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Dynamics on a submanifold: intermediate formalism versus Hamiltonian reduction of Dirac bracket, and integrability

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Abstract We consider the Lagrangian dynamical system forced to move on a submanifold $G_\alpha(q^A)=0$. If for some reason we are interested in knowing the dynamics of all original variables $q^A(t)$, the most economical would be a Hamiltonian formulation on the intermediate phase-space submanifold spanned by reducible variables q^A and an irreducible set of momenta p_i, $[i]=[A]-[\alpha]$. We describe and compare two different possibilities for establishing the Poisson structure and Hamiltonian dynamics on an intermediate submanifold: Hamiltonian reduction of the Dirac bracket and intermediate formalism. As an example of the application of intermediate formalism, we deduce on this basis the Euler–Poisson equations of a spinning body, establish the underlying Poisson structure, and write their general solution in terms of the exponential of the Hamiltonian vector field.

1 Three equivalent Hamiltonian formulations for a system with holonomic constraints

Consider a mechanical system that can be described using a non-singular Lagrangian $L(q^A,\dot q^A)$, defined in configuration space with generalized coordinates $q^A(t)$, $A=1,2,\ldots n$. Suppose the "particle" q^A was then forced to move on a k-dimensional surface $\mathbb{S}^k$ given by the algebraic equations $G_\alpha(q^A)=0$. The task is to construct the Hamiltonian formulation for this theory. There are three possible ways to do this. Let us first briefly describe and compare them.

(A) The first possibility is to work with unconstrained variables. Let x^i, $i=1,2,\ldots,k$ be local coordinates on $\mathbb{S}$. Then, equations of motion follow from the Lagrangian $\tilde L(x^i,\dot x^i)\equiv L(q^A(x^i),\mathrm{d}q^A(x^i)/\mathrm{d}t)$. If $\tilde L$ is also non-singular, we introduce the conjugate momenta p_i for x^i, the Hamiltonian $H(x^i,p_j)$, and the canonical Poisson bracket $\{x^i,p_j\}=\delta^i_j$. Then the Hamiltonian equations are $\dot x^i=\{x^i,H\}$, $\dot p_i=\{p_i,H\}$.

The transition to independent variables x^i is not always desirable. For instance, in the case of a spinning body, the q^A variables are nine elements of an orthogonal 3×3 matrix R_{ij} (therefore, $G_\alpha=0$ reads as $R^TR-\mathbf{1}=0$). To describe a rigid body, we need to know the evolution of q^A and not x^i.

(B) The second possibility is to work with original variables using the Dirac version of Hamiltonian formalism [1–6]. Equations of motion follow from the modified Lagrangian action, where the constraints are taken into account with the help of auxiliary variables $\lambda_\alpha(t)$ as follows [6,7]:

$$S=\int \mathrm{d}t\ L(q^A,\dot q^A)-\lambda_\alpha G_\alpha(q^A). \tag{1}$$

We should pass to the Hamiltonian formulation introducing the conjugate momenta p_A, $p_{\lambda\alpha}$ to all original variables q^A, λ_α. The Hamiltonian equations are then obtained using canonical Poisson brackets $\{q^A,p_B\}=\delta^A{}_B$, $\{\lambda_\alpha,p_{\lambda\beta}\}=\delta_{\alpha\beta}$ and a Hamiltonian of the form $H(q^A,p_B,\lambda_\alpha,p_{\lambda\beta})$. The resulting equations depend on the auxiliary variables λ_α and $p_{\lambda\alpha}$. The systematic method for excluding them is to pass from the canonical to the Dirac bracket. The latter is constructed using second-class constraints

$$G_\alpha(q^A)=0,\quad \Phi_\alpha(q^A,p_B)=0, \tag{2}$$

that appear in the Hamiltonian formulation of the theory (1). Working with the Dirac bracket, all terms with auxiliary variables disappear from the final equations. This gives the Hamiltonian formulation on the phase space with coordinates q^A, p_B.

(C) In the case of a spinning body, a kind of intermediate formulation arises between (A) and (B). The freely spinning

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body can be described by 9 + 3 Euler–Poisson equations[1]

$$\dot{R}_{ij} = -\epsilon_{jkm}\Omega_k R_{im}, \quad I\dot{\mathbf{\Omega}} = [I\mathbf{\Omega}, \mathbf{\Omega}], \tag{3}$$

where I is a numerical 3×3 matrix. They turn out to be the Hamiltonian equations [8–12], with the configuration-space variables assembled into a 3×3 matrix $R_{ij}(t)$, while $\Omega_i(t)$ are three components of momenta. There are nine redundant coordinates R_{ij}, but only three independent momenta Ω_i. So, if in case (A) we worked with unconstrained set (x^i, p_j) and in case (B) with redundant set (q^A, p_B), then now we have an intermediate situation: (q^A, p_j). This gives the most economical Hamiltonian formulation of a theory for which we are interested in knowing the dynamics of all variables q^A.

An intermediate formulation for the theory (1) can be obtained in the Dirac formalism, by first constructing the Dirac bracket (which is a degenerate Poisson structure on original phase space (q^A, p_B)), and then reducing it on the submanifold $\Phi_\alpha = 0$. Let us call it the intermediate submanifold.[2] In the present work we develop an alternative way, allowing us to construct the Poisson structure on this submanifold, without the need for the Dirac bracket. Roughly speaking, this works as follows. For any theory of the form (1) with positive-definite Lagrangian L, we present a universal procedure to find (non-canonical) phase-space coordinates (q^A, π_i, π_α) with special properties. They are constructed using the matrix $G_{\alpha A} \equiv \partial G_\alpha/\partial q^A$ and fundamental solutions of the linear system $G_{\alpha A}x_A = 0$. The intermediate formulation of the theory (1) is obtained by first rewriting the Hamiltonian formulation of unconstrained theory L in terms of new coordinates, and then excluding the variables π_α from all resulting expressions with the help of the constraint $\Phi_\alpha = 0$. In particular, the Poisson structure on the intermediate submanifold turns out to be the canonical Poisson bracket of the original variables (q^A, p_B), first rewritten in terms of new coordinates (q^A, π_B), and then restricted to this submanifold.

As we saw above, an interesting application of the intermediate formalism lies in the branch of spinning body dynamics. This issue is also of interest in modern studies of various aspects related to the construction and behaviour of spinning particles and rotating bodies in external fields beyond the pole–dipole approximation [13–21]. For simple mechanical systems (point particle in an external field or several mutually interacting particles), their equations of motion are postulated based on the analysis of experimental data. Unfortunately, a spinning body turns out to be too complex a system to find its equations in this way. Thus, even writing the equations of motion of a spinning body turns out to be a non-trivial task. At the dawn of the development of mechanics, this was considered one of the central problems, for which several branches of classical mechanics were developed, including Lagrangian mechanics on a submanifold, Hamiltonian mechanics with constraints, symmetry groups and their relation to conservation laws and integrals of motion, and integrable systems. As a result, the basic theory of a rotating body was formulated in the works of Euler, Lagrange, Poisson, Poinsot, and many others [22–25]. However, a didactically systematic formulation and application of these methods to various problems of rigid body dynamics is still regarded as not an easy task [9,10]. For instance, Marsden, Holm and Ratiu in their work [9] in 1998 write: “It was already clear in the last century that certain mechanical systems resist the usual canonical formalism, either Hamiltonian or Lagrangian, outlined in the first paragraph. The rigid body provides an elementary example of this.”

Second-order Lagrangian equations of a spinning body can be obtained as the conditions of extrema of a variational problem, where the body is considered a system of particles subjected to holonomic constraints [11,12]. However, the most convenient for applications turn out to be the equations written in a first-order (Hamiltonian) form (3). Therefore, it is desirable to have a formalism that allows one to deduce these equations starting from the Lagrangian variational problem by direct application of the standard prescriptions of classical mechanics for the passage from Lagrangian to Hamiltonian formulations. The intermediate formalism seems to be the most economical way to do this. It should also be noted that a thorough analysis of the Lagrangian and Hamiltonian formulations reveals some specific properties of the formalism, which are not always taken into account in the literature, when formulating the laws of motion and applying them. In several cases this even leads to the need to revise some classical problems of the dynamics of a spinning body, see [11,12] and references therein.

The remainder of the paper is organized as follows. In Sect. 2, we briefly discuss the dynamics on a surface $G_\alpha(q^A) = 0$ in terms of unconstrained variables, and outline the Liouville integration procedure in a form convenient for later comparison with the integration method based on the Hamiltonian vector field. We then describe Hamiltonian reduction on an intermediate submanifold using a Dirac bracket in Sect. 3. In Sect. 4, we present our intermediate formalism for establishing the Poisson structure and Hamiltonian equations on the intermediate submanifold. In Sect. 5, we present the method for integration of first-order equations using the Hamiltonian vector field. We illustrate the intermediate formalism on a simple example of a point particle forced to move on a sphere in Sect. 6. In Sect. 7, we use the intermediate formalism to establish the Poisson structure that lies behind the Euler–Poisson equations of a spinning body,

[1] We denote the scalar and vector products of the vectors $\mathbf{a}$ and $\mathbf{b}$ by $(\mathbf{a}, \mathbf{b})$, and $[\mathbf{a}, \mathbf{b}]$.

[2] All solutions of the theory (1) lie in the phase-space submanifold $\Phi_\alpha = 0$, $G_\alpha = 0$, hence the term “intermediate”.

and write their general solution in terms of power series with respect to the evolution parameter, and with the coefficients determined by derivatives of the Hamiltonian vector field.

2 Motion on a surface in terms of unconstrained variables and integrability according to Liouville

We assume that the original Lagrangian is non-singular

$$\det\frac{\partial^2 L(q^A,\dot q^A)}{\partial\dot q^A\partial\dot q^B}\equiv\det M_{AB}\neq 0, \tag{4}$$

and that the particle q^A was forced to move on a k-dimensional surface $\mathbb{S}^k$ determined by $n-k$ functionally independent equations

$$G_\alpha(q^A)=0,\quad \alpha=1,2,\ldots,n-k,$$
$$\text{rank}\,\frac{\partial G_\alpha}{\partial q^A}\equiv\text{rank}\,G_{\alpha A}=k. \tag{5}$$

Let x^i, $i=1,2,\ldots,k$ be local coordinates on $\mathbb{S}^k$, and $q^A(x^i)$ be parametric equations of $\mathbb{S}^k$: $G_\alpha(q^A(x^i))\equiv 0$ for any x^i. Then, equations of motion follow from the following unconstrained Lagrangian

$$\tilde L(x^i,\dot x^i)\equiv L(q^A(x^i),\dot q^A(x^i))=L\left(q^A(x^i),\frac{\partial q^A}{\partial x^i}\dot x^i\right), \tag{6}$$

and read as follows:

$$\frac{\partial\tilde L}{\partial x^i}-\frac{d}{dt}\frac{\partial\tilde L}{\partial\dot x^i}=0. \tag{7}$$

By construction, for any solution $x^i(t)$ to the problem (7), the trajectories $q^A(x^i(t))$ lie on the surface (5). This recipe is well-justified [6,7,26] for Lagrangians of the form $T=\frac{1}{2}m_A(\dot q^A)^2-U(q^A)$ with $m_A>0$. For more general Lagrangians, it should be taken as the definition of a particle constrained to a surface.

We add one more technical restriction, assuming that the matrix M_{AB} is positive-definite, that is, $\mathbf{Y}^TM\mathbf{Y}>0$ for any non-zero column $\mathbf{Y}$. Then the matrix

$$M_{ij}\equiv\frac{\partial\tilde L}{\partial\dot x^i\partial\dot x^j}=M_{AB}\frac{\partial q^A}{\partial x^i}\frac{\partial q^B}{\partial x^j}\equiv(Q^T)_{iA}M_{AB}Q_{Bj}, \tag{8}$$

is non-degenerate, see Appendix. In view of this, for positive-definite $L(q^A,\dot q^A)$, the Lagrangian $\tilde L(x^i,\dot x^j)$ is non-singular.

The Hamiltonian formulation in terms of unconstrained variables can be obtained as follows. Introduce the conjugate momenta $p_i=\partial\tilde L/\partial\dot x^i$ for x^i. As $\det M_{ij}\neq 0$, these equations can be resolved with respect to $\dot x^i$, say $\dot x^i=v^i(x^j,p_k)$. Using these equalities, we construct the Hamiltonian by excluding $\dot x^i$ from the expression $H=p_i\dot x^i-\tilde L(x^i,\dot x^j)$. Then, using the canonical Poisson brackets $\{x^i,p_j\}=\delta^i_j$, the Hamiltonian equations of the theory are

$$\dot x^i=\{x^i,H\}=\frac{\partial H}{\partial p_i},\quad \dot p_i=\{p_i,H\}=-\frac{\partial H}{\partial q^i}. \tag{9}$$

If the Hamiltonian does not explicitly depend on time, it is an integral of motion. If, in addition, there are extra $k-1$ integrals of motion, then according to Liouville's theorem, a general solution to equations of motion can be found in quadratures (that is, calculating integrals of some known functions and doing the algebraic operations).

Liouville's theorem. Let the Hamiltonian equations (9) admit k integrals of motion $F_1=H,F_2,F_3.\ldots,F_k$. We assume that they are in involution and functionally independent with respect to momenta

$$\{F_i,F_j\}=0,\quad\text{or}\quad\frac{\partial F_i}{\partial x^a}\frac{\partial F_j}{\partial p_a}=\frac{\partial F_j}{\partial x^a}\frac{\partial F_i}{\partial p_a}, \tag{10}$$

$$\det\frac{\partial F_i(x^j,p_k)}{\partial p_j}\neq 0. \tag{11}$$

Then the equations of motion are integrable in quadratures.

Proof The proof consists in formulating a recipe for constructing the general solution.

(A). Consider the equations $F_i(x^i,p_j)=c_i=\text{const}$ for the constant-level surface of integrals of motion. Due to the condition (11), they can be solved with respect to p_i

$$F_i(x^i,p_j)=c_i,\quad\leftrightarrow\quad p_i=f_i(x^i,c_j),\quad\text{then}$$
$$F_i(x^p,f_j(x^n,c_k))=c_i. \tag{12}$$

We first confirm that the vector function f_i is a gradient of some scalar function. Omitting x^j, which we temporarily regard as parameters, we have $F_i(f_j(c_k))=c_i$, that is, F_i and f_j are mutually inverse transformations. Calculating the derivative of this equality with respect to c_j, we obtain

$$\left.\frac{\partial F_i}{\partial p_a}\right|_{p=f}\frac{\partial f_a}{\partial c_j}=\delta_{ij},\quad\text{then}\quad\frac{\partial f_i}{\partial c_a}\left.\frac{\partial F_a}{\partial p_j}\right|_{p=f}=\delta_{ij},$$
$$\text{and}\quad\det\frac{\partial f_a}{\partial c_j}\neq 0. \tag{13}$$

Contracting Eq. (10) with $\partial f_k/\partial c_i$ and using (13), we obtain the identity

$$\left.\frac{\partial F_j}{\partial x^k}\right|_{p=f}=\frac{\partial f_k}{\partial c_b}\frac{\partial F_b}{\partial x^a}\left.\frac{\partial F_j}{\partial p_a}\right|_{p=f}. \tag{14}$$

Contracting $\partial f_b/\partial c_j$ with the derivative of (12) with respect to x^k and using Eq. (13), we obtain the following expression for the derivative of f_b

$$\frac{\partial f_b}{\partial c_j}\left[\left.\frac{\partial F_j}{\partial x^k}\right|_{p=f}+\left.\frac{\partial F_j}{\partial p_a}\right|_{p=f}\frac{\partial f_a}{\partial x^k}\right]=0,$$
$$\text{then}\quad\frac{\partial f_b}{\partial x^k}=-\frac{\partial f_b}{\partial c_j}\left.\frac{\partial F_j}{\partial x^k}\right|_{p=f}. \tag{15}$$

Together with (14), this implies that $\partial_k f_b$ is an antisymmetric matrix

$$\frac{\partial f_b}{\partial x^k} = -\frac{\partial f_b}{\partial c_j}\frac{\partial F_j}{\partial x^k}\bigg|_{p=f} = -\frac{\partial f_b}{\partial c_j}\frac{\partial f_k}{\partial c_n}\frac{\partial F_n}{\partial x^a}\frac{\partial F_j}{\partial p_a}\bigg|_{p=f}$$
$$= -\frac{\partial f_k}{\partial c_n}\frac{\partial F_n}{\partial x^b}\bigg|_{p=f} = \frac{\partial f_k}{\partial x^b}, \quad \text{or} \quad \partial_i f_j - \partial_j f_i = 0. \tag{16}$$

Then, Eq. (10) implies that the quantities $p_i - f_i(x^k, c_j)$ are in involution

$$\{p_i - f_i, p_j - f_j\} = 0. \tag{17}$$

According to (16), $f_i(x^k)$ is a curl-free vector field, so there is the potential Φ: $f_i(x^k, c_j) = \partial\Phi(x^k, c_j)/\partial x^i$. In the result, we demonstrated that equations of a constant-level surface (12) can be written in the form

$$p_i = \partial_i \Phi(x^j, c_k). \tag{18}$$

(B). According to Stokes' theorem, the line integral of a curl-free field does not depend on the choice of the integration path, and gives the potential

$$\Phi(x^k, c_j) = \int_0^{x^k} f_i(z^i, c_j)\mathrm{d}z^i. \tag{19}$$

(C). Substituting the solution (18) to the equation $H(x^i, p_j) = c_1 \equiv E$ into this equation, we have the identity

$$H\left(x^i, \frac{\partial\Phi(x^i, c_j)}{\partial x^j}\right) = E. \tag{20}$$

Then the function

$$S(t, x^i, c_j) = -Et + \Phi(x^i, c_j), \tag{21}$$

with the property

$$\det\frac{\partial^2 S}{\partial x^i \partial c_j} = \det\frac{\partial f_i}{\partial c_j} \neq 0, \tag{22}$$

by construction obeys the Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + H\left(x^i, \frac{\partial S}{\partial x^j}\right) = 0. \tag{23}$$

According to the theory of canonical transformations (see Sect. 4.7 in [6]), the general solution to the Hamiltonian equations (9) with $2k$ integration constants c_k, b_i can be now obtained by solving the algebraic equations

$$p_i = \frac{\partial S(t, x^j, c_k)}{\partial x^i} = f_i(x^j, c_k),$$
$$b_i = \frac{\partial S(t, x^j, c_k)}{\partial c_i} = -t\delta^i{}_E + \frac{\partial\Phi(x^j, c_k)}{\partial c_i}, \tag{24}$$

with respect to x^i and p_j. The resolvability of the second equation is guaranteed by (22).

As a result, the problem of integrating the Hamiltonian system (9) is reduced to the calculation of line integral (19). In turn, this can be reduced to the calculation of definite integrals. To see this, let us specify the equations (24) to the case of a theory with two configuration-space variables $x^i = (x, y)$ and two integrals of motion $H(x, y, p_x, p_y) = E$ and $F(x, y, p_x, p_y) = c$. Solving these algebraic equations, we obtain $p_x = f_x(x, y, E, c)$ and $p_y = f_y(x, y, E, c)$. Taking the path of integration to be the pair of intervals, $(0, 0) \to (x, 0) \to (x, y)$, we obtain the potential

$$\Phi(x, y, E, c) = \int_0^x f_x(x', 0, E, c)\mathrm{d}x' + \int_0^y f_y(x, y', E, c)\mathrm{d}y'. \tag{25}$$

Then Eq. (24) reads as follows:

$$p_x = f_x(x, y, E, c), \qquad p_y = f_y(x, y, E, c),$$
$$b_x = -t + \int_0^x \frac{\partial f_x(x', 0, E, c)}{\partial E}\mathrm{d}x' + \int_0^y \frac{\partial f_y(x, y', E, c)}{\partial E}\mathrm{d}y',$$
$$b_y = \int_0^x \frac{\partial f_x(x', 0, E, c)}{\partial c}\mathrm{d}x' + \int_0^y \frac{\partial f_y(x, y', E, c)}{\partial c}\mathrm{d}y'. \tag{26}$$

Thus the problem is reduced to the calculation of four definite integrals indicated in these equations.

3 Motion on a surface in terms of original variables

To work with a particle on a surface in terms of original variables, we can use the variational problem with the modified Lagrangian (1), where the constraints are taken into account with the help of auxiliary dynamical variables $\lambda_\alpha(t)$, called Lagrangian multipliers. In all calculations they should be treated on equal footing with $q^A(t)$. In particular, looking for the equations of motion, we take variations with respect to q^A and all λ_α. The variation with respect to λ_α implies

$$G_\alpha(q^A) = 0, \tag{27}$$

that is, the constraints arise as part of the conditions of extrema of the action functional. Thus the presence of λ_α allows q^A to be treated as an unconstrained variable that should be varied independently in obtaining the equations of motion. Taking the variation with respect to q^A, we obtain

$$-\frac{d}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^A} + \frac{\partial L}{\partial q^A} - \lambda_\alpha G_{\alpha A} = 0. \tag{28}$$

Computing the time derivative, these equations read

$$\ddot{q}^A = K^A(q, \dot{q}) - \tilde{M}^{AB} G_{\beta B}\lambda_\beta, \tag{29}$$

where $\tilde{M}^{AB}$ is the inverse of $M_{AB}(q^A, \dot{q}^B)$ and $K^A \equiv \tilde{M}^{AB}\left[-\dot{q}^C \partial^2 L/(\partial\dot{q}^B\partial q^C) + \partial L/\partial q^B\right]$. The theories (29) and (7) turn out to be equivalent, see [6,7].

The auxiliary variables λ_α can be excluded from the system (28) (or (29)) as follows. For any solution $q^A(t)$, the identity $G_\alpha(q^A(t)) = 0$ implies $\dot{G}_\alpha = G_{\alpha A}\dot{q}^A = 0$. Calculating one more derivative, we obtain $G_{\alpha A}\ddot{q}^A + \partial_B\partial_A G_\alpha \dot{q}^A\dot{q}^B = 0$. Using the expression for $\ddot{q}^A$ from (29), we obtain

$$C_{\alpha\beta}\lambda_\beta = G_{\alpha A}K^A + \partial_A\partial_B G_\alpha \dot{q}^A\dot{q}^B, \quad \text{where } C_{\alpha\beta} \equiv G_{\alpha A}\tilde{M}^{AB}G_{\beta B}. \tag{30}$$

According to the Appendix, C has the inverse matrix $\tilde{C}$, so we can separate λ_β as follows:

$$\lambda_\beta = \tilde{C}_{\beta\alpha}[G_{\alpha A}K^A + \partial_A\partial_B G_\alpha \dot{q}^A\dot{q}^B]. \tag{31}$$

Inserting this λ_β into Eq. (28) or (29), we obtain closed equations for determining the physical variables $q^A(t)$.

Comment. If the condition (8) is not satisfied, the invertibility of C is not guaranteed, and we need to continue the analysis of the system (30). The general procedure can be found in Appendix C of [3]. Here we will only show that in a theory with kinematic constraints, the auxiliary variables can always be excluded from the equations for physical variables. Without loss of generality, we can assume that the coordinates q^A were enumerated in such a way that the non-vanishing minor of the matrix $G_{\alpha A}$ is located in the first $n-k$ columns. Then

$$q^A = (q^\alpha, q^i), \quad \det\frac{\partial G_\alpha}{\partial q^\beta} \neq 0. \tag{32}$$

Let us consider the original theory (1) in special coordinates q'^A, adapted to the surface and defined as follows:

$$q^A = (q^\alpha, q^i) \leftrightarrow q'^A = (q'^\alpha = G_\alpha(q^A),\ q'^i = q^i). \tag{33}$$

That is, we take the constraint's functions $G_\alpha(q^A)$ as part of the new coordinates. In the adapted coordinates, our surface is just the hyperplane $q'^\alpha = 0$, and q'^i can be taken as its local coordinates. For the inverse transformation, we obtain

$$q^A = (q^\alpha = \tilde{G}_\alpha(q'^A),\ q^i = q'^i) \quad \text{then} \quad \dot{q}'^\alpha = \frac{\partial\tilde{G}_\alpha}{\partial q'^A}\dot{q}'^A, \quad \dot{q}^i = \dot{q}'^i, \tag{34}$$

where $\tilde{G}_\alpha(q'^A)$ is the solution to equations $q'^\alpha = G_\alpha(q^\alpha, q'^i)$: $G_\alpha(\tilde{G}_\beta(q'^A), q'^i)) = q'^\alpha$. An invertible change of variables can be made directly in the Lagrangian (1), which leads to an equivalent formulation of the original theory, see Sect. 1.4.2 in [6]. Substituting the expressions (34) into (1), we obtain the Lagrangian of the form $L'(q'^A, \dot{q}'^A) - \lambda_\alpha q'^\alpha$, which implies equations of the following structure:

$$\lambda_\alpha = A_\alpha(q'^A, \dot{q}'^A, \ddot{q}'^A), \tag{35}$$

$$B_i(q'^A, \dot{q}'^A, \ddot{q}'^A) = 0, \quad q'^\alpha = 0. \tag{36}$$

That is, we have a closed system (36) for determining $q'^A(t)$, while $\lambda_\alpha(t)$ then can be found algebraically from (35).

For the latter use, observe that

$$M'_{AB} = \frac{\partial^2 L'}{\partial\dot{q}'^A\partial\dot{q}'^B}, \tag{37}$$

and its inverse are positive-definite matrices together with M_{AB}.

Hamiltonian formulation of the theory (1) on phase space (q^A, p_B). Without loss of generality, we assume that equations of the surface $G_\alpha(q^A) = 0$ can be resolved with respect to the first $n-k$-coordinates. Accordingly, the set q^A is divided into two subgroups, q^α and q^i. Greek indices from the beginning of the alphabet run from 1 to $n-k$, while Latin indices from the middle of the alphabet run from 1 to k. Thus

$$\mathbb{S}^k = \left\{ q^A = (q^\alpha, q^i),\ G_\alpha(q^A) = 0, \quad \det\left.\frac{\partial G_\alpha}{\partial q^\beta}\right|_{\mathbb{S}} = n-k,\ \alpha = 1, 2, \ \ldots, n-k \right\}, \tag{38}$$

and our variational problem is (1). Applying the Dirac method, we introduce conjugate momenta $p_A = \partial L/\partial\dot{q}^A$ and $p_{\lambda\alpha} = \partial L/\partial\dot{\lambda}_\alpha$ for all configuration-space variables q^A and λ_α. Conjugate momenta for λ_α are the primary constraints: $p_{\lambda\alpha} = 0$. Since the Lagrangian L was assumed non-singular, the expressions for p_A can be resolved with respect to velocities:

$$p_A = \frac{\partial L}{\partial\dot{q}^A}, \quad \text{then} \quad \dot{q}^A = v^A(q, p). \tag{39}$$

To find the Hamiltonian, we exclude velocities from the expression $H = p_A\dot{q}^A - (L - \lambda_\alpha G_\alpha) + \varphi_\alpha p_{\lambda_\alpha}$, obtaining

$$H = H_0 + \lambda_\alpha G_\alpha(q^A) + \varphi_\alpha p_{\lambda_\alpha}, \quad \text{where } H_0 \equiv p_A v^A(q, p) - L(q^A, v^B(q, p)). \tag{40}$$

By φ_α we denoted the Lagrangian multipliers for the primary constraints. Preservation in time of the primary constraints, $\dot{p}_{\lambda_\alpha} = \{p_{\lambda_\alpha}, H\} = 0$ implies $G_\alpha = 0$ as the secondary constraints. In turn, the equation $dG_\alpha/dt = \{G_\alpha, H\} = \{G_\alpha, H_0\} = 0$ implies tertiary constraints, which should be satisfied by all true solutions

$$\Phi_\alpha \equiv \{G_\alpha, H_0\} = G_{\alpha B}(q)v^B(q, p) = 0, \quad \text{where } G_{\alpha B} \equiv \frac{\partial G_\alpha(q)}{\partial q^B}. \tag{41}$$

The Lagrangian counterpart of these constraints is $\dot{q}^A\partial_A G_\alpha = 0$, and means that for true trajectories, the velocity vector is tangent to the surface $\mathbb{S}^k$. Calculate

$$\operatorname{rank}\frac{\partial\Phi_\alpha}{\partial p_B} = \operatorname{rank}(G_{\alpha A}\tilde{M}^{AB}) = n-k, \quad \text{where } \tilde{M}^{AB} \equiv \frac{\partial v^A(q, p)}{\partial p_B}. \tag{42}$$

Note that $\tilde{M}^{AB}$ is the inverse of the Hessian matrix M_{AB}. This implies that the constraints Φ_α are functionally independent and can be resolved with respect to some $n-k$ momenta of the set p_A. This also implies that the constraints G_β and Φ_α are functionally independent. Calculating their Poisson brackets, we obtain the matrix

$$\{G_\alpha, \Phi_\beta\} = G_{\beta A}(q)\tilde{M}^{AB}G_{\alpha B} \equiv b_{\alpha\beta}. \tag{43}$$

For our Lagrangian with positive-definite M_{AB}, this matrix is non-degenerate, see Appendix.

For the latter use, we introduce the matrix composed of brackets of the constraints $T_I = (G_\alpha, \Phi_\beta)$

$$\triangle_{IJ} = \begin{pmatrix} 0 & b \\ -b^T & c \end{pmatrix}, \quad \triangle^{-1}_{IJ} = \begin{pmatrix} b^{-1T}cb^{-1} & -b^{-1T} \\ b^{-1} & 0 \end{pmatrix}, \tag{44}$$

where the first block corresponds to $\{G_\alpha, G_\beta\} = 0$, and $c_{\alpha\beta} = \{\Phi_\alpha, \Phi_\beta\}$. As b is invertible, the matrix $\triangle_{IJ}$ is invertible, so our constraints G_β and Φ_α are of second class.

Preservation in time of the tertiary constraints gives fourth-stage constraints that involve λ_α, and can be used to find them through q^A and p_B

$$\lambda_\alpha = b^{-1T}_{\alpha\beta}\{\Phi_\beta, H_0\}. \tag{45}$$

Finally, preservation in time of the fourth-stage constraints gives an equation that algebraically determines the Lagrangian multipliers φ_α through other variables

$$\begin{aligned}\varphi_\alpha = b^{-1}_{\alpha\beta}\big[&\{\{\Phi_\alpha, H_0\}, H_0\} + \lambda_\gamma\{\{\Phi_\alpha, H_0\}, G_\gamma\} \\ &-\{b_{\beta\gamma}, H_0\}\lambda_\gamma - \{b_{\beta\gamma}, G_\sigma\}\lambda_\gamma\lambda_\sigma\big].\end{aligned} \tag{46}$$

In the absence of new constraints, the Dirac procedure is completed.

In summary, we revealed the following chain of constraints

$$\begin{aligned}&p_{\lambda\alpha} = 0, \quad G_\alpha(q) = 0, \quad \Phi_\alpha \equiv G_{\alpha B}(q)v^B(q,p) = 0, \\ &\lambda_\alpha = b^{-1T}_{\alpha\beta}\{\Phi_\beta, H_0\},\end{aligned} \tag{47}$$

and determined the auxiliary variables φ_α. Note that the phase-space variable $p_{\lambda\alpha}$ is just a constant, while λ_α is presented through q^A and p_B. So we only need to write the dynamical equations for q^A and p_B. The variables λ_α can be excluded from the Hamiltonian (40) using the constraint (45). In addition, we can omit the term $\varphi_\alpha p_{\lambda\alpha}$, since it does not contribute to Hamiltonian equations for the phase-space variables q^A, p_B. With the resulting Hamiltonian, the equations read as follows:

$$\begin{aligned}\dot{q}^A &= \{q^A, H_0 + b^{-1T}_{\alpha\beta}\{\Phi_\beta, H_0\}G_\alpha\} \\ &= \{q^A, H_0\} + \{q^A, G_\alpha\}b^{-1T}_{\alpha\beta}\{\Phi_\beta, H_0\}, \\ \dot{p}_A &= \{q_A, H_0 + b^{-1T}_{\alpha\beta}\{\Phi_\beta, H_0\}G_\alpha\} \\ &= \{p_A, H_0\} + \{p_A, G_\alpha\}b^{-1T}_{\alpha\beta}\{\Phi_\beta, H_0\}.\end{aligned} \tag{48}$$

Writing the last equalities, we take into account that $G_\alpha = 0$ for true solutions.

Dirac observed that these equations can be rewritten in terms of a canonical Hamiltonian without auxiliary variables

$$H_0(q^A, p_B) = p_A v^A(q,p) - L(q^A, v^B(q,p)), \tag{49}$$

if, instead of a canonical Poisson bracket, we introduce the famous Dirac bracket. Given two phase-space functions $A(q,p)$ and $B(q,p)$, their Dirac bracket is

$$\{A, B\}_D = \{A, B\} - \{A, T_I\}\triangle^{-1}_{IJ}\{T_J, B\}. \tag{50}$$

This has all the properties of the canonical Poisson bracket, including antisymmetry and the Jacobi identity [27]. In addition, its remarkable property is that $T_I = (G_\alpha, \Phi_\beta)$ represent its Casimir functions, that is, the Dirac bracket of any phase-space function with any constraint T_I vanishes: $\{A, T_I\}_D = 0$. The equations constructed with the help of H_0 and the Dirac bracket

$$\dot{q}^A = \{q^A, H_0\}_D, \quad \dot{p}_A = \{p_A, H_0\}_D, \tag{51}$$

differ from (48) by terms proportional to the constraints, and therefore are equivalent. The final equations (51) do not involve the auxiliary variables and are written on the phase space (q^A, p_B). The Dirac bracket determines the Poisson structure of this space.

Hamiltonian reduction to the intermediate submanifold. Using the Dirac formalism, we obtained $2n + 2(n-k)$ equations of our theory written in $2n$-dimensional phase space with coordinates (q^A, p_B). They are the dynamical equations (51) and the constraints $G_\alpha(q^A) = 0$ and $\Phi_\alpha(q^A, p_B) = 0$. All solutions to our equations lie on a $2k$-dimensional submanifold specified by these algebraic constraints. They could be used to exclude $2(n-k)$ variables from the formalism. However, as we saw above, it may be desirable to work with our theory keeping all q^A. Therefore, we exclude only some of the momenta, reducing our theory to the intermediate submanifold of equations $\Phi_\alpha(q^A, p_B) = 0$. Let

$$p_\alpha = f_\alpha(q^A, p_i), \tag{52}$$

be a solution to the constraints $\Phi_\alpha(q^A, p_B) = 0$. The reduction can be done while at the same time keeping the Hamiltonian character of the resulting equations, that is, we establish the Poisson structure and Hamiltonian for our equations on the intermediate submanifold with the coordinates (q^A, p_i). Because of the property that the constraints are composed of Casimir functions, the reduction consists in eliminating the variables p_α from the formalism as follows.

1. It is known [27] that, together with $\Phi_\alpha = 0$, the functions $p_\alpha - f_\alpha(q^A, p_i)$ also represent Casimir functions of the Dirac bracket, so for any phase-space function $A(q^A, p_B)$, we obtain

$$\{A(q^A, p_B), p_\alpha\}_D = \{A(q^A, p_B), f_\alpha(q^A, p_i)\}_D. \tag{53}$$

As a consequence, computation of the Dirac bracket and substitution (52) are commuting operations:

$$\{A(q^A, p_B), B(q^A, p_B)\}_D \\ = \{A(q^A, p_i, f_\alpha), B(q^A, p_i, f_\alpha)\}_D. \tag{54}$$

2. Using (50) and (52), we define the following brackets on the submanifold (q^A, p_i):

$$\{A(q^A, p_i), B(q^A, p_i)\}' \\ = \{A(q^A, p_i), B(q^A, p_i)\}_D\Big|_{p_\alpha = f_\alpha(q^A, p_i)}. \tag{55}$$

Because of the property (53), the brackets $\{,\}'$ obey the Jacobi identity (for the direct proof, see Sect. 4.2 in [27]), and hence determine the Poisson structure on the submanifold (q^A, p_i).
3. Let us replace p_α on $f_\alpha(q^A, p_i)$ in the Hamiltonian (49), denoting the resulting expression by $H_0'(q^A, p_j)$

$$H_0'(q^A, p_j) \\ = \Big[p_A v^A(q, p) - L(q^A, v^B(q, p))\Big]\Big|_{p_\alpha = f_\alpha(q^A, p_i)} \tag{56}$$

Because of the property (53), H_0 can be used in Eq. (51) instead of H, which will give equivalent Hamiltonian equations. Replacing p_α according to (52) on the r.h.s. of these equations, we obtain equivalent equations with the bracket (55)

$$\dot{q}^A = \{q^A, H_0'(q^B, p_j)\}' \quad \dot{p}_i = \{p_i, H_0'(q^B, p_j)\}'. \tag{57}$$

Together with the algebraic equations $G_\alpha = 0$ and $p_\alpha = f_\alpha(q^A, p_i)$, they are equivalent to the original system composed of (51), $G_\alpha = 0$ and $\Phi_\alpha = 0$. This completes the procedure for the reduction to the intermediate submanifold $\Phi_\alpha = 0$.

4 Intermediate formalism

Here we present a more economical way to construct the Hamiltonian formulation of the theory (1) on the intermediate submanifold, which does not require constructing the Dirac bracket and then reducing it to the submanifold.

To this end, we rewrite the obtained Hamiltonian theory (47), (48) in non-canonical phase-space coordinates with special properties. The matrix $G_{\alpha B}(q^A)$ of Eq. (41) is composed of $(n-k)$ linearly independent vector fields $\mathbf{G}_\alpha(q^A)$ orthogonal to the surface $\mathbb{S}^k$ of the configuration space q^A. Let us consider the linear system $G_{\alpha B} x_B = 0$. It has a general solution[3] of the form $x_B = c^i G_{iB}$, where the linearly independent vectors $\mathbf{G}_i$ are fundamental solutions to this system. They have the following structure:

$$\mathbf{G}_i = (G_{i1}(q), G_{i2}(q), \dots, G_{i,n-k}(q), 0, \dots, 1, 0 \dots, 0), \\ \text{then } G_{\alpha B} G_{iB} = 0. \tag{58}$$

By construction, these vector fields form a basis of tangent space to the surface $\mathbb{S}^k$. Together with $\mathbf{G}_\alpha$, they form a basis of tangent space to the entire configuration space. Using the rows $\mathbf{G}_\beta$ and $\mathbf{G}_j$, we construct an invertible matrix G_{BA}, and use it to define the new momenta π_B of the phase space (q^A, p_B) as follows:

$$G_{BA}(q) = \begin{pmatrix} G_{\beta A} \\ G_{jA} \end{pmatrix}, \quad \pi_B = G_{BA}(q) p_A, \\ \text{then } p_A = G^{-1}_{AB}(q)\pi_B \equiv \tilde{G}_{AB}(q)\pi_B. \tag{59}$$

Let us take q^A and π_B as the new phase-space coordinates. Their special property is that both q^A and π_i have vanishing brackets with the original constraints G_α

$$\{q^A, G_\alpha\} = 0, \quad \{\pi_i, G_\alpha\} = 0, \tag{60}$$

where the latter equality is due to Eq. (58).

Let us rewrite our theory in the new variables. Using the canonical brackets $\{q^A, p_B\} = \delta^A{}_B$, we obtain Poisson brackets of the new variables

$$\{q^A, q^B\} = 0, \quad \{q^A, \pi_B\} = G_{BA}(q), \\ \{\pi_A, \pi_B\} = -c_{AB}{}^D(q)\tilde{G}_{DE}(q)\pi_E, \tag{61}$$

where the Lie brackets of basic vector fields $\mathbf{G}_A$ appear

$$c_{AB}{}^D = [\mathbf{G}_A, \mathbf{G}_B]^D \equiv G_{AE}\partial_E G_{BD} - G_{BE}\partial_E G_{AD}, \tag{62}$$
$$c_{ij}{}^k = 0. \tag{63}$$

Therefore, the Lie bracket of the vector fields $\mathbf{G}_A$ determines the Poisson structure of our theory in the sector π_A. The structure functions $c_{ij}{}^k$ vanish for our choice of basic vectors $\mathbf{G}_i$ of special form, see Eq. (58). In particular, the Poisson brackets of the coordinates q^A and π_i are

$$\{q^A, q^B\} = 0, \quad \{q^A, \pi_i\} = G_{iA}(q), \\ \{\pi_i, \pi_j\} = -c_{ij}{}^\alpha(q)\tilde{G}_{\alpha E}(q)\pi_E. \tag{64}$$

The Hamiltonian (40) reads

$$H = H_0 + \lambda_\alpha G_\alpha(q), \\ H_0 = \tilde{G}_{AC}\pi_C v^A(q, \tilde{G}\pi) - L(q^A, v^B(q, \tilde{G}\pi)). \tag{65}$$

Finally, our second-class constraints in the new coordinates are

$$G_\alpha(q) = 0, \quad \Phi_\alpha \equiv G_{\alpha A}(q) v^A(q, \tilde{G}\pi) = 0. \tag{66}$$

[3] To avoid possible confusion, we point out that in the similar equation (41), representing the tertiary constraints, f^A are given functions of q and p.

Let us confirm that the tertiary constraints Φ_α can be resolved with respect to π_α. To this end we compute the matrix $\partial\Phi_\alpha/\partial\pi_\beta$, and show that its determinant is not zero

$$\det\frac{\partial(G_{\alpha A}v^A(q,\tilde{G}\pi))}{\partial\pi_\beta}=\det[G_{\alpha A}\tilde{M}^{AD}(q,\tilde{G}\pi)\tilde{G}_{D\beta}]\neq 0. \tag{67}$$

It is not zero for our class of positive-definite Lagrangians (8); see Appendix. Resolving the constraints $\Phi_\alpha=0$, say

$$\pi_\alpha=f_\alpha(q^A,\pi_i), \tag{68}$$

we use the resulting expressions to exclude π_α from (64) and (65), thus obtaining

$$\{q^A,q^B\}'=0,\quad \{q^\alpha,\pi_i\}'=G_{i\alpha}(q),\quad \{q^j,\pi_i\}'=\delta^j{}_i,$$
$$\{\pi_i,\pi_j\}'=-c_{ij}{}^\alpha[\tilde{G}_{\alpha k}\pi_k+\tilde{G}_{\alpha\beta}f_\beta(q^A,\pi_i)]. \tag{69}$$

$$H_0'(q^A,\pi_j)=\tilde{G}_{AC}\pi_C v^A(q,\tilde{G}\pi)-L(q^A,v^B(q,\tilde{G}\pi))\Big|_{\pi_\alpha=f_\alpha(q^A,\pi_i)}. \tag{70}$$

In general, the brackets (69) are non-linear for both q^A and π_i. Their dependence on the choice of tangent vector fields $\mathbf{G}_i$ to the surface $\mathbb{S}^k$ is encoded in three places: in the brackets $\{q^\alpha,\pi_i\}'$, in the matrix $\tilde{G}$, and in the structure functions $c_{ij}{}^\alpha$, see Eq. (62).

Using these brackets and the Hamiltonian, let us write the following system of equations:

$$\dot{q}^A=\{q^A,H_0'(q^B,\pi_j)\}',\quad \dot{\pi}_i=\{\pi_i,H_0'(q^B,\pi_j)\}'; \tag{71}$$
$$G_\alpha(q)=0,\quad \pi_\alpha=f_\alpha(q^A,\pi_i). \tag{72}$$

Affirmation. The brackets (69) obey the Jacobi identity and hence determine the Poisson structure on the intermediate submanifold $\Phi_\alpha=0$ equipped with the coordinates (q^A,π_i). In addition, Eqs. (71) and (72) represent an equivalent formulation of the original theory (51), (47).

Proof Using the constraints (66), we construct a Dirac bracket on the phase space (q^A,π_B) as follows:

$$\{A,B\}_D=\{A,B\}-\{A,T_I\}\triangle^{-1}_{IJ}\{T_J,B\}. \tag{73}$$

Here, T_I is the set of all constraints: $T_I=(G_\alpha(q),\Phi_\beta(q,\pi))$. In addition, denoting symbolically the blocks $b=\{G,\Phi\}$ and $c=\{\Phi,\Phi\}$, the matrices $\triangle$ and $\triangle^{-1}$ are

$$\triangle=\begin{pmatrix}0 & b\\ -b^T & c\end{pmatrix},\quad \triangle^{-1}=\begin{pmatrix}b^{-1T}cb^{-1} & -b^{-1T}\\ b^{-1} & 0\end{pmatrix}. \tag{74}$$

The constraint's functions (66) are Casimir functions of the Dirac bracket (73). Similarly to the previous section, as Hamiltonian equations of our theory we can take

$$\dot{q}^A=\{q^A,H\}_D,\quad \dot{\pi}_A=\{\pi_A,H\}_D, \tag{75}$$

with H written in Eq. (65). Equation (74) implies the following structure of the Dirac bracket

$$\{A,B\}_D=\{A,B\}-\{A,G\}\triangle'\{G,B\}+\{A,G\}\triangle''\{\Phi,B\}, \tag{76}$$

that is, the last two terms on the r.h.s. involve at least one constraint G_α. Taking into account Eq. (60), we conclude that in the passage from the Poisson bracket (61) to the Dirac bracket (73), the brackets (64) of basic variables q^A and π_i will not be modified, retaining their original form. Excluding π_α from their r.h.s. using (68), we arrive at the brackets (69). Since $\pi_\alpha-f_\alpha(q^A,\pi_i)$ are Casimir functions of the Dirac bracket (73), the brackets (69) obey the Jacobi identity, see Sect. 4.2 in [27] for the direct proof.

To reduce Eq. (75) to the intermediate submanifold $\Phi_\alpha=0$, we proceed in the same way as in the previous section. First, working with Eq. (75), we can omit the terms with constraints in the Hamiltonian (65), and then use (68) in the resulting expression. This gives the Hamiltonian (70), which therefore can be used instead of H in Eq. (75) for q^A and π_i. Second, excluding π_α from the r.h.s. of these equations using (68), they acquire the form (71). This completes the proof of the affirmation.

Another set of non-canonical variables. Instead of (59), we can equally consider the following non-canonical set q^A, π_B:

$$\pi_\alpha=G_{\alpha A}v^A(q,p)\equiv\Phi_\alpha,\quad \pi_i=G_{iA}p_A. \tag{77}$$

That is, we take the third-stage constraints Φ_α as part of the new momenta. Using the adapted coordinates (33), we conclude that the change (77) is invertible with respect to p_A

$$\frac{\partial\pi_A}{\partial p_B}=\begin{pmatrix}G'_{\alpha A}\tilde{M}'^{A\beta} & G'_{\alpha A}\tilde{M}'^{Aj}\\ G'_{i\alpha} & G'_{ij}\end{pmatrix}=\begin{pmatrix}\tilde{M}'^{\alpha\beta} & \tilde{M}'^{\alpha j}\\ 0 & \delta^{ij}\end{pmatrix}. \tag{78}$$

Here, we used that in adapted coordinates $G'_{\alpha A}=(\delta_{\alpha\beta},\mathbf{0})$ and $\tilde{G}'_{D\beta}=(\delta_{\alpha\beta},\mathbf{0})^T$. As $\tilde{M}'^{AB}$ is a positive-definite matrix (see (37)), we have $\det\tilde{M}'^{\alpha\beta}>0$. Together with (78), this implies $\det(\partial\pi_A/\partial p_B)\neq 0$.

Representing p_A through q^A and π_B, we can rewrite the theory in terms of new variables. Our second-class constraints in the new coordinates are

$$G_\alpha(q)=0,\quad \pi_\alpha=0. \tag{79}$$

Using the canonical brackets $\{q^A,p_B\}=\delta^A{}_B$, we obtain the following Poisson brackets for the variables q^A and π_i

$$\{q^A,q^B\}=0,\quad \{q^\alpha,\pi_i\}=G_{i\alpha}(q),$$
$$\{q^j,\pi_i\}=G_{ij}(q)=\delta_{ij},$$
$$\{\pi_i,\pi_j\}=-c_{ij}{}^D(q)p_D(q^A,\pi_i,\pi_\alpha), \tag{80}$$

where the Lie brackets of basic vector fields $\mathbf{G}_i$ appear

$$c_{ij}{}^D=[\mathbf{G}_i,\mathbf{G}_j]^D\equiv G_{iE}\partial_E G_{jD}-G_{jE}\partial_E G_{iD}. \tag{81}$$

As above, the special property of the new variables is that q^A and π_i have vanishing brackets with the original constraints G_α

$$\{q^A, G_\alpha(q)\} = 0, \quad \{\pi_i, G_\alpha(q)\} = \{G_{iA}(q)p_A, G_\alpha(q)\} = -G_{iA}G_{\alpha A} = 0, \tag{82}$$

the latter equality being due to Eq. (58). For this reason, when we pass to the Dirac bracket, the brackets (80) will not be modified, while the brackets of $\pi_\alpha = \Phi_\alpha$ with any phase-space function vanish. The final Hamiltonian is obtained from (40) disregarding the last two terms and substituting $p_A(q^A, \pi_i, \pi_\alpha = 0)$ into the remaining terms

$$H'_0(q^A, \pi_j) = p_A(q^A, \pi_j, 0)v^A(q^B, p_C(q^A, \pi_j, 0)) - L(q^B, p_C(q^A, \pi_j, 0)). \tag{83}$$

The final brackets are (80), where we substitute $\pi_\alpha = 0$ on the r.h.s. of the last equation. Hamiltonian equations are obtained using the final brackets as follows: $\dot{q}^A = \{q^A, H'_0(q^B, \pi_j)\}'$, $\dot{\pi}_i = \{\pi_i, H'_0(q^B, \pi_j)\}'$.

5 Integration of first-order equations using the Hamiltonian vector field

To apply in practice Liouville's theorem discussed in Sect. 2, we need to find the integrals of motion, then solve the algebraic equations (12); then we need to calculate the integrals given in Eq. (26) (for the rigid body they typically are the elliptic integrals), and finally, solve the algebraic equations (26). In this section we present another possibility for integrating first-order equations in terms of power series with respect to t.

Consider the differential operator acting on the space of functions $f(x)$ and defined by formal series

$$e^{h\partial_x} = 1 + h\partial_x + \frac{1}{2}h\partial_x(h\partial_x) + \cdots = \sum_{n=0}^{\infty}\frac{1}{n!}(h\partial_x)^n, \tag{84}$$

where $h = \text{const}$, and $\partial_x = \frac{\partial}{\partial x}$. This obeys the properties $e^{h\partial_x}x = x + h$, $e^{h\partial_x}f(x) = f(e^{h\partial_x}x)$, as can be verified by expansion in power series of both sides of these equalities. There is a generalization of the last equality for the case of a function $h(x)$. For the latter use we introduce the parameter t. Then

$$e^{th(x)\partial_x}f(x) = f(e^{th(x)\partial_x}x), \quad \text{in particular} \quad e^{th(x)\partial_x}h(x) = h(e^{th(x)\partial_x}x). \tag{85}$$

To prove this,[4] let us consider the following Cauchy problem for partial differential equation

$$\partial_t\varphi(t, x) = h(x)\partial_x\varphi(t, x), \quad \varphi(0, x) = f(x), \tag{86}$$

where $h(x)$ and $f(x)$ are given functions. It is known (see Sect. 60 in [28]) that this problem has unique solution $\varphi(t, x)$. The function $e^{th(x)\partial_x}f(x)$ obeys this problem

$$\begin{aligned}\partial_t[e^{th(x)\partial_x}f(x)] &= \partial_t\left[1 + th\partial_x + \frac{t^2}{2!}(h\partial_x)^2 + \cdots\right]f(x)\\ &= [h\partial_x + t(h\partial_x)^2 + \frac{t^2}{2!}(h\partial_x)^3 + \cdots]f(x)\\ &= h\partial_x[e^{th(x)\partial_x}f(x)].\end{aligned} \tag{87}$$

Denoting $e^{th(x)\partial_x}x \equiv y(x)$, we verify that the function $f(e^{th(x)\partial_x}x)$ also obeys this problem

$$\begin{aligned}\partial_t[f(e^{th(x)\partial_x}x)] &= \partial_y f\big|_{y(x)}\,\partial_t[e^{th(x)\partial_x}x]\\ &= \partial_y f\big|_{y(x)}\,\partial_t\left[1 + th\partial_x + \frac{t^2}{2!}(h\partial_x)^2 + \cdots\right]x\\ &= \partial_y f\big|_{y(x)}\left[h\partial_x + t(h\partial_x)^2 + \frac{t^2}{2!}(h\partial_x)^2 + \cdots\right]x\\ &= \partial_y f\big|_{y(x)}\,h\partial_x\left[1 + t(h\partial_x) + \frac{t^2}{2!}(h\partial_x)^2 + \cdots\right]x\\ &= h\,\partial_y f\big|_{y(x)}\,\partial_x y(x) = h\partial_x[f(y(x))]\\ &= h\partial_x[f(e^{th(x)\partial_x}x)].\end{aligned} \tag{88}$$

Since the solution is unique, the two functions must coincide, which proves the equality (85).

As a consequence, the series $z(t, x) = e^{th(x)\partial_x}x$ turns out to be a general solution to the equation

$$\dot{z} = h(z), \tag{89}$$

with x being the integration constant. Indeed, we have

$$\begin{aligned}\dot{z} &= \frac{d}{dt}[e^{th(x)\partial_x}x] = \partial_t\left[1 + th\partial_x + \frac{t^2}{2!}(h\partial_x)^2 + \cdots\right]x\\ &= \left[h\partial_x + t(h\partial_x)^2 + \frac{t^2}{2!}(h\partial_x)^3 + \cdots\right]x\\ &= h + t(h\partial_x)h + \frac{t^2}{2!}(h\partial_x)^2h + \cdots\\ &= e^{th(x)\partial_x}h(x) = h(e^{th(x)\partial_x}x) = h(z),\end{aligned} \tag{90}$$

where the penultimate equality is due to (85).

This observation is immediately generalized for the case of several variables: the functions

$$z^i(t, z_0^j) = e^{th^k(z_0^j)\frac{\partial}{\partial z_0^k}}z_0^i, \tag{91}$$

provide a general solution to the system

$$\dot{z}^i = h^i(z^j). \tag{92}$$

Any Hamiltonian system $\dot{x}^i = \{x^i, H\}$, $\dot{p}_j = \{p_j, H\}$ has this form. Thus its general solution is

$$x^i(t, x_0^j, p_{0k}) = e^{t\{x_0^k, H(x_0, p_0)\}\frac{\partial}{\partial x_0^k} + t\{p_{0k}, H(x_0, p_0)\}\frac{\partial}{\partial p_{0k}}}x_0^i,$$

[4] This proof was suggested by Andrey Pupasov-Maksimov.

$$p_i(t, x_0^j, p_{0k}) = \mathrm{e}^{t\{x_0^k, H(x_0,p_0)\}\frac{\partial}{\partial x_0^k} + t\{p_{0k}, H(x_0,p_0)\}\frac{\partial}{\partial p_{0k}}} p_{0i}. \tag{93}$$

For a generalization of these formulas to the case of the time-dependent Hamiltonian, see [6].

6 Application of intermediate formalism to a toy model

Here, we illustrate the intermediate formalism on the example of a particle on a sphere, obtaining a non-standard Hamiltonian description of this model in a five-dimensional symplectic manifold.

Consider a point particle with coordinates $x_i(t)$ in three-dimensional Euclidean space, forced to move freely on the sphere $\mathbf{x}^2 = c^2$. It can be described by the Lagrangian action

$$S = \int \mathrm{d}t \; \frac{m}{2}\dot{\mathbf{x}}^2 + \lambda(\mathbf{x}^2 - c^2). \tag{94}$$

In the phase space with canonically conjugated coordinates $(\mathbf{x}, \mathbf{p})$, this action implies two second-class constraints $\mathbf{x}^2 - c^2 = 0$ and $(\mathbf{x}, \mathbf{p}) = 0$. The first is analogous to $G_\alpha = 0$ of the general formalism, while the second is analogous to $\Phi_\alpha = 0$ and determines the five-dimensional intermediate submanifold in the phase space. Then the analogue of $G_{\alpha A}$ is the vector $\frac{1}{2}\mathbf{grad}(\mathbf{x}^2 - c^2) = (x_1, x_2, x_3)$. Assuming that we work in the local coordinate chart with $x_3 \neq 0$ fundamental solutions to the equation $(\mathbf{x}, \mathbf{z}) = 0$ are $(1, 0, -\frac{x_1}{x_3})$ and $(0, 1, -\frac{x_2}{x_3})$. The change of variables (59) reads

$$\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = \begin{pmatrix} 1, & 0 & -\frac{x_1}{x_3} \\ 0 & 1 & -\frac{x_2}{x_3} \\ x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}. \tag{95}$$

We obtain

$$\pi_1 = p_1 - \frac{x_1}{x_3}p_3, \quad \pi_2 = p_2 - \frac{x_2}{x_3}p_3, \quad \pi_3 = (\mathbf{x}, \mathbf{p}). \tag{96}$$

Hence, in the new coordinates $\mathbf{x}$ and $\boldsymbol{\pi}$, the intermediate submanifold is just the hyperplane $\pi_3 = 0$. The inverse transformation to (96) is

$$\begin{aligned} p_1 &= \pi_1 - \frac{x_1}{\mathbf{x}^2}\left[x_1\pi_1 + x_2\pi_2 - \pi_3\right], \\ p_2 &= \pi_2 - \frac{x_2}{\mathbf{x}^2}\left[x_1\pi_1 + x_2\pi_2 - \pi_3\right], \\ p_3 &= -\frac{x_3}{\mathbf{x}^2}\left[x_1\pi_1 + x_2\pi_2 - \pi_3\right]. \end{aligned} \tag{97}$$

The next step is to rewrite the canonical Poisson brackets $\{x_i, p_j\} = \delta_{ij}$ and Hamiltonian $H = \frac{1}{2m}\mathbf{p}^2$ in terms of new coordinates, and then substitute $\pi_3 = 0$ in all resulting expressions. Using the expressions (95) and the canonical brackets, we obtain the following non vanishing brackets for the coordinates $(x_1, x_2, x_3, \pi_1, \pi_2)$ of the intermediate submanifold

$$\begin{aligned} &\{x_1, \pi_1\} = 1, \quad \{x_2, \pi_2\} = 1, \quad \{x_3, \pi_1\} = -\frac{x_1}{x_3}, \\ &\{x_3, \pi_2\} = -\frac{x_2}{x_3}. \end{aligned} \tag{98}$$

They do not involve π_3, so they already give the Poisson structure of intermediate manifold. Using Eq. (97) in the canonical Hamiltonian, and then setting $\pi_3 = 0$, we obtain the Hamiltonian reduced to the intermediate submanifold

$$H = \frac{1}{2m}\left\{\pi_1^2 + \pi_2^2 - \frac{(x_1\pi_1 + x_2\pi_2)}{\mathbf{x}^2}\right\}. \tag{99}$$

Equations (98) and (99) represent a Hamiltonian system on a five-dimensional symplectic manifold foliated by the leaves $\mathbf{x}^2 = c^2$, $c \in \mathbb{R}$. The quantity $\mathbf{x}^2$ is a Casimir function of the Poisson structure (98). Thus, any trajectory that passes through a point of the symplectic leaf $\mathbf{x}^2 = c^2$ with given c, lies entirely in this leaf.

7 Rotating asymmetric body in the intermediate formalism

Here, we apply the intermediate formalism to a spinning body. We show that Euler–Poisson equations turn out to be a Hamiltonian system on the intermediate submanifold and deduce the Poisson geometry (112) that lies behind these equations.

Motions of a spinning body can be described [11,12] starting from the Lagrangian action of the form (1)

$$S = \int \mathrm{d}t \; \frac{1}{2}g_{ij}\dot{R}_{ki}\dot{R}_{kj} - \frac{1}{2}\lambda_{ij}\left[R_{ki}R_{kj} - \delta_{ij}\right], \tag{100}$$

where $R^T R - \mathbf{1}$ play the role of G_α of the general formalism. The action is written in the laboratory system with the origin chosen at the centre of mass of the body. $R_{ij}(t)$ is a 3×3 matrix. Its nine elements are the dynamical degrees of freedom which, at the end, describe the rotational motions of the body. The numerical symmetric matrix g_{ij} encodes the distribution of the mass of the body at the initial instant

$$g_{ij} \equiv \sum_{N=1}^{n} m_N x_N^i(0) x_N^j(0), \tag{101}$$

where m_N are masses of the body's particles with position vectors $\mathbf{x}_N(t)$. The mass matrix and inertia tensor are related as follows: $I_{ij} = g_{kk}\delta_{ij} - g_{ij}$. Choosing laboratory axes at $t = 0$ in the directions of axes of inertia, the two tensors acquire a diagonal form, $g_{ij} = g_i\delta_{ij}$, $I_{ij} = I_i\delta_{ij}$. For a non-planar body, g_i are positive numbers [12], so the Hessian matrix of the theory (100) is evidently positive-defined. Therefore, we can apply the intermediate formalism developed in Sects. 3 and 4.

Introducing the conjugate momenta for all dynamical variables $p_{ij} = \partial L/\partial \dot{R}_{ij}$ and $p_{\lambda ij} = \partial L/\partial \dot{\lambda}_{ij}$, we obtain the expression for p_{ij} in terms of velocities

$$p_{ij} = \dot{R}_{ik} g_{kj}, \quad \text{then } \dot{R}_{ij} = p_{ik} g^{-1}_{kj}, \tag{102}$$

and the primary constraints $p_{\lambda ij} = 0$. To construct the final Hamiltonian of the intermediate formalism, we will need only the canonical part $H_0 = p_{ij}\dot{R}_{ij}(p) - L(\dot{R}_{ij}(p))$ of the complete Hamiltonian (40). For the present case, its explicit form is

$$H = \frac{1}{2} g^{-1}_{ij} p_{ki} p_{kj}. \tag{103}$$

The non-vanishing Poisson brackets of canonical variables are as follows (there is no summation over i and j): $\{R_{ij}, p_{ij}\} = 1$, $\{\lambda_{ij}, p_{\lambda ij}\} = 1$. Next, the explicit form of tertiary constraints (41) in our case is

$$\{R_{ki}R_{kj}, H_0\} = \left[R^T p g^{-1} + (R^T p g^{-1})^T\right]_{ij} \equiv \left[R^T R R^{-1} p g^{-1} + (R^T R R^{-1} p g^{-1})^T\right]_{ij} = 0. \tag{104}$$

The surface determined by equations $R^T R = \mathbf{1}$ and (104) is equally determined by $6 + 6$ equations

$$R^T R = \mathbf{1}, \tag{105}$$

$$\Phi_{ij} \equiv \left[R^{-1} p g^{-1} + (R^{-1} p g^{-1})^T\right]_{ij} = 0. \tag{106}$$

We take these Φ_{ij} as analogues of the constraints (41) of the general formalism.

According to the intermediate formalism, we now need to find non-canonical momenta with two properties. First, $9 - 6 = 3$ of them should have vanishing Poisson brackets with the orthogonality constraint, see Eq. (60). Second, the constraints (106) can be used to represent other momenta through these three, see Eq. (68). To achieve this, consider the phase-space functions

$$\mathbb{P}_{ij} \equiv 2(R^{-1}p)_{ij}, \quad \text{then } p_{ij} = \frac{1}{2}(R\mathbb{P})_{ij}. \tag{107}$$

They are constructed from p_{ij} with the use of an invertible matrix, so the transition $(R_{ij}, p_{ij}) \to (R_{ij}, \mathbb{P}_{ij})$ is a change of variables on the phase space. We emphasize that R_{ij} in the action (100) is an arbitrary (not orthogonal!) matrix.

We decompose $\mathbb{P}_{ij}$ on symmetric and antisymmetric parts, $\mathbb{P}_{ij} = S_{ij} - \hat{M}_{ij}$, where $S = R^{-1}p + (R^{-1}p)^T$ and $\hat{M} = R^{-1}p - (R^{-1}p)^T$, and then replace the antisymmetric matrix $\hat{M}$ on an equivalent vector[5]: $\hat{M}_{ij} = \epsilon_{ijk} M_k$, $M_k \equiv \frac{1}{2}\epsilon_{kij}\hat{M}_{ij} = -\epsilon_{kij}(R^{-1}p)_{ij}$. Therefore, the final form of the decomposition is

$$\mathbb{P}_{ij} = S_{ij} - \epsilon_{ijk} M_k, \quad \text{where} \quad S_{ij} = \left[R^{-1}p + (R^{-1}p)^T\right]_{ij}, \quad M_k = -\epsilon_{kij}(R^{-1}p)_{ij}. \tag{108}$$

Accordingly, we consider the following change of variables:

$$(R_{ij},\ p_{ij}) \to (R_{ij},\ S_{ij}, i \le j,\ M_k). \tag{109}$$

The coordinates M_k have the desired properties: their brackets with orthogonality constraint vanish: $\{M_k, R_{pi}R_{pj} - \delta_{ij}\} = 0$, and the variables S_{ij} can be presented through M_k, resolving (106) as follows (there is no summation on i and j in this expression):

$$S_{ij} = \frac{g_i - g_j}{g_i + g_j}\epsilon_{ijk} M_k = \frac{I_j - I_i}{I_k}\epsilon_{ijk} M_k. \tag{110}$$

Therefore, the change of variables (109) is analogous to the change (59) of the general formalism. To obtain the last equality, we used the following relations among elements of diagonal mass matrix and inertia tensor [12]:

$$2g_1 = I_2 + I_3 - I_1, \quad 2g_2 = I_1 + I_3 - I_2, \quad 2g_3 = I_1 + I_2 - I_3, \quad I_1 = g_2 + g_3, \quad I_2 = g_1 + g_3, \quad I_3 = g_1 + g_2, \quad g_i - g_j = I_j - I_i. \tag{111}$$

Computing the canonical Poisson brackets of the new variables R_{ij}, M_k and S_{ij}, we obtain

$$\{R_{ij}, R_{ab}\} = 0, \quad \{M_i, M_j\} = -\epsilon_{ijk}((R^T R)^{-1}\mathbf{M})_k, \quad \{M_i, R_{jk}\} = -\epsilon_{ikm} R^{-1T}_{jm}; \tag{112}$$

$$\begin{aligned} &\{R_{ij}, S_{ab}\} = R^{-1T}_{ia}\delta_{jb} + R^{-1T}_{ib}\delta_{ja}, \\ &\{M_k, S_{ab}\} = -2M_k(R^T R)^{-1}_{ab} + \delta_{ka}((R^T R)^{-1}\mathbf{M})_b + \delta_{kb}((R^T R)^{-1}\mathbf{M})_a, \\ &\{S_{ij}, S_{ab}\} = -(R^T R)^{-1}_{ia}\epsilon_{jbn} M_n - (R^T R)^{-1}_{jb}\epsilon_{jan} M_n + (a \leftrightarrow b). \end{aligned} \tag{113}$$

According to Sect. 4, to reduce our theory on the submanifold $\Phi_{ij} = 0$, it is sufficient to rewrite it in the variables R_{ij}, M_k, S_{ij}, and then, using Eq. (110), to exclude from all resulting expressions the variables S_{ij}. The brackets (112) do not involve S_{ij}, so they already give a Poisson structure of intermediate submanifold $\Phi_{ij} = 0$. Using Eqs. (107) and (108) in the canonical Hamiltonian (103), the latter can be written as follows:

$$H_0 = \frac{1}{8} g^{-1}_{ij}(S_{ai} - \epsilon_{aik} M_k)(S_{aj} - \epsilon_{ajp} M_p) + \frac{1}{8} g^{-1}_{ij}(R^T R - \mathbf{1})_{ab}(S_{ai} - \epsilon_{aik} M_k)(S_{bj} - \epsilon_{bjp} M_p). \tag{114}$$

The second term is proportional to the orthogonality constraint. Therefore, it does not contribute to the Hamiltonian equations for the variables R_{ij} and M_k, and hence it can be omitted. The remaining term can be written as follows:

[5] The phase-space functions M_k, being rewritten back in terms of R_{ij} and $\dot{R}_{ij}$, are just the components of angular momentum in the body-fixed frame [12]: $M_k = R^{-1}_{ki} m_i$, $m_i = \sum_N m_N[\mathbf{x}_N(t), \dot{\mathbf{x}}_N(t)]_i$.

$$H_0 = \frac{1}{8}\sum_j \frac{1}{g_j}(S_{ij} - \epsilon_{ijk}M_k)^2. \tag{115}$$

Using the relations (111), for any chosen $i \neq j$ we obtain $\frac{1}{g_j}(S_{ij} - \epsilon_{ijk}M_k)^2 = \frac{4g_j}{I_k^2}M_k^2$. Using this in (115), we obtain the final form of our Hamiltonian on the intermediate submanifold

$$H_0 = \frac{1}{2}I_{ij}^{-1}M_iM_j. \tag{116}$$

Note that the final expression, which is composed of tensors and vectors, is invariant under rotations. Hence, the Hamiltonian will be of this form in any laboratory system. If the laboratory frame was not adapted with the axes of inertia at the initial instant, the inertia tensor in this expression would be a numerical symmetric matrix with non-vanishing off-diagonal elements.

Using this H_0 with the brackets (112), the Hamiltonian equations $\dot{z} = \{z, H_0\}$ read as follows: $\dot{R}_{ij} = -\epsilon_{jkm}(I^{-1}M)_kR_{im}$, $\dot{\mathbf{M}} = [\mathbf{M}, I^{-1}\mathbf{M}]$. Introducing the phase-space quantity $\Omega_i = I_{ij}^{-1}M_j$, they acquire the standard form of Euler–Poisson equations:

$$\dot{R}_{ij} = -\epsilon_{jkm}\Omega_kR_{im}, \quad I\dot{\mathbf{\Omega}} = [I\mathbf{\Omega}, \mathbf{\Omega}]. \tag{117}$$

By this, we have completed Hamiltonian reduction on the intermediate submanifold (106), showing that Euler–Poisson equations are the Hamiltonian system on this submanifold, with the Poisson structure given by the brackets (112).

The Chetaev bracket is the Dirac bracket. Using the orthogonality constraint on the r.h.s. of the brackets (112), we obtain simpler expressions

$$\{R_{ij}, R_{ab}\} = 0, \quad \{M_i, M_j\} = -\epsilon_{ijk}M_k,$$
$$\{M_i, R_{jk}\} = -\epsilon_{ikm}R_{jm}. \tag{118}$$

By direct computation, it can be verified that they still satisfy the Jacobi identity and lead to the same equations (117). They were suggested by Chetaev [8] as the possible Poisson structure corresponding to the Euler–Poisson equations.

General solution to the Euler–Poisson equations and the motions of a rigid body. Not all solutions to Eq. (117) describe the motions of a spinning body. By construction [11, 12], they should be solved with the universal initial conditions

$$R_{ij}(0) = \delta_{ij}, \quad \Omega_i(0) = \Omega_{0i}, \tag{119}$$

where Ω_{0i} is the initial angular velocity measured in the body-fixed frame. That is, only those trajectories that at some instant of time pass through the unit of $SO(3)$-group can describe possible motions of the body.[6] Let us denote the r.h.s. of Eq. (117) as follows: $H_{ij}(R, \mathbf{\Omega})$ and $H_k(\mathbf{\Omega})$. Then, according to Eq. (91), we can write for their general solution

$$R_{ij}(t, R_{0kp}, \Omega_{0k}) = \mathrm{e}^{tH_{kp}(R_0,\mathbf{\Omega}_0)\frac{\partial}{\partial R_{0kp}} + tH_k(\mathbf{\Omega}_0)\frac{\partial}{\partial \Omega_{0k}}}R_{0ij}. \tag{120}$$

After applying the differential operator in the exponential, R_{0kp} should be replaced on δ_{kp} in each term of the obtained power series. The resulting function $R_{ij}(t, \Omega_{0k})$ will represent the motion of a spinning body, that at $t = 0$ has its inertia axes parallel to the laboratory axes, and the initial angular velocity equal to Ω_{0i}.

8 Conclusion

The most economical Hamiltonian formulation of the theory (1), in which we are interested in knowing the dynamics of all variables q^A, is achieved on the intermediate submanifold of phase space determined by the constraints (41) (or, equivalently, by (68)). We have described and discussed two methods of Hamiltonian reduction to this submanifold. The final result of the reduction using the Dirac bracket is written out in Eqs. (55)–(57). The intermediate formalism gives Eqs. (69)–(72). As we have shown in the last section, the intermediate formalism leads directly to the Euler–Poisson equations of a spinning body.

To further compare the two reductions, let us denote coordinates (q^A, p_B) of the original phase space by z^i, and coordinates (q^A, p_j) of the intermediate submanifold by z^a. The Poisson tensor of the original space is ω^{ij}, the Dirac tensor is ω_D^{ij}, and the Poisson tensor induced on the intermediate submanifold is $\tilde{\omega}^{ab}$. Generally, in the process of reduction using the Dirac bracket, $\omega^{ij} \to \omega_D^{ij} \to \tilde{\omega}^{ab}$, we have $\omega^{ab} \neq \omega_D^{ab} \neq \tilde{\omega}^{ab}$. An alternative possibility, developed in this work, is as follows: In the theory (1) with a positive-definite Lagrangian L, there are phase-space coordinates $z'^i = (q^A, \pi_B)$ such that in the process of reduction $\omega'^{ij} \to \omega_D'^{ij} \to \tilde{\omega}'^{ab}$ we have $\omega'^{ab} = \omega_D'^{ab} \neq \tilde{\omega}'^{ab}$. In view of this, the reduction consists in excluding the redundant momenta (see Eq. (68)) from the block ω'^{ab} of original tensor ω'^{ij}.

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[6] Misunderstanding of this point leads to considerable confusion, see [11,12] and references therein.

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Appendix

Here we enumerate some properties of a positive-definite matrix.

A symmetric real-valued $n \times n$-matrix with the elements M_{ij} is called a positive-definite ($M \succ 0$) if for any non-zero column $\mathbf{Y}$ we have $\mathbf{Y}^T M \mathbf{Y} > 0$. The following affirmations turn out to be equivalent [29]:

1A. $M \succ 0$.

1B. There exists an $n \times n$ positive-definite matrix B such that $M = B^2 \equiv B^T B$.

1C. All principal minors of M are positive numbers. In particular, det $M > 0$.

1D. M is the Gram matrix of some set of p-dimensional linearly independent vectors, say $\mathbf{Z}_i$. That is, $M_{ij} = (\mathbf{Z}_i, \mathbf{Z}_j)$. If Z_{Ai}, $A = 1, 2, \ldots p$ is the matrix formed by the columns $\mathbf{Z}_i$, and we can write $M_{ij} = (Z^T)_{iA} Z_{Aj}$.

1E. All eigenvalues of M are positive numbers.

In addition, there are the following properties:

2A. Diagonal elements of the positive-definite matrix are positive numbers: $M_{ii} > 0$ for any i. Then trace $M > 0$.

2B. The positive-definite matrix is invertible, and its inverse is a positive-definite matrix.

Affirmation. Let rank $Q_{Ai} = k$, where $A = 1, 2, \ldots p$, $i = 1, 2, \ldots k$, $k < p$, and M_{AB} is positive-definite. Then the matrix

$$N_{ij} = (Q^T)_{iA} M_{AB} Q_{Bj}, \tag{121}$$

is non-degenerate, det $N \neq 0$.

Proof Using **1B**, we write $M = B^T B$; then

$$N_{ij} = (BQ)^T_{iA} (BQ)_{Aj}, \tag{122}$$

where, according to **2B**, the matrix B is nondegenerate. Since the columns of Q_{Aj} are linearly independent, the matrix $(BQ)_{Aj}$ is also composed of linearly independent columns. Then (122) means that M_{ij} is the Gram matrix. According to **1D**, it is positive-definite. In particular, det $N > 0$.

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