**THE EUROPEAN
PHYSICAL JOURNAL C**

Regular Article - Theoretical Physics

Pulsating strings in I-brane background

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Abstract We study pulsating strings both on a single stack of NS5-branes and two orthogonal stacks of NS5-branes (the so called I-brane) by using the Polyakov form of the fundamental string action. For the I-brane background, by using a symmetry that decouples the two spheres from the flat geometry, we study pulsating solutions of the string when it pulsates on both the spheres independently and simultaneously. We finally derive the energy of pulsating strings as a function of adiabatic invariant oscillation number in these cases and studied some limiting cases in detail.

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1 Introduction

After its appearance AdS/CFT correspondence [1,2] has turned out to be one of the most powerful tools of modern theoretical physics. The most successful version of the correspondence relies on the duality between type IIB string states in $AdS_5 \times S^5$ and certain operators in a four dimensional Conformal Field Theory (CFT) living on the boundary of AdS_5. But even for the very well studied $\mathcal{N} = 4$ Supersymmetric Yang–Mills (SYM) theory in four dimensions and dual type IIB superstring in the compactified AdS_5 space, finding an exact matching between string states and dual operators from both sides is very difficult. One can check the correspondence beyond the supergravity approximation by studying various classical string solutions in different backgrounds and using the dispersion relation of such strings in the large charge limit, one can look for boundary operators dual to them.

Over the years, a large variety of rotating and spinning strings has been studied in backgrounds arising from the $AdS_5 \times S^5$ string sigma model. This include well known solutions like folded spinning strings [3], spiky strings [4] and giant magnons [5] and their dual gauge theories have been analysed in great details. Pulsating strings were first introduced in [6] and are expected to be dual to highly excited sigma model operators. For example the most general pulsating string in S^5 charged under the isometry group $SO(6)$ will have a dual operator of the form $\mathrm{Tr}(X^{J_1} Y^{J_2} Z^{J_3})$, where X, Y and Z are the chiral scalars and J_i's are the R-charges from the SYM theory. Pulsating string solutions have better stability than non-pulsating ones [7] and they are time-dependent as opposed to the usual rigidly rotating string solution. Pulsating strings have been thoroughly generalized in [8–13], and have been studied in a number of backgrounds having varying degrees of supersymmetry [14–17]. A string rotating and at the same time oscillating have been derived in [18], and the generalization of this was done with extra angular momenta in [19]. Various other developments and explo-

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rations in different backgrounds have been done by different authors [20–28].

In the present paper we will study pulsating string solutions in NS5-brane and I-brane background. In string theory NS5-branes are interesting on their own right as in the near horizon limit the theory on the worldvolume correspond to the little string theory (LST) [29,30]. LST is a nonlocal field theory which has not been understood properly until now, hence it would be a good exercise to analyze the solution in various limits and if possible find out some operators in dual field theory. It has been found that in the near horizon limit the world sheet theory of NS5-brane is exactly solvable, so from the bulk theory point of view the theory is integrable. But, only a very little is known about the boundary theory, hence from that perspective it is rather hard to make definite statements about the exact nature of the theory. We will also study the pulsating string solutions in the I-brane background which arises from the 1 + 1 dimensional intersection of two orthogonal stacks of NS5-branes, with one set of branes lying along $(x^0, x^1, \ldots, x^5)$, and other set lying along $(x^0, x^1, x^6, \ldots, x^9)$ directions [31]. In the near horizon geometry when all five branes are coincident the S-dual picture is given by

$$R^{2,1} \times R_\phi \times SU(2)_{k_1} \times SU(2)_{k_2}$$

where R_ϕ is one combination of the radial directions away from the two sets of NS5-branes, and $R^{2,1}$ (whose coordinates are x^0, x^1) are another combination of two radial directions. The two $SU(2)$s with levels $k_{1,2}$ describe the angular three-spheres corresponding to $(R^4)_{2345}$ and $(R^4)_{6789}$. As mentioned in [31] this background exhibits a higher Poincaré symmetry, ISO(2,1), than the expected ISO(1,1) and twice as many supercharges one might expect.

Various aspects and semiclassical strings on NS5-brane and I-brane have been studied by probing the geometry with both fundamental strings and D1-strings in [32–49]. Very recently we have studied rigidly rotating string in I-brane using Polyakov action of string [49] and we find Polyakov action of string completely decouple the two spheres. This allows us to study the rotating string on both the spheres simultaneously and we find two sets of giant magnon relations for a particular set of values of the parameters while two sets of single spike relations for all other set of values of the parameters. To see whether this type of behaviour also exists for pulsating string solutions, we probe fundamental string in I-brane background with Polyakov action of string. As I-brane arises from two orthogonal stacks of NS5-branes we are expecting that we will obtain two independent class of pulsating string solutions simultaneously on both the spheres. To compare with various relations we also studied pulsating string solutions in NS5-brane background using the same action of strings. The rest of the paper is organized as follows. In Sect. 2, we will briefly review the near-horizon geometry

of NS5-brane and solve the equations of motion for the pulsating strings which are consistent with Virasoro constraints. We also solve for the string profile and find the oscillation numbers along with the energy-oscillation number relation. In Sect. 3, after reviewing the I-brane background briefly we solve the string equations of motion and show that in this case the string can pulsate independently and simultaneously on both the spheres. We also find the string profiles, oscillation numbers and energy-oscillation number relation when energy is equally and unequally distributed among the spheres of I-brane. In Sect. 4 we will summarize our results and comment on further investigations.

2 Pulsating strings in NS5-brane

The classical solution of N NS5-brane is given by the following form of metric, NS-NS two form field and dilaton,

$$
\begin{aligned}
ds^2 &= -dt^2 + \sum_{i=5}^{9} dx_i^2 + H(r)(dr^2 + r^2(d\theta^2 \\
&\quad + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2)), \\
B_{\phi\psi} &= 2N\sin^2\theta d\phi \wedge d\psi, \quad e^{2(\Phi-\Phi_0)} = H(r), \\
&\quad H(r) = 1 + \frac{Nl_s^2}{r^2},
\end{aligned}
\tag{2.1}
$$

where x^i, $i = 5, \ldots, 9$ labels the world-volume directions of NS5-brane, $H(r)$ is the Harmonic function in the transverse space of the NS5-branes and l_s is the string length. In the near horizon limit, $r \to 0$, one can ignore 1 in the $H(r)$, and the solution would look like,

$$
\begin{aligned}
ds^2 &= -dt^2 + \sum_{i=5}^{9} dx_i^2 + Nl_s^2\Big(\frac{dr^2}{r^2} + d\theta^2 \\
&\quad + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2\Big), \\
b_{\phi\psi} &= 2N\sin^2\theta, \quad e^{2(\Phi-\Phi_0)} = \frac{Nl_s^2}{r^2},
\end{aligned}
\tag{2.2}
$$

where $B_{\phi\psi} = b_{\phi\psi}d\phi \wedge d\psi$. To proceed further we will rescale $t \to \sqrt{N}l_s t$, $x_i \to \sqrt{N}l_s x_i$ and introduce the variable ρ that is related to r as,

$$
\rho = \ln\left(\frac{r}{\sqrt{N}l_s}\right),
\tag{2.3}
$$

and setting $l_s = 1$. Under all this operation the final form of the metric and NS-NS two form field becomes,

$$
\begin{aligned}
ds^2 &= N\Big(-dt^2 + \sum_{i=5}^{9} dx_i^2 + d\rho^2 + d\theta^2 + \sin^2\theta d\phi^2 \\
&\quad + \cos^2\theta d\psi^2\Big), \, b_{\phi\psi} = 2N\sin^2\theta.
\end{aligned}
\tag{2.4}
$$

In this section we will restrict to the background metric and NS-NS two form field given by (2.4).

2.1 String equations of motion

To study a fundamental string coupled to NS-NS B-field, we use the Polyakov action with a WZ term,

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \left[\sqrt{-\gamma} \gamma^{\alpha\beta} g_{MN} \partial_\alpha X^M \partial_\beta X^N \right.$$
$$\left. - \epsilon^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N b_{MN} \right], \tag{2.5}$$

where λ is the 't Hooft coupling, $\gamma^{\alpha\beta}$ is the worldsheet metric and $\epsilon^{\alpha\beta}$ is the antisymmetric tensor defined as $\epsilon^{\tau\sigma} = -\epsilon^{\sigma\tau} = 1$.

Variation of the action with respect to X^M gives us the following equations of motion

$$2\partial_\alpha(\eta^{\alpha\beta}\partial_\beta X^N g_{KN}) - \eta^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N \partial_K g_{MN}$$
$$- 2\partial_\alpha(\epsilon^{\alpha\beta}\partial_\beta X^N b_{KN})$$
$$+ \epsilon^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N \partial_K b_{MN} = 0, \tag{2.6}$$

and variation with respect to the metric gives the two Virasoro constraints,

$$g_{MN}(\partial_\tau X^M \partial_\tau X^N + \partial_\sigma X^M \partial_\sigma X^N) = 0, \tag{2.7}$$
$$g_{MN}(\partial_\tau X^M \partial_\sigma X^N) = 0. \tag{2.8}$$

We use the conformal gauge (i.e. $\sqrt{-\gamma}\gamma^{\alpha\beta} = \eta^{\alpha\beta}$) with $\eta^{\tau\tau} = -1$, $\eta^{\sigma\sigma} = 1$ and $\eta^{\tau\sigma} = \eta^{\sigma\tau} = 0$) to solve the equations of motion.

For studying a generic class of pulsating strings we use the ansatz,

$$t = t(\tau) = \kappa\tau, \quad x_i = v_i\sigma, \quad \rho = \mu\sigma, \quad \theta = \theta(\tau),$$
$$\phi = m\sigma, \quad \psi = \psi(\tau). \tag{2.9}$$

Solving t and ψ equations we obtain,

$$\dot{t} = \kappa,$$
$$\dot{\psi} = \frac{c_1 + 2m\sin^2\theta}{\cos^2\theta}. \tag{2.10}$$

The equations for x_i, ρ, and ϕ satisfies trivially. The other non-trivial contribution comes from θ equation, which gives,

$$\ddot{\theta} = \left[3m^2 - \frac{(c_1 + 2m)^2}{\cos^4\theta} \right] \sin\theta\cos\theta. \tag{2.11}$$

Integrating the above equation we get,

$$\dot{\theta}^2 = c_2 + 3m^2\sin^2\theta - \frac{(c_1 + 2m)^2}{\cos^2\theta}, \tag{2.12}$$

where $c_{1,2}$ are the integrating constants.

Now, the Virasoro constraint $g_{MN}(\partial_\tau X^M \partial_\tau X^N + \partial_\sigma X^M \partial_\sigma X^N) = 0$, gives

$$\dot{\theta}^2 = \alpha^2 + 4m^2 + 4c_1m + 3m^2\sin^2\theta - \frac{(c_1 + 2m)^2}{\cos^2\theta}, \tag{2.13}$$

where $\alpha^2 = \kappa^2 - \mu^2 - \sum v_i^2$. The other Virasoro $g_{MN}\partial_\tau X^M \partial_\sigma X^N = 0$ is trivially satisfied. Comparing the Virasoro constraint (2.13) with the equation of motion (2.12), we get the following relation between various constants,

$$c_2 = \alpha^2 + 4m^2 + 4c_1m. \tag{2.14}$$

To determine the conserved charges associated to the string motion, we start from the full form of the sigma model action in background (2.4),

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \left[\dot{t}^2 + \sum_{i=5}^9 v_i^2 \right.$$
$$\left. + \mu^2 - \dot{\theta}^2 + m^2\sin^2\theta - \dot{\psi}^2\cos^2\theta + 4m\dot{\psi}\sin^2\theta \right]. \tag{2.15}$$

From this action we can easily determine the conserved charges using the Noether procedure, and they can be written as,

$$E = -\int \frac{\partial\mathcal{L}}{\partial\dot{t}} d\sigma = \sqrt{\lambda}\kappa,$$
$$J = \int \frac{\partial\mathcal{L}}{\partial\dot{\psi}} d\sigma = \sqrt{\lambda}c_1. \tag{2.16}$$

Here E is the energy and J is the angular momenta on the sphere. Rescaling them as,

$$\mathcal{E} = \frac{E}{\sqrt{\lambda}} = \kappa, \quad \mathcal{J} = \frac{J}{\sqrt{\lambda}} = c_1. \tag{2.17}$$

Now expressing κ and c_1 in terms of these conserved charges and for simplicity using $\sum v_i^2 = 1$, we can express the constraint equation (2.14) as,

$$c_2 = \mathcal{E}^2 - 1 - \mu^2 + 4m^2 + 4m\mathcal{J}. \tag{2.18}$$

2.2 String profile

From (2.12) we find as θ varies from 0 to $\pi/2$, $\dot{\theta}^2$ varies from $\mathcal{E}^2 - 1 - \mu^2 - \mathcal{J}^2$ to infinity. This looks like the equation of motion of a particle moving in an effective potential $V(\theta)$, where θ varies from a minimal to a maximal value. Note that the θ equation can be written in the form,

$$\ddot{x} + 3m^2[-(R_- + R_+)x + 2x^3] = 0, \tag{2.19}$$

where $x = \sin\theta$. Equation (2.19) can be compared with Duffing oscillator equation ($\ddot{x} + \gamma\dot{x} + \omega_0^2 x + \epsilon x^3 = f_0\cos(\omega t)$)

without damping and driving force if $(R_- + R_+)$ is negative. Integrating (2.19), we get,

$$\dot{x}^2 = 3m^2 x^2 (1 - x^2) + c_2 (1 - x^2) - (\mathcal{J} + 2m)^2$$
$$= 3m^2 (x^2 - R_-)(R_+ - x^2),$$ (2.20)

where

$$R_\pm = \frac{-(c_2 - 3m^2) \pm \sqrt{(c_2 + 3m^2)^2 - 12m^2(\mathcal{J} + 2m)^2}}{6m^2}.$$
  (2.21)

With proper scaling of the variables, we can write the solution of (2.20) in terms of standard Jacobi elliptic function,[1] provided the initial condition $x(0) = 0$:

$$\sin\theta(\tau) = \sqrt{\frac{-R_+ R_-}{R_+ - R_-}} sd\left(\sqrt{3(R_+ - R_-)}m\tau, \sqrt{\frac{R_+}{R_+ - R_-}}\right)$$
  (2.22)

Using the property of Jacobi functions that $sd(z|m) = sd(z + 4\mathbb{K}(m)|m)$ and as usual taking only the real period, we can find the condition for time-periodic solution for θ is,

$$0 < \frac{R_+}{R_+ - R_-} < 1.$$ (2.23)

This translate to the following inequality

$$c_2 > (\mathcal{J} + 2m)^2$$ (2.24)

which gives a constraint on the conserved charges so that the string has a pulsating motion. Using this inequality in (2.18) we get,

$$\mathcal{E}^2 - 1 - \mu^2 - \mathcal{J}^2 > 0.$$ (2.25)

This condition (2.25) is in tune with our earlier observation about the limits of the oscillation in the $\dot{\theta}^2$ equations.

Now we will present some of the string profiles under variation of one particular parameter while keeping the other fixed. In these plots worldsheet time τ is along vertical direction and worldsheet coordinate σ with closed boundary condition is along the horizontal direction. A slice along vertical direction represent a closed circular string at a particular worldsheet time. These plots showing with worldsheet time the circular string grows from zero size to a maximum extent and then again contracts to zero size in a periodic fashion. Figure 1 shows the string profile in NS5-brane for different values of winding number (m) while energy $(\mathcal{E})$, angular momenta $(\mathcal{J})$ and other parameter (μ) remaining constant. These figures clearly indicate that with the increase of m, number of lobes increases within a certain value of worldsheet time τ. From these plots we can guess the number of lobes $= (m + 1)/2$. We can also see with the increase in the

number of lobes the amplitude decreases for a given value of energy. To show another effect of m we presented the topview of Fig. 1 in Fig. 2. These plots clearly showing the string folds m times under closed boundary condition, that is why m is called the winding number. Figure 3 shows the plot of string profile with different values of energies while keeping angular momenta, winding number and the parameter (μ) fixed. These plots clearly indicate the amplitude of the string profile increases with the increase of energy. We can also note that although we keep $m = 3$ fixed but the number of lobes increases with energy within a certain value of the worldsheet time. Hence, number of lobes depends on both the energy and the winding number m. String profiles are not sensitive to angular momenta and other parameter μ unless they take such values that the inequality (2.25) breaks down, this leads to unphysical structure as the string profile breaks down as shown in Fig. 4.

We can also find the dynamics of the string along the ψ direction by integrating $\frac{d\psi}{d\theta}$ which have the form,

$$\frac{d\psi}{d\theta} = \frac{\mathcal{J} + 2m \sin^2\theta}{\sqrt{3}m \cos\theta \sqrt{(\sin^2\theta - R_-)(R_+ - \sin^2\theta)}}.$$ (2.26)

These can be integrated to find ψ in terms of standard elliptic integrals,

$$\psi(\tau) = \frac{-1}{\sqrt{-3R_-}} \left[\frac{\mathcal{J} + 2m}{m} \Pi\left(R_+, \arcsin\left(\frac{\sin\theta(\tau)}{\sqrt{R_+}}\right), \frac{R_+}{R_-}\right) \right.$$
$$\left. + 2F\left(\arcsin\left(\frac{\sin\theta(\tau)}{\sqrt{R_+}}\right), \frac{R_+}{R_-}\right) \right],$$ (2.27)

where $F(\varphi, k)$ and $\Pi(n, \varphi, k)$ are the incomplete elliptic integrals of first kind and third kind respectively.

2.3 Oscillation number

Now we will use Bohr–Sommerfeld like quantization procedure for the pulsating strings in this background. The oscillation numbers can be written using the canonical momenta conjugate to θ as follows,

$$N = \sqrt{\lambda}\mathcal{N} = \frac{\sqrt{\lambda}}{2\pi} \oint d\theta \, \Pi_\theta$$
$$= \frac{\sqrt{\lambda}}{2\pi} \oint d\theta \sqrt{c_2 + 3m^2 \sin^2\theta - \frac{(\mathcal{J} + 2m)^2}{\cos^2\theta}}.$$ (2.28)

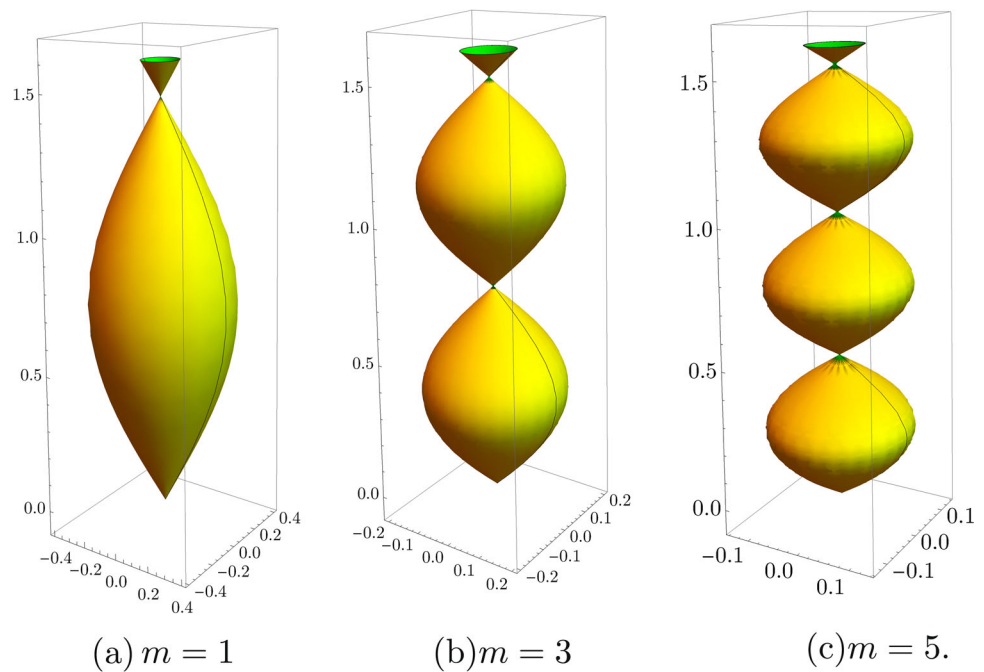Taking $\sin\theta = x$ we can choose the proper limits and transform the above integral to,

$$\mathcal{N} = \frac{2\sqrt{3}m}{\pi} \int_0^{\sqrt{R_+}} \frac{\sqrt{(x^2 - R_-)(R_+ - x^2)}}{1 - x^2} dx.$$ (2.29)
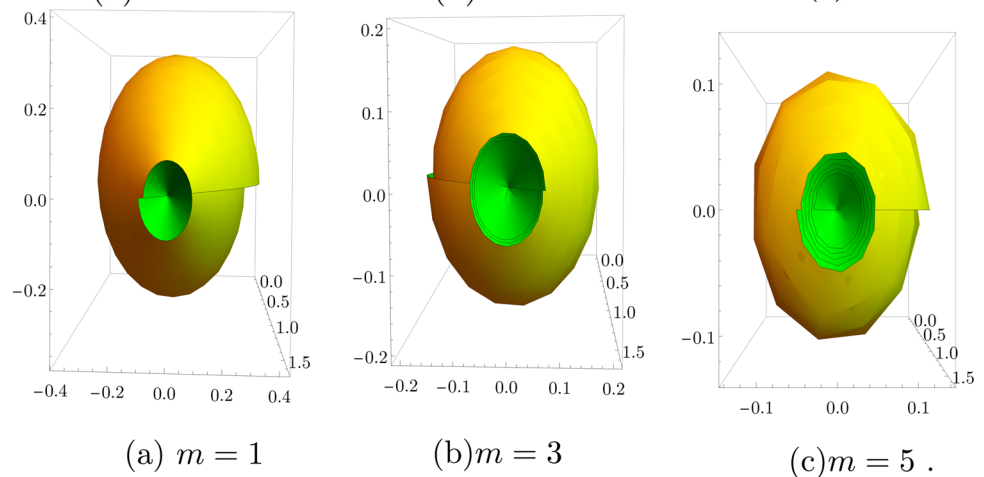
We can directly compute the integral to find,

$$\mathcal{N} = \frac{2\sqrt{3}m}{\pi\sqrt{-R_-}} \left[(1 - R_+) K\left(\frac{R_+}{R_-}\right) - R_- E\left(\frac{R_+}{R_-}\right) \right]$$

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[1] In the notation we follow $sd(z|m)$ is the solution of $w''(z)^2 + w(z)(2m(1 - m)w^2(z) - 2m + 1) = 0$.

Fig. 1 Pulsating string profiles with $\mathcal{E} = 1.7$ and **a** $m = 1$, **b** $m = 3$, **c** $m = 5$

(a) $m = 1$ (b)$m = 3$ (c)$m = 5.$

Fig. 2 Pulsating string profiles topview $\mathcal{E} = 1.7$, **a** $m = 1$, **b** $m = 3$, **c** $m = 5$

(a) $m = 1$ (b)$m = 3$ (c)$m = 5$.

$$+(R_- - 1)(R_+ - 1)\,\Pi\left(R_+ \Big| \frac{R_+}{R_-}\right)\bigg], \qquad (2.30)$$

where $K(k)$, $E(k)$ and $\Pi(n, k)$ are standard complete elliptic integrals of first, second and third kind respectively. Instead of working with this, we can made the expression a little simpler by taking the partial derivative of $\mathcal{N}$ with respect to m,

$$\begin{aligned}
\frac{\partial \mathcal{N}}{\partial m} &= \frac{2\sqrt{3}}{\pi} \int_0^{\sqrt{R_+}} \frac{x^2 dx}{\sqrt{(x^2 - R_-)(R_+ - x^2)}} \\
&\quad - \frac{4(\mathcal{J} + 2m)}{\sqrt{3}\pi m} \int_0^{\sqrt{R_+}} \frac{dx}{(1 - x^2)\sqrt{(x^2 - R_-)(R_+ - x^2)}} \\
&= \frac{2\sqrt{-3R_-}}{\pi}\left[K\left(\frac{R_+}{R_-}\right) - E\left(\frac{R_+}{R_-}\right)\right] \\
&\quad - \frac{4(\mathcal{J} + 2m)}{\sqrt{-3R_-}\pi m}\left[\Pi\left(R_+ \Big| \frac{R_+}{R_-}\right)\right]. \qquad (2.31)
\end{aligned}$$

In the short string limit, i.e. when both the energy and angular momentum of the string are small, we can expand the above expression in $\mathcal{E}$ and $\mathcal{J}$ to get the oscillation number

$$\mathcal{N} = \mathcal{A}(\mathcal{J}) + \mathcal{E}^2 \mathcal{B}(\mathcal{J}) + \mathcal{O}(\mathcal{E}^4). \qquad (2.32)$$

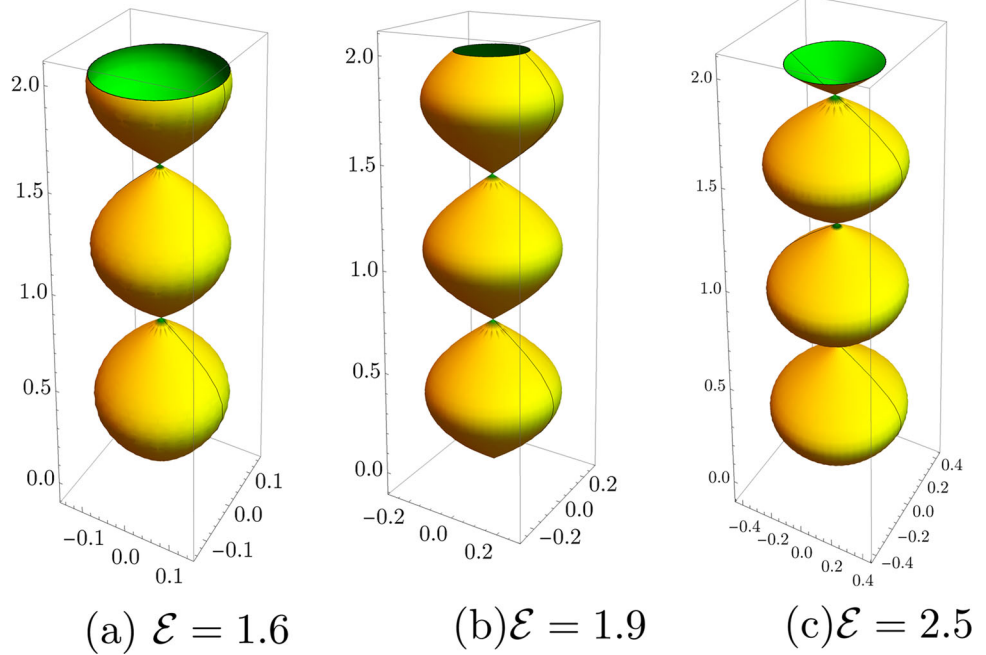Inverting (2.32) we can write the energy-oscillation number relation as,

$$\mathcal{E} = [\mathcal{B}(\mathcal{J})]^{-1/2}\sqrt{\mathcal{N} - \mathcal{A}(\mathcal{J})} + \mathcal{O}(\mathcal{N} - \mathcal{A}(\mathcal{J}))^{3/2} \quad (2.33)$$

where,

$$\begin{aligned}
\mathcal{A}(\mathcal{J}) &= \left(4m - \frac{15(\mu^2 + 1)}{2m} - \frac{159\mu^2}{4m^3}\right) \\
&\quad - \left(6\log m - \frac{153(\mu^2 + 1)}{4m^2} - \frac{33435\mu^2}{64m^4}\right)\mathcal{J} \\
&\quad - \left(\frac{51}{2m} + \frac{213(\mu^2 + 1)}{m^3} + \frac{81465\mu^2}{16m^5}\right)\mathcal{J}^2
\end{aligned}$$

Fig. 3 Pulsating string profiles with $m = 3$ and **a** $\mathcal{E} = 1.6$, **b** $\mathcal{E} = 1.9$, **c** $\mathcal{E} = 2.5$

(a) $\mathcal{E} = 1.6$

(b) $\mathcal{E} = 1.9$

(c) $\mathcal{E} = 2.5$

$$+ \left(\frac{267}{4m^2} + \frac{81015(\mu^2 + 1)}{64m^4} + \frac{1422975\mu^2}{32m^6} \right) \mathcal{J}^3$$
$$+ \mathcal{O}(\mathcal{J}^4), \tag{2.34}$$

and

$$\mathcal{B}(\mathcal{J}) = \left(\frac{15}{2m} + \frac{159\left(\mu^2 + 1\right)}{4m^3} + \frac{10995\mu^2}{16m^5} \right)$$
$$- \left(\frac{153}{4m^2} + \frac{33435\left(\mu^2 + 1\right)}{64m^4} + \frac{911955\mu^2}{64m^6} \right) \mathcal{J}$$
$$+ \left(\frac{213}{m^3} + \frac{81465\left(\mu^2 + 1\right)}{16m^5} + \frac{10899675\mu^2}{56m^7} \right) \mathcal{J}^2$$
$$- \left(\frac{81015}{64m^4} + \frac{1422975\left(\mu^2 + 1\right)}{32m^6} \right.$$
$$\left. + \frac{283301175\mu^2}{128m^8} \right) \mathcal{J}^3 + \mathcal{O}(\mathcal{J}^4). \tag{2.35}$$

We will compare these relations (2.34), (2.35) along with the relation (2.33) with the corresponding relations of I-brane in the next Sect. 3.

3 Pulsating strings in I-brane

The geometry of I-brane arises when k_1 number of NS5-branes lying along $(0, 1, \ldots, 5)$ intersect k_2 number of NS5-branes lying along $(0, 1, 6, \ldots, 9)$ directions in $(0,1)$-plane. If the branes are coincident, then the type IIB supergravity solution is given by the following metric, three form NS-NS

fields and dilaton [31], as,

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + H_1(y) \sum_{\alpha=2}^{5} (dy^\alpha)^2$$
$$+ H_2(z) \sum_{p=6}^{9} (dz^p)^2,$$
$$H_{\alpha\beta\gamma} = -\epsilon_{\alpha\beta\gamma\delta} \partial^\delta H_1(y),$$
$$H_{mnp} = -\epsilon_{mnpq} \partial^q H_2(z), \quad e^{2\Phi} = H_1(y) H_2(z), \tag{3.1}$$

where the harmonic functions are $H_1(y) = 1 + \frac{k_1 l_s^2}{y^2}$ and $H_2(z) = 1 + \frac{k_2 l_s^2}{z^2}$ with $y = \sqrt{\sum_{\alpha=2}^{5} (y^\alpha)^2}$ and $z = \sqrt{\sum_{p=6}^{9} (z^p)^2}$. In the near horizon limit ($\frac{k_1 l_s^2}{y^2} \gg 1$ and $\frac{k_2 l_s^2}{z^2} \gg 1$), the metric and the NS-NS two form fields can be written as,

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + k_1 l_s^2 \frac{dr_1^2}{r_1^2} + k_1 l_s^2 d\Omega_1^2$$
$$+ k_2 l_s^2 \frac{dr_2^2}{r_2^2} + k_2 l_s^2 d\Omega_2^2,$$
$$B_{\phi_1\psi_1} = 2k_1 l_s^2 \sin^2 \theta_1 d\phi_1 \wedge d\psi_1,$$
$$B_{\phi_2\psi_2} = 2k_2 l_s^2 \sin^2 \theta_2 d\phi_2 \wedge d\psi_2, \tag{3.2}$$

where the three spheres $d\Omega_1$, $d\Omega_2$ are the volume elements on the sphere along $(y^2, \ldots, y^5)$ and $(z^6, \ldots, z^9)$ directions respectively, and are given by,

$$d\Omega_1^2 = d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 + \cos^2 \theta_1 d\psi_1^2,$$
$$d\Omega_2^2 = d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2 + \cos^2 \theta_2 d\psi_2^2.$$

To proceed further we choose $l_s = 1$, $k_1 = k_2 = N$, rescale $x^0 \rightarrow \sqrt{N}t$, $x^1 \rightarrow \sqrt{N}x$ and make the following change of variables

$$\rho_1 = \ln \frac{r_1}{\sqrt{N}}, \quad \rho_2 = \ln \frac{r_2}{\sqrt{N}}. \tag{3.3}$$

The final form of the metric and the background NS-NS field are given by,

$$\begin{aligned}
ds^2 = N(&-dt^2 + dx^2 + d\rho_1^2 + d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 \\
&+ \cos^2\theta_1 d\psi_1^2 + d\rho_2^2 + d\theta_2^2 \\
&+ \sin^2\theta_2 d\phi_2^2 + \cos^2\theta_2 d\psi_2^2), \\
b_{\phi_1\psi_1} = &2N\sin^2\theta_1, \quad b_{\phi_2\psi_2} = 2N\sin^2\theta_2.
\end{aligned} \tag{3.4}$$

3.1 String equations of motion

For studying a generic class of pulsating strings in I-brane background (3.4) using the Polyakov action of string as before, we use the following ansatz,

$$\begin{aligned}
&t = t(\tau) = \kappa\tau, \quad x = \sigma, \quad \rho_1 = \mu_1\sigma, \quad \rho_2 = \mu_2\sigma, \\
&\theta_1 = \theta_1(\tau), \\
&\theta_2 = \theta_2(\tau), \quad \phi_1 = m_1\sigma, \quad \phi_2 = m_2\sigma, \quad \psi_1 = \psi_1(\tau), \\
&\psi_2 = \psi_2(\tau).
\end{aligned} \tag{3.5}$$

Solving t, ψ_1 and ψ_2 equations we obtain,

$$\begin{aligned}
&\dot{t} = \kappa, \\
&\dot{\psi}_1 = \frac{c_1 + 2m_1\sin^2\theta_1}{\cos^2\theta_1}, \\
&\dot{\psi}_2 = \frac{c_2 + 2m_2\sin^2\theta_2}{\cos^2\theta_2}.
\end{aligned} \tag{3.6}$$

The equations for x, $\rho_{1,2}$ and $\phi_{1,2}$ are satisfied trivially. The other non-trivial contributions come from $\theta_{1,2}$ equations, solving them we get,

$$\ddot{\theta}_1 = \left[3m_1^2 - \frac{(c_1 + 2m_1)^2}{\cos^4\theta_1}\right]\sin\theta_1\cos\theta_1, \tag{3.7}$$

$$\ddot{\theta}_2 = \left[3m_2^2 - \frac{(c_2 + 2m_2)^2}{\cos^4\theta_2}\right]\sin\theta_2\cos\theta_2. \tag{3.8}$$
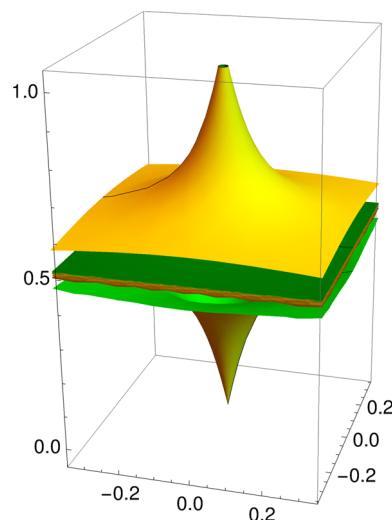
Integrating the above equations, we get,

$$\dot{\theta}_1^2 = c_3 + 3m_1^2\sin^2\theta_1 - \frac{(c_1 + 2m_1)^2}{\cos^2\theta_1}, \tag{3.9}$$

$$\dot{\theta}_2^2 = c_4 + 3m_2^2\sin^2\theta_2 - \frac{(c_2 + 2m_2)^2}{\cos^2\theta_2}, \tag{3.10}$$

where $c_{1,2,3,4}$ are constants.

Now, the Virasoro constraint $g_{MN}(\partial_\tau X^M \partial_\tau X^N + \partial_\sigma X^M \partial_\sigma X^N) = 0$, gives

$$\begin{aligned}
\dot{\theta}_1^2 + \dot{\theta}_2^2 = &\alpha^2 + 4m_1^2 + 4m_2^2 + 4c_1m_1 + 4c_2m_2 \\
&+ 3m_1^2\sin^2\theta_1 + 3m_2^2\sin^2\theta_2
\end{aligned}$$

Fig. 4 Plot shows unphysical structure when the inequality condition is not obeyed by the solution

$$-\frac{(c_1 + 2m_1)^2}{\cos^2\theta_1} - \frac{(c_2 + 2m_2)^2}{\cos^2\theta_2}, \tag{3.11}$$

where $\alpha^2 = \kappa^2 - 1 - \mu_1^2 - \mu_2^2$. The other Virasoro $g_{MN}\partial_\tau X^M \partial_\sigma X^N = 0$ is trivially satisfied. Adding (3.9) and (3.10) we get,

$$\begin{aligned}
\dot{\theta}_1^2 + \dot{\theta}_2^2 = &c_3 + c_4 + 3m_1^2\sin^2\theta_1 + 3m_2^2\sin^2\theta_2 \\
&- \frac{(c_1 + 2m_1)^2}{\cos^2\theta_1} - \frac{(c_2 + 2m_2)^2}{\cos^2\theta_2},
\end{aligned} \tag{3.12}$$

Comparing the Virasoro constraint (3.11) with the equation of motion (3.12), we get the following relation between various constants,

$$c_3 + c_4 = \alpha^2 + 4m_1^2 + 4m_2^2 + 4c_1m_1 + 4c_2m_2. \tag{3.13}$$

To determine the conserved charges associated to the string motion, we write down the full form of the sigma model action in background (3.4),

$$\begin{aligned}
S = -\frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma \Big[&\dot{t}^2 + 1 + \mu_1^2 + \mu_2^2 - \dot{\theta}_1^2 \\
&-\dot{\theta}_2^2 + m_1^2\sin^2\theta_1 + m_2^2\sin^2\theta_2 \\
&-\dot{\psi}_1^2\cos^2\theta_1 - \dot{\psi}_2^2\cos^2\theta_2 \\
&+4m_1\dot{\psi}_1\sin^2\theta_1 + 4m_2\dot{\psi}_2\sin^2\theta_2\Big].
\end{aligned} \tag{3.14}$$

From this action we can easily determine the conserved energy (E) and angular momenta (J_1) and (J_2) of the two orthogonal spheres, as following,

$$E = -\int \frac{\partial\mathcal{L}}{\partial\dot{t}}d\sigma = \sqrt{\lambda}\kappa,$$

$$J_1 = \int \frac{\partial\mathcal{L}}{\partial\dot{\psi}_1}d\sigma = \sqrt{\lambda}c_1,$$

$$J_2 = \int \frac{\partial \mathcal{L}}{\partial \dot{\psi}_2} \, d\sigma = \sqrt{\lambda} c_2. \tag{3.15}$$

As before, rescaling them as,

$$\mathcal{E} = \frac{E}{\sqrt{\lambda}} = \kappa, \quad \mathcal{J}_1 = \frac{J_1}{\sqrt{\lambda}} = c_1, \quad \mathcal{J}_2 = \frac{J_2}{\sqrt{\lambda}} = c_2. \tag{3.16}$$

Now expressing κ, c_1 and c_2 in terms of these conserved charges we can express the constraint equation (3.13) as,

$$c_3 + c_4 = \mathcal{E}^2 - 1 - \mu_1^2 - \mu_2^2 + 4m_1^2 + 4m_2^2 + 4m_1 \mathcal{J}_1$$
$$+ 4m_2 \mathcal{J}_2. \tag{3.17}$$

As we have only the condition (3.17), this is not sufficient to separate c_3 and c_4 in terms of the conserved charges. But, to plot the string profile and to expand the oscillation numbers in terms of $\mathcal{E}$ we need to separate c_3 and c_4. In the following, we will separate them by hand using some arguments on the energy distributions among the spheres of I-brane.

From the form of the constraint (2.18) of NS5-brane and from the ansatz (3.5) of I-brane, we can easily guess $\mu_1, m_1, \mathcal{J}_1$ belongs to the sphere $d\Omega_1$ while $\mu_2, m_2, \mathcal{J}_2$ belongs to the other sphere $d\Omega_2$ of I-brane. First we will distribute the remaining quantity $(\mathcal{E}^2 - 1)$ equally to both the spheres and we will call this situation as the energy is equally distributed among the spheres of I-brane. So, for equal energy distribution case we separate c_3 and c_4 as,

$$c_3 = \frac{\mathcal{E}^2 - 1}{2} - \mu_1^2 + 4m_1^2 + 4m_1 \mathcal{J}_1,$$
$$c_4 = \frac{\mathcal{E}^2 - 1}{2} - \mu_2^2 + 4m_2^2 + 4m_2 \mathcal{J}_2, \tag{3.18}$$

such that the constraint (3.17) is satisfied.

Next we will distribute energy unequally among the spheres of I-brane by considering a factor f $(0 < f < 1)$. In this case we will distribute f of the total energy to the sphere $d\Omega_1$ and rest of it to the sphere $d\Omega_2$, hence, we can separate c_3 and c_4 respecting the constraint (3.17) as,

$$c_3 = f(\mathcal{E}^2 - 1) - \mu_1^2 + 4m_1^2 + 4m_1 \mathcal{J}_1,$$
$$c_4 = (1 - f)(\mathcal{E}^2 - 1) - \mu_2^2 + 4m_2^2 + 4m_2 \mathcal{J}_2. \tag{3.19}$$

3.2 String profile

From (3.9) we find as θ_1 varies from 0 to $\pi/2$, $\dot{\theta}_1^2$ varies from $c_3 - (\mathcal{J}_1 + 2m_1)^2$ to infinity which looks like the equations of motion of a particle moving in an effective potential $V(\theta_1)$, where θ_1 rotates between a minimal and a maximal value. Note that the θ_1 equation can be written as,

$$\ddot{x}_1 + 3m_1^2[-(R_{1-} + R_{1+})x_1 + 2x_1^3] = 0, \tag{3.20}$$

where $x_1 = \sin\theta_1$.

Again from (3.10) we find that $\dot{\theta}_2^2$ varies from $c_4 - (\mathcal{J}_2 + 2m_2)^2$ to infinity as θ_2 varies from 0 to $\pi/2$. This again looks like the particle moving in another effective potential $V(\theta_2)$,

and θ_2 rotates between a minimal and a maximal value. Using $x_2 = \sin\theta_2$, θ_2 equation can be written as,

$$\ddot{x}_2 + 3m_2^2[-(R_{2-} + R_{2+})x_2 + 2x_2^3] = 0. \tag{3.21}$$

Thus, if both $(R_{1-} + R_{1+})$ and $(R_{2-} + R_{2+})$ are negative quantities, then we get two independent Duffing oscillator equations without damping and deriving terms. Integrating (3.20) and (3.21) we get,

$$\dot{x}_1^2 = 3m_1^2(x_1^2 - R_{1-})(R_{1+} - x_1^2), \tag{3.22}$$
$$\dot{x}_2^2 = 3m_2^2(x_2^2 - R_{2-})(R_{2+} - x_2^2), \tag{3.23}$$

where

$$R_{1\pm} = \frac{-(c_3 - 3m_1^2) \pm \sqrt{(c_3 + 3m_1^2)^2 - 12m_1^2(\mathcal{J}_1 + 2m_1)^2}}{6m_1^2},$$

$$R_{2\pm} = \frac{-(c_4 - 3m_2^2) \pm \sqrt{(c_4 + 3m_2^2)^2 - 12m_2^2(\mathcal{J}_2 + 2m_2)^2}}{6m_2^2}. \tag{3.24}$$

On further integrating (3.22) and (3.23), we can express the solutions in terms of Jacobi functions provided the initial conditions $x_1(0) = x_2(0) = 0$:

$$\sin\theta_1(\tau) = \sqrt{\frac{-R_{1+}R_{1-}}{R_{1+} - R_{1-}}} \, sd\left(\sqrt{3(R_{1+} - R_{1-})}m_1\tau, \right.$$
$$\left. \sqrt{\frac{R_{1+}}{R_{1+} - R_{1-}}}\right) \tag{3.25}$$

$$\sin\theta_2(\tau) = \sqrt{\frac{-R_{2+}R_{2-}}{R_{2+} - R_{2-}}} \, sd\left(\sqrt{3(R_{2+} - R_{2-})}m_2\tau, \right.$$
$$\left. \sqrt{\frac{R_{2+}}{R_{2+} - R_{2-}}}\right). \tag{3.26}$$

Thus in this case we obtain two copies of string profiles arising from the two spheres of I-brane. These profiles are independent of each other and can exist simultaneously. We also note that each profile is exactly equivalent to the profile (2.22) obtained for the NS5-brane.

Again, using the property of Jacobi functions and taking only the real period, we find the condition for time-periodic solution for θ_1 and θ_2 is,

$$0 < \frac{R_{1+}}{R_{1+} - R_{1-}} < 1, \quad 0 < \frac{R_{2+}}{R_{2+} - R_{2-}} < 1. \tag{3.27}$$

This translates to the following inequality,

$$\mathcal{E}^2 - 1 - \mu_1^2 - \mu_2^2 - \mathcal{J}_1^2 - \mathcal{J}_2^2 > 0, \tag{3.28}$$

which is the constraint on the conserved charges so that the string has pulsating motion. Hence, we obtained the pulsating string profile which can pulsate on both the spheres independently and simultaneously. The condition (3.28) is also in tune with limits of $\dot{\theta}_1^2$ and $\dot{\theta}_2^2$ of (3.9) and (3.10).
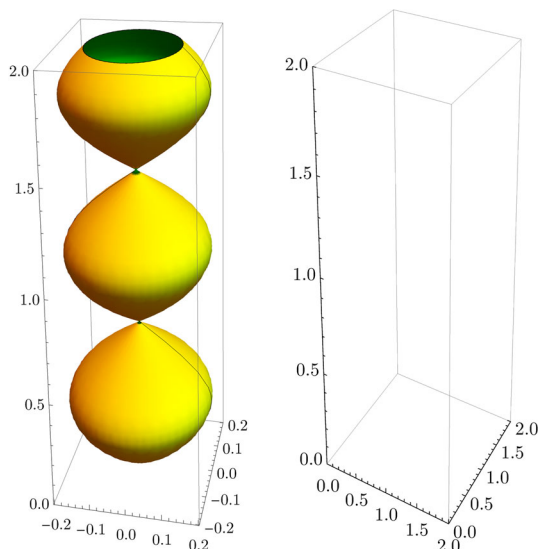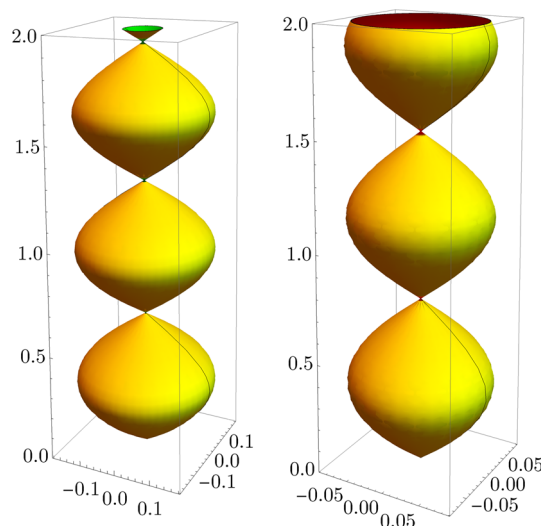
Fig. 5 Pulsating string profile on I-brane for $f = 1$ with $\mathcal{E} = 2$, $m_1 = m_2 = 3$, $\mathcal{J}_1 = \mathcal{J}_2 = 0.05$, $\mu_1 = \mu_2 = 1$
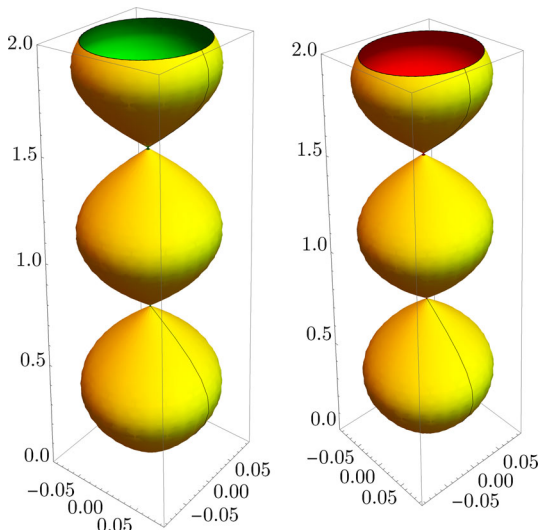
Fig. 6 Pulsating string profile for $f = 0.5$ with $\mathcal{E} = 2$, $m_1 = m_2 = 3$, $\mathcal{J}_1 = \mathcal{J}_2 = 0.5$, $\mu_1 = \mu_2 = 1$

Fig. 7 Pulsating string profile for $f = 0.7$ with $\mathcal{E} = 2.5$, $m_1 = m_2 = 3$, $\mathcal{J}_1 = \mathcal{J}_2 = 0.5$, $\mu_1 = \mu_2 = 1$

Figure 5 shows the string profile of I-brane background for $f = 1$, in this case the energy is fully distributed to the sphere $d\Omega_1$ and there is no energy available to the other sphere $d\Omega_2$. In this case, the string have the pulsating motion only on the sphere $d\Omega_1$ and as there is no energy available to the other sphere $d\Omega_2$, string don't have any pulsating motion there although we have used non-zero values of angular momenta and winding numbers. Instead, if we had used $f = 0$ then we would see the string to pulsate only on the sphere $d\Omega_2$. Thus, if we switch off the energy in one of the sphere of I-brane, then the string profile behave similar to that of NS5-brane on the other sphere. In Fig. 6 we presented the string profile for $f = 0.5$, in this case energy is equally distributed among

the spheres and we see the string have pulsating motion on both the spheres simultaneously. We can also note for equal energy distribution case as the winding numbers m_1 and m_2 are equal, the string is pulsating simultaneously on both the sphere with same amplitude. Instead, if we had used $m_1 \neq m_2$, then we would expect string to pulsate with different amplitude by forming unequal number of lobes as we saw in case of NS5-branes. Figure 7 shows string profile for $f = 0.7$, here 70 percent of energy is distributed to the sphere $d\Omega_1$ and 30 percent to the sphere $d\Omega_2$. We see with more energy the string on the sphere $d\Omega_1$ pulsate with larger amplitude, simultaneously the string is pulsating with smaller amplitude on the other sphere $d\Omega_2$ as less energy is available there. We can also see that first sphere have larger number of lobes compared to the second sphere for same extent of worldsheet time instead of same value of winding numbers $m_1 = m_2$.

We can also find the dynamics of the string along the ψ_1 and ψ_2 direction by integrating $\frac{d\psi_1}{d\theta_1}$ and $\frac{d\psi_2}{d\theta_2}$ which have the form,

$$
\frac{d\psi_1}{d\theta_1} = \frac{\mathcal{J}_1 + 2m_1 \sin^2 \theta_1}{\sqrt{3} m_1 \cos \theta_1 \sqrt{(\sin^2 \theta_1 - R_{1-})(R_{1+} - \sin^2 \theta_1)}},
$$

$$
\frac{d\psi_2}{d\theta_2} = \frac{\mathcal{J}_2 + 2m_2 \sin^2 \theta_2}{\sqrt{3} m_2 \cos \theta_2 \sqrt{(\sin^2 \theta_2 - R_{2-})(R_{2+} - \sin^2 \theta_2)}}. \quad (3.29)
$$

These can be integrated to find ψ_1 and ψ_2 in terms of standard elliptic integrals,

$$
\psi_1(\tau) = \frac{-1}{\sqrt{-3R_{1-}}} \left[\frac{\mathcal{J}_1 + 2m_1}{m_1} \Pi\left(R_{1+}, \arcsin\left(\frac{\sin \theta_1(\tau)}{\sqrt{R_{1+}}} \right), \frac{R_{1+}}{R_{1-}} \right) \right.
$$
$$
\left. + 2F\left(\arcsin\left(\frac{\sin \theta_1(\tau)}{\sqrt{R_{1+}}} \right), \frac{R_{1+}}{R_{1-}} \right) \right], \quad (3.30)
$$

$$\psi_2(\tau) = \frac{-1}{\sqrt{-3R_{2-}}}\left[\frac{\mathcal{J}_2 + 2m_2}{m_2}\Pi\left(R_{2+}, \arcsin\left(\frac{\sin\theta_2(\tau)}{\sqrt{R_{2+}}}\right), \frac{R_{2+}}{R_{2-}}\right)\right.$$
$$\left. + 2F\left(\arcsin\left(\frac{\sin\theta_2(\tau)}{\sqrt{R_{2+}}}\right), \frac{R_{2+}}{R_{2-}}\right)\right]. \tag{3.31}$$

3.3 Oscillation number

As we did in the previous section here also we will use Bohr–Sommerfeld like quantization procedure and write down the oscillation numbers using the canonical momenta conjugate to θ_1 and θ_2 as follows,

$$N_1 = \sqrt{\lambda}\mathcal{N}_1 = \frac{\sqrt{\lambda}}{2\pi}\oint d\theta_1\Pi_{\theta_1}$$
$$= \frac{\sqrt{\lambda}}{2\pi}\oint d\theta_1\sqrt{c_3 + 3m_1^2\sin^2\theta_1 - \frac{(\mathcal{J}_1 + 2m_1)^2}{\cos^2\theta_1}}, \tag{3.32}$$

$$N_2 = \sqrt{\lambda}\mathcal{N}_2 = \frac{\sqrt{\lambda}}{2\pi}\oint d\theta_2\Pi_{\theta_2}$$
$$= \frac{\sqrt{\lambda}}{2\pi}\oint d\theta_2\sqrt{c_4 + 3m_2^2\sin^2\theta_2 - \frac{(\mathcal{J}_2 + 2m_2)^2}{\cos^2\theta_2}}. \tag{3.33}$$

Again, taking $\sin\theta_1 = x_1$ and $\sin\theta_2 = x_2$ and choosing the proper limits and the above integrals transform to,

$$\mathcal{N}_1 = \frac{2\sqrt{3}m_1}{\pi}\int_0^{\sqrt{R_{1+}}}\frac{\sqrt{(x_1^2 - R_{1-})(R_{1+} - x_1^2)}}{1 - x_1^2}dx_1, \tag{3.34}$$

$$\mathcal{N}_2 = \frac{2\sqrt{3}m_2}{\pi}\int_0^{\sqrt{R_{2+}}}\frac{\sqrt{(x_2^2 - R_{2-})(R_{2+} - x_2^2)}}{1 - x_2^2}dx_2. \tag{3.35}$$

Directly computing the integrals, we find,

$$\mathcal{N}_1 = \frac{2\sqrt{3}m_1}{\pi\sqrt{-R_{1-}}}\left[\left(1 - R_{1+}\right)K\left(\frac{R_{1+}}{R_{1-}}\right) - R_{1-}E\left(\frac{R_{1+}}{R_{1-}}\right)\right.$$
$$\left. + (R_{1-} - 1)(R_{1+} - 1)\,\Pi\left(R_{1+}|\frac{R_{1+}}{R_{1-}}\right)\right],$$

$$\mathcal{N}_2 = \frac{2\sqrt{3}m_2}{\pi\sqrt{-R_{2-}}}\left[\left(1 - R_{2+}\right)K\left(\frac{R_{2+}}{R_{2-}}\right) - R_{2-}E\left(\frac{R_{2+}}{R_{2-}}\right)\right.$$
$$\left. + (R_{2-} - 1)(R_{2+} - 1)\,\Pi\left(R_{2+}|\frac{R_{2+}}{R_{2-}}\right)\right].$$

As before, in order to make the expressions little simpler we take the partial derivative of $\mathcal{N}_1$ with respect to m_1,

$$\frac{\partial\mathcal{N}_1}{\partial m_1} = \frac{2\sqrt{3}}{\pi}\int_0^{\sqrt{R_{1+}}}\frac{x_1^2 dx_1}{\sqrt{(x_1^2 - R_{1-})(R_{1+} - x_1^2)}}$$

$$-\frac{4(\mathcal{J}_1 + 2m_1)}{\sqrt{3}\pi m_1}\int_0^{\sqrt{R_{1+}}}\frac{dx_1}{(1 - x_1^2)\sqrt{(x_1^2 - R_{1-})(R_{1+} - x_1^2)}}$$
$$= \frac{2\sqrt{-3R_{1-}}}{\pi}\left[K\left(\frac{R_{1+}}{R_{1-}}\right) - E\left(\frac{R_{1+}}{R_{1-}}\right)\right]$$
$$-\frac{4(\mathcal{J}_1 + 2m_1)}{\sqrt{-3R_{1-}}\pi m_1}\left[\Pi\left(R_{1+}|\frac{R_{1+}}{R_{1-}}\right)\right]. \tag{3.36}$$

Again, taking the partial derivative of $\mathcal{N}_2$ with respect to m_2 we get,

$$\frac{\partial\mathcal{N}_2}{\partial m_2} = \frac{2\sqrt{3}}{\pi}\int_0^{\sqrt{R_{2+}}}\frac{x_2^2 dx_2}{\sqrt{(x_2^2 - R_{2-})(R_{2+} - x_2^2)}}$$
$$-\frac{4(\mathcal{J}_2 + 2m_2)}{\sqrt{3}\pi m_2}\int_0^{\sqrt{R_{2+}}}\frac{dx_2}{(1 - x_2^2)\sqrt{(x_2^2 - R_{2-})(R_{2+} - x_2^2)}}$$
$$= \frac{2\sqrt{-3R_{2-}}}{\pi}\left[K\left(\frac{R_{2+}}{R_{2-}}\right) - E\left(\frac{R_{2+}}{R_{2-}}\right)\right]$$
$$-\frac{4(\mathcal{J}_2 + 2m_2)}{\sqrt{-3R_{2-}}\pi m_2}\left[\Pi\left(R_{2+}|\frac{R_{2+}}{R_{2-}}\right)\right]. \tag{3.37}$$

In the following we will discuss the expansion of the above expressions under short string limit for different kinds of energy distribution situations.

3.3.1 Equal energy distribution case

Assuming the energy is equally distributed on both the spheres. In the short string limit, i.e. when both the energy and angular momentum of the string are small, we can expand the above expressions as,

$$\mathcal{N}_1 = \mathcal{A}_1(\mathcal{J}_1) + \mathcal{E}^2\mathcal{B}_1(\mathcal{J}_1) + \mathcal{O}(\mathcal{E}^4), \tag{3.38}$$
$$\mathcal{N}_2 = \mathcal{A}_2(\mathcal{J}_2) + \mathcal{E}^2\mathcal{B}_2(\mathcal{J}_2) + \mathcal{O}(\mathcal{E}^4). \tag{3.39}$$

Adding (3.38) and (3.39) and inverting the added series to get the expression for energy in terms of the conserved quantities as,

$$\mathcal{E} = [\mathcal{B}_1(\mathcal{J}_1) + \mathcal{B}_2(\mathcal{J}_2)]^{-1/2}\sqrt{\mathcal{N}_1 + \mathcal{N}_2 - \mathcal{A}_1(\mathcal{J}_1) - \mathcal{A}_2(\mathcal{J}_2)}$$
$$+ \mathcal{O}(\mathcal{N}_1 + \mathcal{N}_2 - \mathcal{A}_1(\mathcal{J}_1) - \mathcal{A}_2(\mathcal{J}_2))^{3/2} \tag{3.40}$$

where

$$\mathcal{A}_1(\mathcal{J}_1) = \left(4m_1 - \frac{15}{4m_1} - \frac{15\mu_1^2}{2m_1} - \frac{159\mu_1^2}{8m_1^3}\right)$$
$$- \left(6\log m_1 - \frac{153}{8m_1^2} - \frac{153\mu_1^2}{4m_1^2} - \frac{33435\mu_1^2}{128m_1^4}\right)\mathcal{J}_1$$
$$- \left(\frac{51}{2m_1} + \frac{213}{2m_1^3} + \frac{213\mu_1^2}{m_1^3} + \frac{81465\mu_1^2}{32m_1^5}\right)\mathcal{J}_1^2$$
$$+ \left(\frac{267}{4m_1^2} + \frac{81015}{128m_1^4} + \frac{81015\mu_1^2}{64m_1^4}\right)$$

$$+\frac{1422975\mu_1^2}{64m_1^6}\Big)\mathcal{J}_1^3 + \mathcal{O}(\mathcal{J}_1^4), \tag{3.41}$$

$$\mathcal{A}_2(\mathcal{J}_2) = \left(4m_2 - \frac{15}{4m_2} - \frac{15\mu_2^2}{2m_2} - \frac{159\mu_2^2}{8m_2^3}\right)$$
$$- \left(6\log m_2 - \frac{153}{8m_2^2} - \frac{153\mu_2^2}{4m_2^2} - \frac{33435\mu_2^2}{128m_2^4}\right)\mathcal{J}_2$$
$$- \left(\frac{51}{2m_2} + \frac{213}{2m_2^3} + \frac{213\mu_2^2}{m_2^3} + \frac{81465\mu_2^2}{32m_2^5}\right)\mathcal{J}_2^2$$
$$+ \left(\frac{267}{4m_2^2} + \frac{81015}{128m_2^4} + \frac{81015\mu_2^2}{64m_2^4}\right.$$
$$\left.+ \frac{1422975\mu_2^2}{64m_2^6}\right)\mathcal{J}_2^3 + \mathcal{O}(\mathcal{J}_2^4), \tag{3.42}$$

$$\mathcal{B}_1(\mathcal{J}_1) = \left(\frac{15}{4m_1} + \frac{159}{16m_1^3} + \frac{159\mu_1^2}{8m_1^3} + \frac{10995\mu_1^2}{64m_1^5}\right)$$
$$- \left(\frac{153}{8m_1^2} + \frac{33435}{256m_1^4} + \frac{33435\mu_1^2}{128m_1^4}\right.$$
$$\left.+ \frac{911955\mu_1^2}{256m_1^6}\right)\mathcal{J}_1 + \left(\frac{213}{2m_1^3} + \frac{81465}{64m_1^5} + \frac{81465\mu_1^2}{32m_1^5}\right.$$
$$\left.+ \frac{10899675\mu_1^2}{224m_1^7}\right)\mathcal{J}_1^2 - \left(\frac{81015}{128m_1^4}\right.$$
$$+ \frac{1422975}{128m_1^6} + \frac{1422975\mu_1^2}{64m_1^6} + \frac{283301175\mu_1^2}{512m_1^8}\right)$$
$$\times \mathcal{J}_1^3 + \mathcal{O}(\mathcal{J}_1^4), \tag{3.43}$$

$$\mathcal{B}_2(\mathcal{J}_2) = \left(\frac{15}{4m_2} + \frac{159}{16m_2^3} + \frac{159\mu_2^2}{8m_2^3} + \frac{10995\mu_2^2}{64m_2^5}\right)$$
$$- \left(\frac{153}{8m_2^2} + \frac{33435}{256m_2^4} + \frac{33435\mu_2^2}{128m_2^4}\right.$$
$$\left.+ \frac{911955\mu_2^2}{256m_2^6}\right)\mathcal{J}_2 + \left(\frac{213}{2m_2^3} + \frac{81465}{64m_2^5} + \frac{81465\mu_2^2}{32m_2^5}\right.$$
$$\left.+ \frac{10899675\mu_2^2}{224m_2^7}\right)\mathcal{J}_2^2 - \left(\frac{81015}{128m_2^4}\right.$$
$$+ \frac{1422975}{128m_2^6} + \frac{1422975\mu_2^2}{64m_2^6} + \frac{283301175\mu_2^2}{512m_2^8}\right)$$
$$\times \mathcal{J}_2^3 + \mathcal{O}(\mathcal{J}_2^4). \tag{3.44}$$

3.3.2 Unequal energy distribution case

In this case expanding oscillation numbers $\mathcal{N}_1$ and $\mathcal{N}_2$ when the corresponding energy and angular momentum are small

and inverting the added series, we obtain the similar kind of relation as,

$$\mathcal{E} = [\mathcal{B}_1(\mathcal{J}_1) + \mathcal{B}_2(\mathcal{J}_2)]^{-1/2}\sqrt{\mathcal{N}_1 + \mathcal{N}_2 - \mathcal{A}_1(\mathcal{J}_1) - \mathcal{A}_2(\mathcal{J}_2)}$$
$$+ \mathcal{O}(\mathcal{N}_1 + \mathcal{N}_2 - \mathcal{A}_1(\mathcal{J}_1) - \mathcal{A}_2(\mathcal{J}_2))^{3/2} \tag{3.45}$$

but now,

$$\mathcal{A}_1(\mathcal{J}_1) = \left(4m_1 - \frac{15f}{2m_1} - \frac{15\mu_1^2}{2m_1} - \frac{159f\mu_1^2}{4m_1^3}\right)$$
$$- \left(6\log m_1 - \frac{153f}{4m_1^2} - \frac{153\mu_1^2}{4m_1^2} - \frac{33435f\mu_1^2}{64m_1^4}\right)\mathcal{J}_1$$
$$- \left(\frac{51}{2m_1} + \frac{213f}{m_1^3} + \frac{213\mu_1^2}{m_1^3} + \frac{81465f\mu_1^2}{16m_1^5}\right)\mathcal{J}_1^2$$
$$+ \left(\frac{267}{4m_1^2} + \frac{81015f}{64m_1^4} + \frac{81015\mu_1^2}{64m_1^4}\right.$$
$$\left.+ \frac{1422975f\mu_1^2}{32m_1^6}\right)\mathcal{J}_1^3 + \mathcal{O}(\mathcal{J}_1)^4, \tag{3.46}$$

$$\mathcal{A}_2(\mathcal{J}_2) = \left(4m_2 - \frac{15(1-f)}{2m_2} - \frac{15\mu_2^2}{2m_2} - \frac{159(1-f)\mu_2^2}{4m_2^3}\right)$$
$$- \left(6\log m_2 - \frac{153(1-f)}{4m_2^2} - \frac{153\mu_2^2}{4m_2^2}\right.$$
$$\left.- \frac{33435(1-f)\mu_2^2}{64m_2^4}\right)\mathcal{J}_2$$
$$- \left(\frac{51}{2m_2} + \frac{213\mu_2^2}{m_2^3} + \frac{213(1-f)}{m_2^3} + \frac{81465(1-f)\mu_2^2}{16m_2^5}\right)\mathcal{J}_2^2$$
$$+ \left(\frac{267}{4m_2^2} + \frac{81015(1-f)}{64m_2^4} + \frac{81015\mu_2^2}{64m_2^4}\right.$$
$$\left.+ \frac{1422975(1-f)\mu_2^2}{32m_2^6}\right)\mathcal{J}_2^3 + \mathcal{O}(\mathcal{J}_2)^4, \tag{3.47}$$

$$\mathcal{B}_1(\mathcal{J}_1) = \left(\frac{15f}{2m_1} + \frac{159f^2}{4m_1^3} + \frac{159f\mu_1^2}{4m_1^3} + \frac{10995f^2\mu_1^2}{16m_1^5}\right)$$
$$- \left(\frac{153f}{4m_1^2} + \frac{33435f^2}{64m_1^4} + \frac{33435f\mu_1^2}{64m_1^4}\right.$$
$$\left.+ \frac{911955f^2\mu_1^2}{64m_1^6}\right)\mathcal{J}_1 + \left(\frac{213f}{m_1^3}\right.$$
$$+ \frac{81465f^2}{16m_1^5} + \frac{81465f\mu_1^2}{16m_1^5} + \frac{10899675f^2\mu_1^2}{56m_1^7}\right)\mathcal{J}_1^2$$
$$- \left(\frac{81015f}{64m_1^4} + \frac{1422975f^2}{32m_1^6}\right.$$
$$\left.+ \frac{1422975f\mu_1^2}{32m_1^6} + \frac{283301175f^2\mu_1^2}{128m_1^8}\right)\mathcal{J}_1^3 + \mathcal{O}(\mathcal{J}_1)^4. \tag{3.48}$$

$$\mathcal{B}_2(\mathcal{J}_2) = \left(\frac{15(1-f)}{2m_2} + \frac{159(1-f)^2}{4m_2^3} + \frac{159(1-f)\mu_2^2}{4m_2^3}\right.$$
$$\left.+ \frac{10995(1-f)^2\mu_2^2}{16m_2^5}\right)$$
$$- \left(\frac{153(1-f)}{4m_2^2} + \frac{33435(1-f)^2}{64m_2^4}\right.$$

$$+\frac{33435(1-f)\mu_2^2}{64m_2^4}+\frac{911955(1-f)^2\mu_2^2}{64m_2^6}\Bigg)\mathcal{J}_2$$

$$+\Bigg(\frac{213(1-f)}{m_2^3}+\frac{81465(1-f)^2}{16m_2^5}$$

$$+\frac{81465(1-f)\mu_2^2}{16m_2^5}+\frac{10899675(1-f)^2\mu_2^2}{56m_2^7}\Bigg)\mathcal{J}_2^2$$

$$-\Bigg(\frac{81015(1-f)}{64m_2^4}+\frac{1422975(1-f)^2}{32m_2^6}$$

$$+\frac{1422975(1-f)\mu_2^2}{32m_2^6}$$

$$+\frac{283301175(1-f)^2\mu_2^2}{128m_2^8}\Bigg)\mathcal{J}_2^3+\mathcal{O}(\mathcal{J}_2)^4. \qquad (3.49)$$

Now we will discuss how the most general relations (3.46)–(3.49) along with (3.45) reduces to different cases for different values of f. First we consider $f = 1$ that is when energy is fully distributed to the sphere $d\Omega_1$ and no energy is there to excite the string in the other sphere $d\Omega_2$ as demonstrated in Fig. 5. In this case as the string don't have any kind of motion in the sphere $d\Omega_2$, it won't be unlogical if we consider $\mathcal{J}_2 = 0$, in addition if we choose $\mu_2 = \sqrt{\frac{8}{15}}m_2$ then we can easily check from (3.47) and (3.49) we obtain $\mathcal{A}_2(0) = 0$ and $\mathcal{B}_2(0) = 0$. These will in turn imply $\mathcal{N}_2 = 0$. Also, the relations (3.46) and (3.48) will reduces to (2.34) and (2.35) respectively. Hence, in this case the energy-oscillation number relation (3.45) exactly reduces to the energy-oscillation number relation of NS5-brane (2.33).

One can obtain, similar kind of reduction for $f = 0$ when the energy is fully distributed to the second sphere and no energy is available to the first sphere. In this case string will not have any kind of motion in the first sphere and we can obtain $\mathcal{A}_1(0) = \mathcal{B}_1(0) = \mathcal{N}_1 = 0$ and (3.47) and (3.49) reduces to (2.34) and (2.35) respectively. Thus for $f = 1$ or $f = 0$, i.e., when we switch off the energy in one of the sphere of I-brane then not only the string profile but also the energy-oscillation number relation of I-brane reduces to that of NS5-brane.

When the energy is equally distributed among the spheres that is for $f = 0.5$ as demonstrated in Fig. 6, one can easily verify that the expressions (3.46), (3.47), (3.48) and (3.49) reduces to (3.41), (3.42), (3.43) and (3.44) respectively. Hence, we can conclude the relations of the most general case of unequally energy distribution given by the relations (3.46)–(3.49) are consistent with the corresponding relations of equally energy distribution.

4 Summary and conclusion

In this paper, first we have obtained the pulsating string solution i.e., oscillating behaviour under some constraint among various conserved charges and parameters in the NS5-brane background. We have shown the string profile is only sensitive to the energy and the winding number m. We have found that for a given energy, number of lobes increase with increasing winding number m, but the amplitude of pulsating string decreases. Also these lobes are of m fold. Initially, we assumed the string worldsheet has a fixed surface area for a given value of energy. To verify this assumption we estimate the surface area of different plots of Fig. 1 in Appendix A using the formula of area of revolution of a curve. We find the surface area of Fig. 1a is approximately twice the surface area of Fig. 1b and 4.2 times the surface area of Fig. 1c. But Fig. 1b is 3 fold, and Fig. 1c is 5 fold, so here definitely the surface area of the worldsheet is not fixed for a particular value of energy but increases with the increase of winding number m. Then, why do we get different lobes for the same energy as shown in Fig. 1?

It is easy to understand having larger number of lobes represent higher frequency, but here as the energy is constant all three plots of Fig. 1 having different frequencies are degenerate. At first it might seem to violate Einstein's relation $E = h\nu$, but note that here the plots are actually periodic surfaces. For these surfaces we have amplitude in both the horizontal directions, and so far what we are calling amplitude is actually amplitude squared which can be considered as intensity and intensity (I) is given by $I = nh\nu$, where n is number of particles (circular strings here). To keep the total energy constant here the diagram with larger number of lobes have lower intensity that is, although frequency is high but less number of circular strings involved there which keep the total energy constant. Thus, with a given energy these circular strings can oscillate with different frequencies but these different frequency states will involve different number of them. Again, from Fig. 3 we found with increasing energy intensity increases. This is because higher intensity involves more number of circular strings giving rise to higher energy. From the same figure, we have also seen for the same value of winding number m, number of lobes increases with the energy. This is because with higher energy the frequency of these circular strings also increases giving rise to larger number of lobes. Thus, energy can be increased by involving more number of circular strings as well as by increasing frequency of these circular strings. For comparing with the results of I-brane background we also determine the oscillation number and expand it under short string limit and obtain the energy oscillation number relation for pulsating string in NS5-brane background.

Next, we have obtained pulsating string solution in I-brane background. Here we have found two copies of similar kind string profiles arising from the two spheres and not surprisingly each profile is equivalent to that of string profile in NS5-brane background. We have argued that in the near horizon geometry of I-brane background, string

can have pulsating motion simultaneously and independently on both the spheres. In order to plot string profile and expand oscillation number we distribute energy equally and unequally among the spheres. When energy is equally distributed among the spheres the string was shown to be pulsating on both the spheres simultaneously with same amplitude for $m_1 = m_2$ or in other words in this case the circular strings have same intensity on both the spheres. When energy is unequally distributed, the string was shown to pulsate with larger amplitude or circular strings have higher intensity on the sphere bearing higher energy and simultaneously with smaller amplitude or circular string have lower intensity on the sphere bearing lower energy. We have also shown when we cut off the energy from one of the spheres of I-brane then the string pulsates only on the other sphere. Finally , we have also expanded the oscillation number under short string limit and obtain the energy-oscillation number relation. This general energy-oscillation number relation get reduced to the energy-oscillation number relation of NS5-brane when we switch off energy in one of the sphere of I-brane and also reduce to equally energy distributed energy-oscillation number relation when we distribute energy equally to the spheres.

It would be interesting to look for possible dual gauge theory operators of these pulsating string solutions. Recently, in [50] the dynamics of a strongly coupled field theory on I-brane was investigated holographically. In our case, even though it is not very clear what kind of dual gauge theory it would correspond to, it is worth exploring following [50] and look for operators which may correspond to these solutions in the string theory side. As an extension of this work, one can further study the perturbation of these pulsating string solutions. We hope to come up with some of these in near future.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: This is a theory paper and all the data generated or analysed during this study are included in this published article].

Appendix A: Calculation of surface area of the worldsheet of Fig. 1

The surface area of Fig. 1 can be estimated by considering the area of revolution of sinusoidal curves. We know the area of revolution of the curve $f(x)$ is given by

$$A_3 \approx 3 * 2\pi \int_0^{\pi/6} 0.1 \sin 6x \sqrt{1 + (0.6 \cos 6x)^2} dx = 0.664206$$

where a is the other end point of the curve $f(x)$. For Fig. 1a, the function is $f_1(x) = 0.4 \sin 2x$, $f_1'(x) = 0.8 \cos 2x$ and we integrate from 0 to $\pi/2$ (approximately). Using above equation we estimate the surface area of Fig. 1a as A_1,

$$A_1 \approx 2\pi \int_0^{\pi/2} 0.4 \sin 2x \sqrt{1 + (0.8 \cos 2x)^2} \, dx = 2.76015. \tag{4.1}$$

For Fig. 1b, the function is $f_2(x) = 0.2 \sin 4x$, $f_2'(x) = 0.8 \cos 4x$ and here we integrate from 0 to $\pi/4$ (approximately). Here, the surface area A_2 is,

$$A_2 \approx 2 * 2\pi \int_0^{\pi/4} 0.2 \sin 4x \sqrt{1 + (0.8 \cos 4x)^2} dx = 1.38008 \tag{4.2}$$

As there are two lobes we multiplied a factor of 2 in front. For Fig. 1c, the function is $f_3(x) = 0.1 \sin 6x$ and $f_3'(x) = 0.6 \cos 6x$ and in this case we integrate from 0 to $\pi/6$ (approximately). The surface area A_3 in this case is given by,

$$A_3 \approx 3 * 2\pi \int_0^{\pi/6} 0.1 \sin 6x \sqrt{1 + (0.6 \cos 6x)^2} dx = 0.664206 \tag{4.3}$$

Hence, $A_1 = 2A_2$ and $A_1 \approx 4.2A_3$.

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