but not Ricci flat. It describes an asymptotically flat spacetime. Hence, it is also appropriate to call it the asymptotically flat Buchdahl-inspired metric.

For the region $r > r_s$, the metric of the asymptotically flat Buchdahl-inspired metric reads

$$ds^2 = \left(1 - \frac{r_s}{r}\right)^{\tilde{k}} \left\{ -\left(1 - \frac{r_s}{r}\right) dt^2 + \frac{\rho^4(r)\, dr^2}{r^4 \left(1 - \frac{r_s}{r}\right)} + \rho^2(r) d\Omega^2 \right\}, \tag{16}$$

where the function $\rho(r)$ is

$$\rho(r) = \zeta\, r_s \frac{\left(1 - \frac{r_s}{r}\right)^{\frac{\zeta - 1}{2}}}{1 - \left(1 - \frac{r_s}{r}\right)^{\zeta}} \tag{17}$$

and the dimensionless parameters are

$$\tilde{k} := \frac{k}{r_s}, \quad \zeta := \sqrt{1 + 3\tilde{k}^2}. \tag{18}$$

### 2.4 Asymptotically flat vacuum solution in "q" coordinate

In [18], one of us reported yet another expression for the asymptotically flat Buchdahl-inspired metric, Eq. (16). For the region $q \geqslant q_+$ (which are defined in Eq. (20)):

$$ds^2 = \left(\frac{q - q_+}{q - q_-}\right)^{\frac{\tilde{k} - 1}{\zeta}} \left\{ -\left(\frac{q - q_+}{q - q_-}\right)^{\frac{2}{\zeta}} dt^2 + dq^2 + (q - q_+)(q - q_-) d\Omega^2 \right\} \tag{19}$$

with

$$q_{\pm} := \frac{r_s}{2} \left(-1 \pm \zeta\right), \quad \zeta = \sqrt{1 + 3\tilde{k}^2}. \tag{20}$$

### 2.5 Asymptotically flat vacuum solution in isotropic coordinate

The metric (16) can be transformed into an isotropic form [21],

$$ds^2 = \left|\frac{\bar{r} - r_s/4}{\bar{r} + r_s/4}\right|^{\frac{2}{\zeta}(\zeta + \tilde{k} - 1)} \left\{ -\left|\frac{\bar{r} - r_s/4}{\bar{r} + r_s/4}\right|^{\frac{2}{\zeta}(2 - \zeta)} dt^2 + \zeta^2 \left(1 + \frac{r_s}{4\bar{r}}\right)^4 \left(d\bar{r}^2 + \bar{r}^2 d\Omega^2\right) \right\}, \tag{21}$$

which is symmetric with respect to a reciprocal coordinate transformation, per

$$\frac{4\bar{r}}{r_s} \leftrightarrows \frac{r_s}{4\bar{r}}. \tag{22}$$

### 2.6 Asymptotically flat vacuum solution in Morris–Thorne form

In Ref. [21], the metric (16)–(18) was brought into the Morris–Thorne form [22,23]:

$$ds^2 = -e^{2\Phi(R)} dt^2 + \frac{dR^2}{1 - \frac{b(R)}{R}} + R^2 d\Omega^2, \tag{23}$$

$$e^{2\Phi(R)} = y^{\frac{2}{\zeta}(\tilde{k} + 1)}, \tag{24}$$

$$1 - \frac{b(R)}{R} = \frac{1}{4y^2} \left( (y^2 + 1) + \frac{\tilde{k} - 1}{\zeta} (1 - y^2) \right)^2 \geq 0, \tag{25}$$

$$R = (\zeta r_s) \frac{y^{\frac{\tilde{k} - 1}{\zeta} + 1}}{1 - y^2}, \tag{26}$$

$$y := \left(1 - \frac{r_s}{r}\right)^{\frac{\zeta}{2}} \in (0, 1), \quad \zeta = \sqrt{1 + 3\tilde{k}^2}, \tag{27}$$

with $\Phi(R)$ and $b(R)$ being the redshift and shape functions, respectively. Note that the relation $y(R)$ is implicit by inverting Eq. (26).

### 2.7 A more generic Morris–Thorne form

In Ref. [24], we generalized the metric in (23)–(27) by making two modifications: (i) replacing $\tilde{k}$ with $\eta$ in the redshift function (see below); (ii) treating $\tilde{k}$, $\eta$, and $\zeta$ as *independent* parameters (in contrast to Eq. (27), where $\zeta = \sqrt{1 + 3\tilde{k}^2}$). The generalized metric is expressed as

$$ds^2 = -e^{2\Phi(R)} dt^2 + \frac{dR^2}{1 - \frac{b(R)}{R}} + R^2 d\Omega^2 \tag{28}$$

with

$$e^{2\Phi(R)} = y^{\frac{2}{\zeta}(\eta + 1)}, \tag{29}$$

$$1 - \frac{b(R)}{R} = \frac{1}{4y^2} \left( (y^2 + 1) + \frac{\tilde{k} - 1}{\zeta} (1 - y^2) \right)^2 \geq 0, \tag{30}$$

$$R = (\zeta r_s) \frac{y^{\frac{\tilde{k} - 1}{\zeta} + 1}}{1 - y^2}; \quad y \in (0, 1). \tag{31}$$

Here we would like to present several remarks about the properties of the solution in the generic Morris–Thorne form.

*Remark 1* The metric (28)–(31) recovers the metric (23)–(27) when $\eta = \tilde{k}$ and $\zeta = \sqrt{1 + 3\tilde{k}^2}$. Additionally, it recovers the Campanelli–Lousto metric in Brans–Dicke gravity [25–28] when $\zeta = 1$.

*Remark 2* Although the metric (28)–(31) seems to have four parameters $\left\{\tilde{k}, \eta, \zeta, r_s\right\}$, it effectively depends on only *three* parameters $\alpha := \frac{\eta+1}{\zeta}$, $\beta := \frac{\tilde{k}-1}{\zeta}$, $r'_s := \zeta r_s$, per

$$ds^2 = -e^{2\Phi(R)}dt^2 + \frac{dR^2}{1-\frac{b(R)}{R}} + R^2 d\Omega^2, \tag{32}$$

$$e^{2\Phi(R)} = y^{2\alpha} \tag{33}$$

$$1 - \frac{b(R)}{R} = \frac{1}{4y^2}\left((y^2+1) + \beta(1-y^2)\right)^2 \geq 0, \tag{34}$$

$$R = r'_s \frac{y^{\beta+1}}{1-y^2} \text{ for } y \in (0,1), \tag{35}$$

with the relation $y(R)$ being implicit in Eq. (35).

*Remark 3* The Schwarzschild metric corresponds to metric (32)–(35) when $\alpha = 1$, $\beta = -1$ and $\zeta = 1$. Meanwhile, the asymptotically flat Buchdahl-inspired metric corresponds to $\left\{\alpha = \frac{\tilde{k}+1}{\sqrt{1+3\tilde{k}^2}}, \beta = \frac{\tilde{k}-1}{\sqrt{1+3\tilde{k}^2}}\right\}$, obeying the relation $\alpha^2 + \alpha\beta + \beta^2 = 1$.

*Remark 4* Regardless of $\alpha$, when $\beta < -1$, metric (32)–(35) yields a wormhole because the function $R(y)$ in (35) produces a minimum at $y_0 = \sqrt{\frac{\beta+1}{\beta-1}} \in (0,1)$. The two (symmetric) asymptotically flat sheets that are glued together at the "throat" $y_0$ are both defined in the range $y \in [y_0, 1)$.

## 3 Geodesics and classical tests of the general Buchdahl-inspired metrics

In this section, we will present the geodesics evolution of a massive/massless particle orbiting the Buchdahl-inspired metrics. From the geodesic evolution, we are able to calculate some observational quantities so that we can use data to constrain them. We claim that the data from the solar system experiments can provide stronger constraints than those obtained upon using the stellar stars orbiting the supermassive black hole in the galactic center, so we will consider the solar system experiments first.

### 3.1 Geodesics in the general Buchdahl-inspired metrics

In this subsection, we give the general Buchdahl-inspired metrics in the standard coordinates and consider the geodesics of both massless and massive objects in this metric. In contrast to the metric given in (16), here we treat the parameters $\tilde{k}$, $\eta$, and $\zeta$ as independent parameters. Using the coordinates $(t, R, \theta, \phi)$, the metric is given by

$$ds^2 = -e^{2\Phi(R)}dt^2 + \frac{dR^2}{1-\frac{b(R)}{R}} + R^2 d\Omega^2. \tag{36}$$

In the weak field approximation, this metric can be expressed as

$$ds^2 \simeq -\left(1 - \frac{(1+\eta)r_s}{R} + \frac{(1+\eta)(\tilde{k}+\eta)r_s^2}{2R^2}\right)dt^2 + \left(1 + \frac{(1-\tilde{k})r_s}{R}\right)dR^2 + R^2 d\Omega^2. \tag{37}$$

The radius $r_s$ can be related to the ADM mass of the solution via $r_s = 2GM$, with $M$ being the ADM mass and $G$ being the gravitational constant. Comparing this weak field expansion with the Newtonian limit, we can relate $G$ with the Newtonian gravitational constant $G_N$ as $(1+\eta)G = G_N$.

Let us first consider the evolution of a particle in the general Buchdahl-inspired metric (36). For a massive/massless particle, if we ignore self-gravitational effects, its evolution is governed by the following geodesics

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0, \tag{38}$$

where $\lambda$ denotes the affine parameter of the geodesics and $\Gamma^\mu_{\nu\rho}$ represent the Christoffel symbols of the general Buchdahl-inspired metric. Considering that the general Buchdahl-inspired metric is static and spherically symmetric, it has two Killing vectors, $\xi^\mu_t = \{\partial_t, 0, 0, 0\}$ and $\xi^\mu_\phi = \{0, 0, \partial_\phi, 0\}$, which leads to two constants of motion $E$ and $L$ (conserved energy and angular momentum), i.e.,

$$E = -g_{\mu\nu}\xi^\mu_t \frac{dx^\nu}{d\lambda} = -g_{tt}\frac{dt}{d\lambda}, \tag{39}$$

$$L = g_{\mu\nu}\xi^\mu_\phi \frac{dx^\nu}{d\lambda} = g_{\phi\phi}\frac{d\phi}{d\lambda}. \tag{40}$$

For geodesics, we also have $g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \varepsilon$, with $\varepsilon = -1$ for timelike geodesics which describes evolution of massive particle, and $\varepsilon = 0$ for null geodesics which describes the evolution of massless particle. Then, using (39) and (40), we obtain

$$g_{RR}\left(\frac{dR}{d\lambda}\right)^2 + g_{\theta\theta}\left(\frac{d\theta}{d\lambda}\right)^2 = \varepsilon - g_{tt}\left(\frac{dt}{d\lambda}\right)^2 - g_{\phi\phi}\left(\frac{d\phi}{d\lambda}\right)^2 = \varepsilon - \frac{E^2}{g_{tt}} - \frac{L^2}{g_{\phi\phi}}. \tag{41}$$

Without loss of generality, we consider the evolution of the particle in the equatorial plane, i.e., we can set $\theta = \pi/2$ and $d\theta/d\lambda = 0$. Then we can simplify the above equation into the form

$$\left(\frac{dR}{d\lambda}\right)^2 = E^2 - V_{\text{eff}}(R), \tag{42}$$

where $V_{\text{eff}}(R)$ denotes the effective potential of the particle,

$$V_{\text{eff}}(R) = E^2 - \left(\varepsilon - \frac{E^2}{g_{tt}} - \frac{L^2}{g_{\phi\phi}}\right)\frac{1}{g_{RR}}. \tag{43}$$

For later convenience, it is useful to give the derivative of $r$ with respect to $\phi$, which can be obtained using (40) in Eq. (42) and is given by

$$\left(\frac{\mathrm{d}R}{\mathrm{d}\phi}\right)^2 = \left(\varepsilon - \frac{E^2}{g_{tt}} - \frac{L^2}{g_{\phi\phi}}\right)\frac{g_{\phi\phi}^2}{L^2 g_{RR}}. \tag{44}$$

This equation is the starting point for later calculations of the light deflection angle, gravitational time delay, and perihelion advance in the general Buchdahl-inspired metric in the following subsections.

### 3.2 Light deflection angle

The precise measurements of the deflection of the light passing by the Sun play an essential role in the establishment of GR. These data can also be used for constraining any possible derivation of the deflection angle in many modified gravities from that in GR. Here, our purpose is to calculate the possible effects of the general Buchdahl-inspired metric on the deflection angle of the light and then constrain them using the most recent measurements.

For the propagation of the light in the general Buchdahl-inspired metric (36), we have $\varepsilon = 0$. Introducing the impact parameter

$$b \equiv \frac{L}{E}, \tag{45}$$

Eq. (44) can be transformed into

$$\frac{\mathrm{d}\phi}{\mathrm{d}R} = \pm\sqrt{\frac{g_{RR}}{g_{\phi\phi}}}\left(-\frac{g_{\phi\phi}}{b^2 g_{tt}} - 1\right)^{-1/2}, \tag{46}$$

where $\pm$ represents the cases with increasing and decreasing $R$, respectively. In general, for a bending light that does not fall into the object described by the general Buchdahl-inspired metric (36), the range of the allowed $R$ is determined by the condition $\frac{\mathrm{d}R}{\mathrm{d}\lambda} \geq 0$. In the general Buchdahl-inspired metric, this implies that the allowed range of $R$ should be $R_0 \leq R < +\infty$, with $R_0$ denoting the closest approach of the light to the Sun. $R_0$ is a root of $\frac{\mathrm{d}R}{\mathrm{d}\lambda} = 0$, and thus we have

$$b^2 = -\frac{g_{\phi\phi}(R_0)}{g_{tt}(R_0)}. \tag{47}$$

Then the deflection of the angle of the light can be calculated using

$$\Delta\phi = 2\int_{R_0}^{+\infty}\frac{\mathrm{d}\phi}{\mathrm{d}R}\mathrm{d}R - \pi. \tag{48}$$

Since we are considering the deflection of the light by the Sun, it is convenient to employ the weak field approximation by expanding the above integral in terms of $r_s/R$, which gives

$$\Delta\phi \simeq \frac{4GM}{R_0}\frac{2+\eta-k}{2} \simeq \frac{4G_{\mathrm{N}}M}{R_0}\frac{2+\eta-k}{2(1+\eta)}, \tag{49}$$

where $G_{\mathrm{N}} = (1+\eta)G$ is the Newtonian gravitational constant. For a special Buchdahl-inspired metric, as given in (16) with $\eta = \tilde{k}$ and $\zeta = \sqrt{1+3\tilde{k}^2}$, we have

$$\Delta\phi \simeq \frac{4G_{\mathrm{N}}M}{R_0}(1-\tilde{k}), \tag{50}$$

where $G_{\mathrm{N}} = (1+\tilde{k})G$ is the Newtonian gravitational constant for the Buchdahl-inspired metric.

Now we consider the light deflected by the Sun. We can express the deflection angle $\Delta\phi$ in terms of the $\Delta\phi^{\mathrm{GR}} = 1.75''$ as

$$\frac{\Delta\phi}{\Delta\phi^{\mathrm{GR}}} = 1 - \frac{\eta+\tilde{k}}{2(1+\eta)}. \tag{51}$$

The deflection angles of the light from distant sources by the Sun have been measured in many experiments over the past 100 years. The most precise measurement to date was carried out using the very-long-baseline interferometry technique [29]. Using the result of this measurement, we can constrain the parameter $\frac{\eta+\tilde{k}}{2(1+\eta)}$ in the general Buchdahl-inspired metric to be

$$-5.0\times10^{-5} < \frac{\eta+\tilde{k}}{2(1+\eta)} < 2.5\times10^{-4} \quad (68\%\ \text{C.L.}). \tag{52}$$

For a special Buchdahl-inspired metric (16) with $\eta = \tilde{k}$ and $\zeta = \sqrt{1+3\tilde{k}^2}$, this constraint leads to a constraint on the parameter $\tilde{k}$ as

$$-5.0\times10^{-5} < \tilde{k} < 2.5\times10^{-4} \quad (68\%\ \text{C.L.}). \tag{53}$$

### 3.3 Gravitational time delay

Gravitational time delay is an important phenomenon in that light or radio waves can take more time to travel if they pass by a massive object, like the Sun or a planet. This phenomenon can be precisely measured by sending a radar signal from Earth or a spacecraft passing through the Sun and reflecting off another planet or spacecraft. The effects of the general Buchdahl-inspired metric (36) on the gravitational time delay can be derived from Eq. (46), from which we obtain

$$\begin{aligned}\frac{\mathrm{d}t}{\mathrm{d}R} &= \frac{\mathrm{d}t}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}R} = \frac{\mathrm{d}\phi}{\mathrm{d}R}\frac{\mathrm{d}t/\mathrm{d}\lambda}{\mathrm{d}\phi/\mathrm{d}\lambda}\\ &= \pm\frac{1}{b}\sqrt{-\frac{g_{RR}}{g_{tt}}}\left(\frac{1}{b^2} + \frac{g_{tt}}{g_{\phi\phi}}\right)^{-1/2}.\end{aligned} \tag{54}$$

Considering a radio wave traveling from the Sun to the point $R_A$, the time spent during this process can be calculated from the integral

$$t(R_A) = \frac{1}{b}\int_{R_0}^{R_A}\sqrt{-\frac{g_{RR}}{g_{tt}}}\left(\frac{1}{b^2}+\frac{g_{tt}}{g_{\phi\phi}}\right)^{-1/2}\mathrm{d}R. \tag{55}$$

Here, $R_0$ is the closest approach of the radio wave to the Sun, which can be determined by Eq. (47). In the weak field approximation, we have

$$t(r_A) \simeq \sqrt{R_A^2-R_0^2}+G_\mathrm{N}M\sqrt{\frac{R_A-R_0}{R_A+R_0}} +\frac{2+\eta-\tilde{k}}{1+\eta}G_\mathrm{N}M\operatorname{arccosh}\left(\frac{R_A}{R_0}\right). \tag{56}$$

The first term is the travel time of light in flat spacetime, and the remaining part contains both the contributions to the travel time in the Schwarzschild metric and the new effects of the general Buchdahl-inspired metric. When $\tilde{k} = 0 = \eta$, the above expression exactly reduces to the Schwarzschild result.

There are two different cases in the experiments measuring the gravitational time delay by sending a radar wave from Earth or spacecraft which is then reflected off another planet or spacecraft. One is the inferior conjunction case, in which the planet (or spacecraft, denoted by B) which reflects the radar signal is located between the Earth (or spacecraft, denoted by A) and the Sun. The calculation of the time delay due to the general Buchdahl-inspired metric of this case is simple and can be obtained by

$$\Delta t_I \simeq 2\frac{2+\eta-\tilde{k}}{1+\eta}G_\mathrm{N}M\ln\frac{R_A}{R_B} = \Delta t_I^\mathrm{GR}\frac{2+\eta-\tilde{k}}{2(1+\eta)}, \tag{57}$$

or alternatively,

$$\frac{\Delta t_I}{\Delta t_I^\mathrm{GR}} = \frac{2+\eta-\tilde{k}}{2(1+\eta)}. \tag{58}$$

The other is the superior conjunction case, in which the planet that reflects the radar signal and the Earth are on opposite sides of the Sun. For this case, the gravitational time delay is given by

$$\Delta t_S \simeq \frac{2+\eta-\tilde{k}}{2(1+\eta)}4G_\mathrm{N}M\left[1+\ln\frac{4R_AR_B}{R_0^2}\right]. \tag{59}$$

The most precise results related to the gravitational time delay were obtained from the Cassini experiments [30]. This result ruled out many modified gravity theories that predicted larger deviations from GR. Here we would like to use its results to constrain the parameters $\tilde{k}$ and $\eta$ in the general Buchdahl-inspired metric. The Cassini experiment was conducted in June 2002, and the test of the gravitational time delay was achieved in the measurement of the frequency shift of radio waves to and from the Cassini spacecraft as they passed near the Sun. In the superior conjunction case, the relative change in frequency is related to the time delay $\Delta t_S$ via

$$\delta\nu = \frac{\nu(t)-\nu_0}{\nu_0} = \frac{d}{\mathrm{d}t}\Delta t_S, \tag{60}$$

where $\nu_0$ denotes the frequency of the radio waves emitted from Earth, and $\nu(t)$ is the frequency of the radio wave reflected back to Earth at $t$. Using Eq. (59), we have

$$\delta\nu \simeq -\frac{2+\eta-\tilde{k}}{2(1+\eta)}\frac{8G_\mathrm{N}M}{R_0}\frac{\mathrm{d}R_0(t)}{\mathrm{d}t} = \frac{2+\eta-\tilde{k}}{2(1+\eta)}\delta\nu^\mathrm{GR}. \tag{61}$$

Alternatively,

$$\frac{\delta\nu}{\delta\nu^\mathrm{GR}} = \frac{2+\eta-\tilde{k}}{2(1+\eta)}. \tag{62}$$

Using the measurement performed in the Cassini experiment [30], we can constrain the parameters $\tilde{k}$ and $\eta$ in the general Buchdahl-inspired metric to be

$$-4.4\times10^{-5} < \frac{\eta+\tilde{k}}{2(1+\eta)} < 2\times10^{-6}. \tag{63}$$

This constraint is stronger than that obtained by the observations of the deflection angle.

For a special Buchdahl-inspired metric (16) with $\eta = \tilde{k}$ and $\zeta = \sqrt{1+3\tilde{k}^2}$, the above constraint leads to a constraint on the parameter $\tilde{k}$ as

$$-4.4\times10^{-5} < \tilde{k} < 2\times10^{-6}. \tag{64}$$

### 3.4 Perihelion advance

Now we consider the orbit's perihelion advance for a massive particle moving in the general Buchdahl-inspired metric. For a massive particle, we have $\varepsilon = -1$. We still start with Eq. (44), introducing a new variable $x = 1/R$, which leads to

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\phi}\right)^2 = x^4\left[-1-\frac{E^2}{g_{tt}}-\frac{L^2}{g_{\phi\phi}}\right]\frac{g_{\phi\phi}^2}{L^2g_{RR}}. \tag{65}$$

Taking the derivative on both sides of the above equation with respect to $\phi$ and expanding the equation about the small parameter $r_s$ (in the weak field approximation), we obtain

$$\frac{\mathrm{d}^2x}{\mathrm{d}\phi^2}+x-\frac{G_\mathrm{N}M}{L^2} \simeq 3G_\mathrm{N}Mx^2-3\frac{\eta+\tilde{k}}{1+\eta}G_\mathrm{N}Mx^2 +2\frac{\tilde{k}+\eta}{1+\eta}\frac{G_\mathrm{N}^2M^2}{L^2}x. \tag{66}$$

The right-hand side of the above equation can be treated as perturbations to Newtonian gravity, which contains two parts. The first part $3G_\mathrm{N}Mx^2$ represents the correction from the Schwarzschild metric in GR, and the last two terms (the second and the third term on the right-hand side of the above equation) are the new effects from the general Buchdahl-inspired metric. When the perturbations are absent, the above equation has an exact solution $x_0(\phi)$ for a bounded orbit,

$$x_0(\phi) = \frac{G_N M}{L^2}(1 + e\cos\phi), \tag{67}$$

which describes an elliptical orbit with the eccentricity $e$. The perturbations from GR and the general Buchdahl-inspired metric lead to the small derivation to the exact elliptical orbit. Thus we can write the orbit which contains the effects of the perturbations in the form of

$$x(\phi) = x_0(\phi) + x_1(\phi), \tag{68}$$

where $x_0$ is the elliptical orbit with the eccentricity $e$ given by Eq. (67), and $x_1(\phi)$ is the small correction to the elliptical orbit, which satisfies

$$\frac{d^2x_1}{d\phi^2} + x_1 \simeq 3G_N M x_0^2 - 3\frac{\eta + \tilde{k}}{1+\eta}G_N M x_0^2 + 2\frac{\tilde{k}+\eta}{1+\eta}\frac{G_N^2 M^2}{L^2}x_0. \tag{69}$$

Using $x_0(\phi)$ in Eq. (67), we obtain

$$\frac{d^2x_1}{d\phi^2} + x_1 \simeq A_0 + A_1\cos\phi + A_2\cos^2\phi, \tag{70}$$

where

$$A_0 = \frac{3 + 2\eta - \tilde{k}}{1+\eta}\frac{G_N^3 M^3}{L^4}, \tag{71}$$

$$A_1 = \frac{6 + 2\eta - 4\tilde{k}}{1+\eta}e\frac{G_N^3 M^3}{L^4}, \tag{72}$$

$$A_2 = 3\frac{1-\tilde{k}}{1+\eta}e^2\frac{G_N^3 M^3}{L^4}. \tag{73}$$

The solution of $x_1$ is given by

$$x_1 = A_0 + \frac{A_2}{2} - \frac{A_2}{6}\cos(2\phi) + \frac{A_1}{2}\phi\sin\phi. \tag{74}$$

Only the last term contributes to the perihelion advance of a massive particle moving in the general Buchdahl-inspired metric. For this reason, we can drop other terms and write the solution of $x(\phi)$ as

$$x \simeq \frac{G_N M}{L^2}(1 + e\cos\phi) + \frac{A_1}{2}\phi\sin\phi = \frac{G_N M}{L^2}\left[1 + e\cos\left(\phi - \frac{\delta\phi_0}{2\pi}\phi\right)\right], \tag{75}$$

where

$$\delta\phi_0 \simeq \frac{6\pi G_N^2 M^2}{L^2}\left(1 - \frac{2}{3}\frac{\eta + \tilde{k}}{1+\eta}\right). \tag{76}$$

This expression represents the angular shift of the perihelia per orbit.

For an ellipse described by Eq. (67), we can relate the angular momentum $L$ of the massive particle to the semi-major axis $a_0$ of the ellipse as

$$a_0 = \frac{L^2}{G_N M(1-e^2)}. \tag{77}$$

Thus we obtain

$$\Delta\phi = \frac{6\pi G_N M}{a_0(1-e^2)}\left(1 - \frac{2}{3}\frac{\eta + \tilde{k}}{1+\eta}\right). \tag{78}$$

Alternatively,

$$\frac{\Delta\phi}{\Delta\phi^{GR}} \simeq 1 - \frac{2}{3}\frac{\eta + \tilde{k}}{1+\eta}. \tag{79}$$

It is evident that the above expression reduces to the Schwarzschild result by taking $\tilde{k} = 0 = \eta$.

There are several observations of the perihelion advance that can be used to constrain the parameters $\eta$ and $\tilde{k}$ in the general Buchdahl-inspired metric. Now we consider three different observations related to the phenomenon of the perihelion advance in very different scales, i.e., the perihelion advances of the laser-ranging satellites orbiting Earth [31], of Mercury orbiting the Sun [32], and of the S2 star orbiting the supermassive black hole in the central region of our Milky Way galaxy [33].

Let us first consider the measured perihelion advance of the LAGEOS satellites around Earth. Using 13 years of tracking data from the LAGEOS satellites, the precession of the periapsis of the LAGEOS 2 satellite was measured as [31]

$$\frac{\Delta\phi}{\Delta\phi^{GR}} = 1 + (0.28 \pm 2.14) \times 10^{-3}. \tag{80}$$

From this result, we obtain

$$-1.8 \times 10^{-3} < \frac{1}{2}\frac{\eta + \tilde{k}}{1+\eta} < 1.4 \times 10^{-3}. \tag{81}$$

This bound corresponds to a constraint on the parameter $\tilde{k}$

$$-1.8 \times 10^{-3} < \tilde{k} < 1.4 \times 10^{-3} \tag{82}$$

for the special Buchdahl-inspired metric with $\eta = \tilde{k}$ and $\zeta = \sqrt{1 + 3\tilde{k}^2}$.

We now turn to consider the observation of the anomalous perihelion advance for Mercury. The most accurate measurement of the perihelion advance was performed by the MESSENGER mission [32], which measured the perihelion advance for Mercury as

$$\Delta\phi = (42.9799 \pm 0.009)''/\text{century}. \tag{83}$$

With this measurement, the bound on the $\frac{\eta+\tilde{k}}{1+\eta}$ arising from the general Buchdahl metric can be computed using the experimental error $0.009''$/century, which yields

$$-1.6\times 10^{-5} < \frac{1}{2}\frac{\eta+\tilde{k}}{1+\eta} < 1.6\times 10^{-5}. \tag{84}$$

This bound is better than that from the LAGEOS satellites by two orders of magnitude. Again, from the above constraint, we can obtain the constraint on the parameter $\tilde{k}$ for the special Buchdahl-inspired metric, with $\eta = \tilde{k}$ and $\zeta = \sqrt{1+3\tilde{k}^2}$, i.e.,

$$-1.6\times 10^{-5} < \tilde{k} < 1.6\times 10^{-5}. \tag{85}$$

Here we should mention that the measurement of the anomalous perihelion advance for Mercury will be improved significantly in the near future from the joint European–Japanese BepiColombo project, which was launched in October 2018 [34,35]. It is expected that this mission will improve the accuracy of the perihelion advance to $10^{-4}$ per century. This is one order of magnitude better than the current accuracy of the MESSENGER mission [32]. Thus, with the BepiColombo project, we expect to improve the constraints on the parameters $\frac{\eta+\tilde{k}}{1+\eta}$ arising from the general Buchdahl-inspired metric or the parameter $\tilde{k}$ in the special Buchdahl-inspired spacetime to $-10^{-6} \lesssim \frac{\eta+\tilde{k}}{1+\eta}$ or $\tilde{k} \lesssim 10^{-6}$, which is much more restricted than that obtained from the Cassini experiment.

Finally, let us consider the S2 star orbiting the central black hole of the Milky Way galaxy. Comparing the above two measurements, the observations of the S2 star provide a very different environment from test gravity in the strong gravity regime. The Schwarzschild precession of the S2 star was recently measured by the GRAVITY collaboration [33], which gives

$$\frac{\Delta\phi}{\Delta\phi^{\mathrm{GR}}} = 1.1 \pm 0.19, \tag{86}$$

where

$$\Delta\phi^{\mathrm{GR}} = 12' \tag{87}$$

per orbit period from the prediction of GR. For the effect of the general Buchdahl-inspired metric on the precession, this observation leads to

$$-0.21 < \frac{1}{2}\frac{\eta+\tilde{k}}{1+\eta} < 0.067, \tag{88}$$

which corresponds to

$$-0.21 < \tilde{k} < 0.067 \tag{89}$$

for the special Buchdahl-inspired metric with $\eta = \tilde{k}$ and $\zeta = \sqrt{1+3\tilde{k}^2}$.

### 3.5 Including rotation: shadow investigation

An exact stationary axisymmetric vacuum solution for pure $R^2$ gravity, up to a conformal factor, was derived in [19]

$$\begin{aligned} \mathrm{d}s^2 = A(q,\theta;a)\Bigg[&-\frac{\Delta(q)-a^2\sin^2\theta}{\rho^2}\mathrm{d}t^2 \\ &+\frac{\rho^2}{\Delta(q)}\mathrm{d}q^2+\rho^2\mathrm{d}\theta^2+\frac{2a\sin^2\theta}{\rho^2}[\Delta(q)-r^2(q)-a^2] \\ &\times\mathrm{d}t\,\mathrm{d}\phi+\frac{\Sigma}{\rho^2}\sin^2\theta\mathrm{d}\phi^2\Bigg], \end{aligned} \tag{90}$$

where $r^2(q) = (q-q_+)^{\frac{2q_+}{q_+-q_-}}(q-q_-)^{\frac{-2q_-}{q_+-q_-}}$, $\rho^2(q,\theta) = r^2(q)+a^2\cos^2\theta$, $\Delta(q) = (q-q_+)(q-q_-)+a^2$, and $\Sigma(q,\theta) = [r^2(q)+a^2]^2 - \Delta(q)a^2\sin^2\theta$. The conformal factor $A(q,\theta;a)$, not needed for shadow investigation, was determined numerically. The remaining parameters are given by

$$q_+ = \frac{r_s}{2}\left[\sqrt{1+3\tilde{k}^2}-1\right], \quad q_- = -\frac{r_s}{2}\left[\sqrt{1+3\tilde{k}^2}+1\right]. \tag{91}$$

Using the Event Horizon Telescope collaboration results [1–4], we modeled the central black hole M87* by the rotating metric (90), depending on the mass $M = (1+\tilde{k})r_s/2$, rotation parameter $a$, and the dimensionless parameter $\tilde{k}$. Considering the shadow angular size and assuming that $M$ and $a$ parameters are those of M87*, we obtained

$$-0.155 \le \tilde{k} \le 0.004. \tag{92}$$

## 4 Geodetic precession of spinning objects in the general Buchdahl-inspired metric

In this section, we calculate the geodetic precession of spinning objects in the general Buchdahl-inspired metric. In curved spacetime, the evolution of a spinning particle follows two equations, the geodesics equation

$$\frac{du^\mu}{\mathrm{d}\lambda} + \Gamma^\mu_{\nu\lambda}u^\nu u^\lambda = 0, \tag{93}$$

and the parallel transport equation,

$$\frac{\mathrm{d}s^\mu}{\mathrm{d}\lambda} + \Gamma^\mu_{\nu\lambda}s^\nu u^\lambda = 0, \tag{94}$$

where $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$ is the four-velocity of the particle, and $s^\mu$ denotes the four-spin vector. $u^\mu$ and $s^\nu$ satisfy the following orthogonal condition

$$u^\mu s_\mu = 0. \tag{95}$$

The four-spin vector also satisfies the normalization condition

$$s^\mu s_\mu = 1. \tag{96}$$

Without loss of generality, we consider the evolution of the particle in the equatorial plane, i.e., we can set $\theta = \pi/2$ and $\mathrm{d}\theta/\mathrm{d}\lambda = 0$. Let us study the test spinning particle moving in a circular orbit, i.e., $\dot{R} = 0$, and its four-velocity $u^\mu$ can be expressed as

$$u^t = \dot{t} = -\frac{E}{g_{tt}}, \quad u^\phi = \dot{\phi} = \frac{l}{g_{\phi\phi}}. \tag{97}$$

Then the angular velocity of the spinning particle is written as

$$\Omega = \frac{u^\phi}{u^t} = -\frac{l}{E}\frac{g_{tt}}{g_{\phi\phi}}. \tag{98}$$

The stable circular orbit in the equatorial plane requires

$$E^2 - V_{\rm eff}(R) = 0 \quad \text{and} \quad \frac{dV_{\rm eff}}{\mathrm{d}R} = 0. \tag{99}$$

Solving these two equations, we have

$$E = \sqrt{\frac{g_{tt}^2(R) g'_{\phi\phi}(R)}{g'_{tt}(R) g_{\phi\phi}(R) - g_{tt}(R) g'_{\phi\phi}(R)}}, \tag{100}$$

$$l = \sqrt{\frac{-g_{\phi\phi}^2(R) g'_{tt}(R)}{g'_{tt}(R) g_{\phi\phi}(R) - g_{tt}(R) g'_{\phi\phi}(R)}}, \tag{101}$$

$$\Omega = \sqrt{-\frac{g'_{tt}(R)}{g'_{\phi\phi}(R)}}. \tag{102}$$

Along this circular orbit, the parallel transport equation (94) can be cast in the form

$$\frac{\mathrm{d}s^t}{\mathrm{d}\lambda} + \frac{1}{2}\frac{g'_{tt}(R)}{g_{tt}(R)} u^t s^R = 0, \tag{103}$$

$$\frac{\mathrm{d}s^R}{\mathrm{d}\lambda} - \frac{1}{2}\frac{g'_{tt}(R)}{g_{RR}(R)} u^t s^t - \frac{1}{2}\frac{g'_{\phi\phi}(R)}{g_{RR}(R)} u^\phi s^\phi = 0, \tag{104}$$

$$\frac{\mathrm{d}s^\theta}{\mathrm{d}\lambda} = 0, \tag{105}$$

$$\frac{\mathrm{d}s^\phi}{\mathrm{d}\lambda} + \frac{1}{2}\frac{g'_{\phi\phi}}{g_{\phi\phi}} u^\phi s^R = 0. \tag{106}$$

Differentiating (103) with respect to the affine parameter $\lambda$ and converting $\lambda \to t$ using the relation $\mathrm{d}t = u^t \mathrm{d}\lambda$, we arrive at a second-order ordinary differential equation of $s^R$,

$$\frac{\mathrm{d}^2 s^R}{\mathrm{d}t^2} + \frac{1}{4}\left[\frac{g'^2_{\phi\phi}(R)}{g_{RR}(R) g_{\phi\phi}(R)}\Omega^2 + \frac{g'^2_{tt}(R)}{g_{tt}(R) g_{\phi\phi}(R)}\right] s^R = 0. \tag{107}$$

This equation admits an exact solution

$$s^R(t) = s^R(0)\cos(\omega_g t), \tag{108}$$

where

$$\omega_g = \frac{1}{2}\sqrt{\frac{g'^2_{\phi\phi}(R)}{g_{RR}(R) g_{\phi\phi}(R)}\Omega^2 + \frac{g'^2_{tt}(R)}{g_{tt}(R) g_{RR}(R)}}, \tag{109}$$

represents the oscillating frequency pertaining to the spin four-vector $s^\mu$. With the solution of $s^R$, the other three components $s^t$, $s^\theta$, and $s^\phi$ can be immediately solved, giving

$$s^t(t) = -\frac{1}{2}\frac{g'_{tt}(R)}{g_{tt}(R)} s^R(0)\sin(\omega_g t), \tag{110}$$

$$s^\theta(t) = 0, \tag{111}$$

$$s^\phi(t) = -\frac{1}{2}\frac{g'_{\phi\phi}(R)}{g_{\phi\phi}(R)} \Omega s^R(0)\sin(\omega_g t). \tag{112}$$

In obtaining these solutions, we have used initial conditions $s^t(0) = s^\theta(0) = s^\phi(0) = 0$, which means that spin vector $s^\mu$ was initially directed along the radial direction.

Comparing Eqs. (109) and (102), we can obviously observe that the two frequencies, the oscillating frequency $\omega_g$ of rotation of the spin vector and the orbital frequency $\Omega$ of a massive spinning particle along the circular orbit, are different. This difference leads to a precession of the spin vector. This is the phenomenon called *geodetic precession*. For one complete period of the circular orbit, the angle of the geodetic precession can be expressed as

$$\Delta\Theta = 2\pi\left(1 - \frac{\omega_g}{\Omega}\right) \simeq \frac{3\pi G_{\rm N} M}{R}\left(1 - \frac{2}{3}\frac{\eta + \tilde{k}}{1+\eta}\right). \tag{113}$$

When $\tilde{k} = 0 = \eta$, the above result reduces to the geodetic precession of the Schwarzschild metric.

The geodetic precession can be tested and measured using the gyroscopes in near-Earth artificial satellites. One such experiment is the Gravity Probe B experiment, which was spaced at an altitude of 642 km and had an orbital period of 97.65 min. According to GR, the geodetic effect induces a precession of the gyroscope spin axis by 6606.1 milliarcseconds (mas) per year. Gravity Probe B measures this effect to be [36]

$$\Delta\Theta = (6601.8 \pm 18.3)\text{mas/year}, \tag{114}$$

which leads to

$$-2.6\times 10^{-3} < \frac{\eta + \tilde{k}}{2(1+\eta)} < 2.6\times 10^{-3}. \tag{115}$$

For the special Buchdahl-inspired metric, the above result gives

$$-2.6\times 10^{-3} < \tilde{k} < 2.6\times 10^{-3}. \tag{116}$$

If we treat the Earth–Moon system as a gyroscope orbiting the Sun, its geodetic precession due to the gravitational field of the Sun has also been measured by using the lunar laser ranging data. Recent measurement of the geodetic precession yields a relative deviation from GR as [37]

$$\frac{\Delta\Theta - \Delta\Theta^{\rm GR}}{\Delta\Theta^{\rm GR}} = -0.0019 \pm 0.0064, \tag{117}$$

**Table 1** Summary of estimates for bounds of the parameter $\frac{\eta+\tilde{k}}{1+\eta}$ arising in the general Buchdahl-inspired metric (36) and the parameter $\tilde{k}$ in the special Buchdahl-inspired metric from several observations

| Experiments/observations | Constraints on $\frac{\eta+\tilde{k}}{2(1+\eta)}$ and $\tilde{k}$ | Datasets |
|---|---|---|
| Light deflection | $(-5.0,\ 25) \times 10^{-5}$ | VLBI observation of quasars [29] |
| Time delay | $(-44,\ 2) \times 10^{-6}$ | Cassini experiment [30] |
| Perihelion advance | $(-1.8,\ 1.4) \times 10^{-3}$ | LAGEOS satellites [31] |
| | $(-1.6,\ 1.6) \times 10^{-5}$ | MESSENGER mission [32] |
| | $(-0.21,\ 0.067)$ | Observation of S2 star at galactic center [33] |
| Geodetic precession | $(-2.6,\ 2.6) \times 10^{-3}$ | Gravity Probe B [36] |
| | $(-3.4,\ 6.2) \times 10^{-3}$ | Lunar laser ranging data [37] |
| Shadow (rotating solution) | $(-0.155,\ 0.004)$ | Event Horizon Telescope collaboration [1–4] |

which gives

$$-3.4 \times 10^{-3} < \frac{\eta+\tilde{k}}{2(1+\eta)} < 6.2 \times 10^{-3}. \tag{118}$$

Again, this bound corresponds to

$$-3.4 \times 10^{-3} < \tilde{k} < 6.2 \times 10^{-3}, \tag{119}$$

for the special Buchdahl-inspired metric, with $\eta = \tilde{k}$ and $\zeta = \sqrt{1+3\tilde{k}^2}$.

## 5 Conclusion

In this paper, we study the observational constraints that can be imposed on the asymptotically flat Buchdahl-inspired solution. For this purpose, we theoretically calculate the effects of the parameter $\tilde{k}$ on several solar system experiments and black hole observations. Specifically, we calculate in detail the deflection angle of light by the Sun, gravitational time delay, perihelion advance, and geodetic procession for massless and massive objects in the Buchdahl-inspired spacetime. With these theoretical predictions, we derive the constraints on the parameter $\tilde{k}$ in the asymptotically flat Buchdahl-inspired spacetime by comparing our theoretical calculations with observations. Our results are summarized in Table 1. In addition, we provide different comparisons of parameters from modified gravity and general relativity. For instance, Eq. (51) provides a comparison of the deflection angle of light between the two theories; Eq. (58) provides a comparison of gravitational time delay; Eq. (79) and (86) provide a similar comparison for perihelion and periastron precession between theories, while Eq. (117) gives a comparison of geodetic precession as predicted by the two theories.

It is worth mentioning here that the measurement of the gravitational time delay by the Cassini experiment provides the most sensitive tool to constrain the parameter $\tilde{k}$ in the solar system. Another important constraint comes from observing the perihelion advance for Mercury by the MESSENGER mission. As we mentioned, the measurement of the anomalous perihelion advance for Mercury will be improved significantly in the near future from the joint European–Japanese BepiColombo project, which was launched in October 2018 [34,35]. It is expected that this mission will improve the accuracy of the perihelion advance to $10^{-4}$ per century, which can be used to improve the constraints on the parameters $\frac{\eta+\tilde{k}}{1+\eta}$ arising from the general Buchdahl-inspired metric or the parameter $\tilde{k}$ in the special Buchdahl-inspired spacetime to $-10^{-6} \lesssim \frac{\eta+\tilde{k}}{1+\eta}$ or $\tilde{k} \lesssim 10^{-6}$, which is much more restricted than that obtained from the Cassini experiment.

In contrast to GR, pure $\mathcal{R}^2$ gravity does not adhere to Birkhoff's theorem. As a higher-derivative characteristic, the Buchdahl parameter $\tilde{k}$ of its vacuum solution exterior to a star is system-dependent. Therefore, our empirical tests as presented in this article were carried out under this premise. We have focused on the exterior vacuum solution, deferring the theoretical determination of $\tilde{k}$ for future exploration [38].

The determination of $\tilde{k}$ is, in principle, contingent on the composition—specifically, the equation of state and the distribution of matter within the host star. Typically, this inquiry involves matching the interior and exterior solutions across the star's surface. An alternative approach entails deriving a set of Tolman–Oppenheimer–Volkoff (TOV) equations governing the pressure and density of the star material. By numerically solving these equations in conjunction with the metric components, the exterior vacuum configuration of a star can be obtained based on a presumed equation of state and conditions at the star's center [39–41]. Progress has recently been made on this front by one of us in reducing the TOV equations for $f(\mathcal{R})$ gravity to a single integro-differential equation. This simplification has enabled our investigation into the interior–exterior matching for the Buchdahl-inspired solution, with detailed findings to be reported separately [38].

While most test results presented in Table 1 align with GR, the cases for S2 star and M87*, which may qualify as in a strong field regime, show large deviations for $\tilde{k}$ from 0, albeit with error bars too large for definitive conclusions; see Eqs. (89) and (92). As investigated in [21], a $\tilde{k}$ value in the range $(-1, 0)$ has been associated with the potential formation of wormholes. Theoretically, such spacetime configurations could support the possibility of closed timelike curves, recently explored in [42]. Consequently, future tests of the Buchdahl-inspired solution and pure $\mathcal{R}^2$ gravity in strong field regimes may be warranted.

**Acknowledgements** Tao Zhu is supported by the Zhejiang Provincial Natural Science Foundation of China under Grants No. LR21A050001 and No. LY20A050002, the National Natural Science Foundation of China under Grants No. 12275238 and No. 11675143, the National Key Research and Development Program of China under Grant No. 2020YFC2201503, and the Fundamental Research Funds for the Provincial Universities of Zhejiang in China under Grant No. RF-A2019015.

**Data availability** This manuscript has no associated data. [Authors' comment: The present investigation did not generate new data. Hence no data available.]



## References

1. K. Akiyama et al. [Event Horizon Telescope], First M87 event horizon telescope results. I. The shadow of the supermassive black hole. Astrophys. J. Lett. **875**, L1 (2019). arXiv:1906.11238 [astro-ph.GA]
2. K. Akiyama et al. [Event Horizon Telescope], First M87 event horizon telescope results. IV. Imaging the central supermassive black hole. Astrophys. J. Lett. **875**, L4 (2019)
3. K. Akiyama et al. [Event Horizon Telescope], First M87 event horizon telescope results. V. Physical origin of the asymmetric ring. Astrophys. J. Lett. **875**, L5 (2019)
4. K. Akiyama et al. [Event Horizon Telescope], First M87 event horizon telescope results. VI. The shadow and mass of the central black hole. Astrophys. J. Lett. **875**, L6 (2019)
5. B.P. Abbott et al. [LIGO Scientific and Virgo], Observation of gravitational waves from a binary black hole merger. Phys. Rev. Lett **116**(6), 061102 (2016). https://doi.org/10.1103/PhysRevLett.116.061102. arXiv:1602.03837 [gr-qc]
6. B.P. Abbott et al. [LIGO Scientific and Virgo], Properties of the binary black hole merger GW150914. Phys. Rev. Lett. **116**(24), 241102 (2016). arXiv:1602.03840 [gr-qc]
7. H.A. Buchdahl, Non-linear Lagrangians and cosmological theory. Mon. Not. R. Astron. Soc. **150**, 1 (1970)
8. T.P. Sotiriou, V. Faraoni, $f(R)$ theories of gravity. Rev. Mod. Phys. **82**, 451–497 (2010). arXiv:0805.1726 [gr-qc]
9. A. De Felice, S. Tsujikawa, $f(R)$ theories. Living Rev. Relativ. **13**, 3 (2010). arXiv:1002.4928 [gr-qc]
10. C.G. Boehmer, E. Jensko, Modified gravity: a unified approach to metric-affine models. J. Math. Phys. **64**, 082505 (2023). arXiv:2301.11051 [gr-qc]
11. A. Salvio, A. Strumia, Agravity. J. High Energy Phys. **06**, 080 (2014). arXiv:1403.4226 [hep-ph]
12. M.B. Einhorn, D.R. Timothy Jones, Naturalness and dimensional transmutation in classically scale-invariant gravity. J. High Energy Phys. **03**, 047 (2015). arXiv:1410.8513 [hep-th]
13. H.K. Nguyen, Beyond Schwarzschild-de Sitter spacetimes: I. A new exhaustive class of metrics inspired by Buchdahl for pure $\mathcal{R}^2$ gravity in a compact form. Phys. Rev. D **106**, 104004 (2022). arXiv:2211.01769 [gr-qc]
14. T. Damour, G. Esposito-Farèse, Nonperturbative strong-field effects in tensor-scalar theories of gravitation. Phys. Rev. Lett. **70**, 2220 (1993)
15. L. Alvarez-Gaume, A. Kehagias, C. Kounnas, D. Lüst, A. Riotto, Aspects of quadratic gravity. Fortschr. Phys. **64**, 176 (2016). arXiv:1505.07657 [hep-th]
16. H.K. Nguyen, Emerging Newtonian potential in pure $R^2$ gravity on a de Sitter background. J. High Energy Phys. **08**, 127 (2023). arXiv:2306.03790 [gr-qc]
17. H.A. Buchdahl, On the gravitational field equations arising from the square of the Gaussian curvature. Nuovo Cimento **23**, 141 (1962)
18. H.K. Nguyen, Beyond Schwarzschild-de Sitter spacetimes: II. An exact non-Schwarzschild metric in pure $\mathcal{R}^2$ gravity and new anomalous properties of $\mathcal{R}^2$ spacetimes. Phys. Rev. D **107**, 104008 (2023). arXiv:2211.03542 [gr-qc]
19. M. Azreg-Aïnou, H.K. Nguyen, A stationary axisymmetric vacuum solution for pure $\mathcal{R}^2$ gravity. Phys. Scr. **98**, 125025 (2023). arXiv:2304.08456 [gr-qc]
20. H.K. Nguyen, Non-triviality of asymptotically flat Buchdahl-inspired metrics in pure $R^2$ gravity. arXiv:2305.12037 [gr-qc]
21. H.K. Nguyen, M. Azreg-Aïnou, Traversable Morris–Thorne–Buchdahl wormholes in quadratic gravity. Eur. Phys. J. C **83**, 626 (2023). arXiv:2305.04321 [gr-qc]
22. M.S. Morris, K.S. Thorne, Wormholes in spacetime and their for interstellar travel: a tool for teaching general relativity. Am. J. Phys. **56**, 5 (1988)
23. M.S. Morris, K.S. Thorne, U. Yurtsever, Wormholes, time machines, and the weak energy condition. Phys. Rev. Lett. **61**, 1446 (1988)
24. H.K. Nguyen, M. Azreg-Aïnou, Revisiting weak energy conditions and wormholes in Brans–Dicke gravity. arXiv:2305.15450 [gr-qc]
25. M. Campanelli, C. Lousto, Are black holes in Brans–Dicke theory precisely the same as in general relativity? Int. J. Mod. Phys. D **2**, 451 (1993). arXiv:gr-qc/9301013
26. L. Vanzo, S. Zerbini, V. Faraoni, Campanelli–Lousto and veiled spacetimes. Phys. Rev. D **86**, 084031 (2012). arXiv:1208.2513 [gr-qc]
27. A.G. Agnese, M. La Camera, Wormholes in the Brans–Dicke theory of gravitation. Phys. Rev. D **51**, 2011 (1995)
28. A.G. Agnese, M. La Camera, Schwarzschild metrics, quasi-universes and wormholes, in *Frontiers of Fundamental Physics 4*. ed. by B.G. Sidharth, M.V. Altaisky (Springer, Boston, 2001). arXiv:astro-ph/0110373
29. E. Fomalont, S. Kopeikin, G. Lanyi, J. Benson, Progress in measurements of the gravitational bending of radio waves using the

VLBA. Astrophys. J. **699**, 1395–1402 (2009). arXiv:0904.3992 [astro-ph.CO]
30. B. Bertotti, L. Iess, P. Tortora, A test of general relativity using radio links with the Cassini spacecraft. Nature **425**, 374 (2003)
31. D.M. Lucchesi, R. Peron, Accurate measurement in the field of the earth of the general-relativistic precession of the LAGEOS II pericenter and new constraints on non-Newtonian gravity. Phys. Rev. Lett. **105**, 231103 (2010). arXiv:1106.2905 [gr-qc]
32. R.S. Park, W.M. Folkner, A.S. Konopliv, J.G. Williams, D.E. Smith, M.T. Zuber, Precession of Mercury's perihelion from ranging to the MESSENGER spacecraft. Astrophys. J. **153**, 121 (2017)
33. R. Abuter et al. [GRAVITY Collaboration], Detection of the Schwarzschild precession in the orbit of the star S2 near the Galactic centre massive black hole. Astron. Astrophys. **636**, L5 (2020)
34. C.M. Will, New general relativistic contribution to Mercury's perihelion advance. Phys. Rev. Lett. **120**, 191101 (2018). arXiv:1802.05304 [gr-qc]
35. J. Benkhoff, J. van Casteren, H. Hayakawa, M. Fujimoto, H. Laakso, M. Novara, P. Ferri, H.R. Middleton, R. Ziethe, BepiColombo, comprehensive exploration of Mercury: mission overview and science goals. Planet. Space Sci. **58**, 2 (2010)
36. C.W.F. Everitt et al. [Gravity Probe B re- sults], Gravity probe B: final results of a space experiment to test general relativity. Phys. Rev. Lett. **106**, 221101 (2011)
37. J.G. Williams, S.G. Turyshev, D.H. Boggs, Progress in lunar laser ranging tests of relativistic gravity. Phys. Rev. Lett. **93**, 261101 (2004)
38. H.K. Nguyen, B. Chauvineau, Tolman–Oppenheimer–Volkoff equation in $f(\mathcal{R})$ gravity: a new formulation and its theoretical properties (in preparation)
39. R. Kase, S. Tsujikawa, Neutron stars in f(R) gravity and scalar-tensor theories. JCAP **09**, 054 (2019). arXiv:1906.08954 [gr-qc]
40. K. Yagi, M. Stepniczka, Neutron stars in scalar-tensor theories: analytic scalar charges and universal relations. Phys. Rev. D **104**, 044017 (2021). arXiv:2105.01614 [gr-qc]
41. M. Orellana, F. García, F.A.T. Pannia, G.E. Romero, Structure of neutron stars in $R$-squared gravity. Gen. Relativ. Gravit. **45**, 771–783 (2013). arXiv:1301.5189 [astro-ph.CO]
42. H.K. Nguyen, F.S.N. Lobo, Closed timelike curves induced by a Buchdahl-inspired vacuum spacetime in $R^2$ gravity. Universe **9**, 467 (2023). arXiv:2310.19829 [gr-qc]