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The Brans–Dicke field in non-metricity gravity: cosmological solutions and conformal transformations

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Abstract We consider the Brans–Dicke theory in non-metricity gravity, which belongs to the family of symmetric teleparallel scalar–tensor theories. Our focus lies in exploring the implications of the conformal transformation, as we derive the conformal equivalent theory in the Einstein frame, distinct from the minimally coupled scalar field theory. The fundamental principle of the conformal transformation suggests the mathematical equivalence of the related theories. However, to thoroughly analyze the impact on physical variables, we investigate the spatially flat Friedmann–Lemaître–Robertson–Walker geometry, defining the connection in the non-coincidence gauge. We construct exact solutions for the cosmological model in one frame and compare the physical properties in the conformal related frame. Surprisingly, we find that the general physical properties of the exact solutions remain invariant under the conformal transformation. Finally, we construct, for the first time, an analytic solution for the symmetric teleparallel scalar–tensor cosmology.

1 Introduction

Symmetric teleparallel general relativity (STGR) [1] represents an alternative gravitational theory, considered equivalent to General Relativity (GR). In STGR, the fundamental geometric elements consist of the metric tensor $g_{\mu\nu}$ and the symmetric, flat connection $\Gamma^{\lambda}_{\mu\nu}$ with the covariant derivative ∇_{λ}, leading to $\nabla_{\lambda} g_{\mu\nu} \neq 0$. While GR defines autoparallels using the Levi–Civita connection for the metric tensor $g_{\mu\nu}$, STGR emphasizes the non-metricity component, crucial for the theory's description. The equivalence between these two gravitational theories becomes evident upon a study of the gravitational Lagrangians [2]. In GR, the Lagrangian function involves the Ricci scalar constructed by the Levi–Civita connection $\mathring{R}$, whereas in STGR, the corresponding Lagrangian is defined by the non-metricity scalar Q. The Ricci scalar and the non-metricity scalar differ by a boundary term $B = \mathring{R} - Q$ [1–3]. Consequently, the variation of the two distinct Lagrangians yields the same physical theory. However, this equivalence breaks down when introducing matter non-minimally coupled to gravity [4–8], or nonlinear terms of the gravitational scalars in the Action Integral [9,10].

In $f(Q)$-gravity [9,10], a straightforward extension of the STGR theory, the gravitational Lagrangian takes the form of a nonlinear function f of the non-metricity scalar Q. These nonlinear terms introduce additional degrees of freedom, leading to modifications in the gravitational field equations that give rise to new phenomena [11]. In the context of cosmology, $f(Q)$ has been proposed as a solution to the dark energy problem [12–17] and has been utilized to explain cosmic acceleration [18–21].

In the symmetric teleparallel theory of gravity, the presence of a flat geometry defined by the connection $\Gamma^{\lambda}_{\mu\nu}$ allows for the existence of a coordinate system known as the coincidence gauge, where the covariant derivative can be represented as a partial derivative. This implies that in the symmetric teleparallel theory of gravity, the inertial effects can be distinguished from gravity [9]. Consequently, the choice of the connection as the starting point in the symmetric teleparallel theory leads to the formulation of distinct gravitational theories [3]. As a result, self-accelerating solutions can naturally emerge both in the early and late universe [22]. The impact of different connections on the existence of cosmological solutions has been extensively explored in [22], while the scenario of static spherically symmetric spacetimes has been considered in [23,24]. The reconstruction of the cosmological history was derived in [25–27]. Specifically, the phase-space analysis was studied, for the field equations for the four different connections which describe the

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Friedmann–Lemaître–Robertson–Walker (FLRW) geometry [3]. For similar studies see also [28,29]. Quantum cosmology in $f(Q)$-gravity investigated in [30], while in [31] a minisuperspace description is presented from where it follows that the $f(Q)$-theory can be described by two scalar fields. The first scalar field corresponds to the degrees of freedom associated with the higher-order derivatives of the theory, whereas the second scalar field is linked to the connection defined in the non-coincidence gauge. For further investigations into $f(Q)$-gravity and its generalizations, we recommend referring to the works cited in [32–43] and the references provided therein.

Scalar fields non-minimally coupled to gravity have found extensive application in gravitational physics within the framework of General Relativity, such as in scalar-curvature theories [44,45], or in the context of teleparallelism, specifically scalar–tensor theories [46,47]. The Brans–Dicke theory [48] represents one of the earliest scalar-curvature theories, formulated with the intention of establishing a gravitational theory that adheres to $\hat{\omega}_m$'s principle [49]. This model is defined in the Jordan frame [50], where the presence of a matter source is essential for the existence of physical space. In contrast, General Relativity is defined in the Einstein frame, enabling the existence of physical space even in the absence of a matter term. The Brans–Dicke parameter is a characterized constant of the theory which indicates the coupling between the scalar and the gravitation Lagrangian [51]. When the Brans–Dicke parameter vanishes, the theory is equivalent to the $f(R)$-gravity, where the non-minimally coupled scalar field attributes the higher-order degrees of freedom [52].

Although the scalar-curvature theory is initially defined in the Jordan frame, a geometrical mapping exists that enables the transformation of the theory into the Einstein frame. Consequently, the scalar-curvature theory can be interpreted in the form equivalent to General Relativity, involving a minimally coupled scalar field. This geometric mapping is a conformal transformation, establishing a connection between the solution trajectories of the two frames [53]. However, the physical properties of the solution trajectories are not invariant under the application of the conformal transformation. For example singular solutions does not remain singulars after the application of the conformal transformation, for more details see the discussion [54–56] and references therein. More recently, the Hamiltonian inequivalence between the Jordan and Einstein frames has been explored in [57–59].

In this study we are interested to study the effects of the conformal transformation on the physical properties of cosmological solutions on the Brans–Dicke analogue in symmetric teleparallel scalar–tensor theory [4]. It is known that $f(Q)$-gravity is equivalent to a specific family of symmetric teleparallel scalar–tensor models, and we use the analogy of the Brans–Dicke model with the $f(R)$-gravity in order to introduce the non-metricity Brans–Dicke theory. We focus in the cosmological scenario of a spatially flat FLRW geometry. Moreover, we consider the case in which the connection is defined in the non-coincidence gauge and the gravitational theory is equivalent to a multiscalar field model. While the mathematical application of the conformal transformation in non-metricity theory has been previously explored in [5], no concrete conclusions were drawn regarding the physical properties of the solutions under the conformal transformation. More recently, in [60], several exact cosmological solutions were identified in the non-metricity scalar–tensor theory for the non-coincidence gauge. Within this work, we aim to determine exact and analytic solutions for the non-metricity Brans–Dicke cosmological theory, subsequently comparing the physical properties of the solutions between the Jordan and the "Einstein" frames. As we shall see in the following, the conformal equivalent theory is not defined in the Einstein frame, because a coupling between the scalar field and another geometric invariant it follows [5]. However, in order to stand out the two theories, we shall call the one to defined in the (pseudo)-Einstein frame.

The structure of the paper is outlined as follows.

In Sect. 2 we discuss the fundamental properties and definitions of symmetric teleparallel gravity. Additionally, we explore $f(Q)$-theory and the symmetric teleparallel scalar–tensor theory of gravity. We demonstrate that $f(Q)$-theory can be reformulated as a non-metricity scalar–tensor theory. Furthermore, we present the utilization of conformal transformations and the derivation of the conformal equivalent theory in Sect. 3. In Sect. 4, we introduce the extension of the Brans–Dicke field in non-metricity gravity. Here, we introduce a novel parameter ω, akin to the Brans–Dicke parameter of scalar-curvature theory. As $\omega \to 0$, the gravitational Action characterizes the $f(Q)$-theory, similarly to how the Brans–Dicke field characterizes $f(R)$-gravity in the same limit. Within this gravitational model, we consider a spatially flat FLRW background geometry, and for the connection defined in the non-coincidence gauge, we present the field equations in both the Jordan frame and the Einstein frame.

To explore the effects of the conformal transformation on the physical properties of solution trajectories within the conformal equivalent theories, Sect. 5 is dedicated to deriving precise solutions for the field equations. We conduct a comparative analysis of the physical properties between the two frames. It is observed that singular scaling solutions in one frame correspond to singular scaling solutions in the other frame, displaying identical asymptotic behaviour. Additionally, for the non-singular de Sitter solution, it is established that the asymptotic behaviour of physical properties remains unchanged under the application of the conformal transformation. Moreover, in Sect. 6, we introduce an analytical solution for the scalar–tensor theory in non-metricity gravity for the first time. The analysis reveals that this universe originates from a Big Rip singularity, transitions into an era characterized by an ideal gas, and ultimately converges towards a de Sitter universe as a future attractor. Notably, the observed behaviour of the physical parameters remains consistent regardless of the frame in which the theory is defined. Finally, our findings are summarized in Sect. 7.

2 Symmetric teleparallel gravity

Let M^n be a manifold defined by the metric tensor, $g_{\mu\nu}$, and the derivative ∇_λ, defined by the generic connection $\Gamma^\lambda_{\mu\nu}$ with conditions, the $\Gamma^\lambda_{\mu\nu}$ to inherit the symmetries of the metric tensor $g_{\mu\nu}$; that is, if X is a Killing vector of $g_{\mu\nu}$, i.e. $\mathcal{L}_X g_{\mu\nu}$, then $\mathcal{L}_X \Gamma^\lambda_{\mu\nu} = 0$, in which $\mathcal{L}_X$ is the Lie derivative with respect the vector field X. Furthermore, for the connection $\Gamma^\lambda_{\mu\nu}$ it holds that the Riemann tensor $R^\kappa{}_{\lambda\mu\nu}$ and torsion tensor $\mathrm{T}^\lambda_{\mu\nu}$ are always zero; that is,

$$R^\kappa{}_{\lambda\mu\nu} \equiv \frac{\partial \Gamma^\kappa{}_{\lambda\nu}}{\partial x^\mu} - \frac{\partial \Gamma^\kappa{}_{\lambda\mu}}{\partial x^\nu} + \Gamma^\sigma{}_{\lambda\nu}\Gamma^\kappa{}_{\mu\sigma} - \Gamma^\sigma{}_{\lambda\mu}\Gamma^\kappa{}_{\mu\sigma} = 0, \quad (1)$$

$$\mathrm{T}^\lambda_{\mu\nu} \equiv \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} = 0. \quad (2)$$

In symmetric teleparallel theory of gravity only the non-metricity tensor survives, defined as [1]

$$Q_{\lambda\mu\nu} = \nabla_\lambda g_{\mu\nu}, \quad (3)$$

that is,

$$Q_{\lambda\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^\lambda} - \Gamma^\sigma{}_{\lambda\mu} g_{\sigma\nu} - \Gamma^\sigma{}_{\lambda\nu} g_{\mu\sigma}. \quad (4)$$

We define the disformation tensor

$$L^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(Q_{\mu\nu\sigma} + Q_{\nu\mu\sigma} - Q_{\sigma\mu\nu} \right) \quad (5)$$

and the non-metricity conjugate tensor [3]

$$P^\lambda_{\mu\nu} = \frac{1}{4} \left(-2L^\lambda{}_{\mu\nu} + Q^\lambda g_{\mu\nu} - Q'^\lambda g_{\mu\nu} - \delta^\lambda_{(\mu} Q_{\nu)} \right), \quad (6)$$

where now the non-metricity vectors Q^λ and Q'^λ are defined as

$$Q_\lambda = Q_\lambda{}^\mu{}_\mu, \; Q'_\lambda = Q^\mu{}_{\lambda\mu}, \quad (7)$$

and

$$P^\lambda = P^\lambda_{\mu\nu} g^{\mu\nu} = \frac{(n-2)}{4} \left(Q^\lambda - Q'^\lambda \right).$$

The non-metricity scalar is defined as

$$Q = Q_{\lambda\mu\nu} P^{\lambda\mu\nu}$$

and the gravitational Action Integral in STGR is given by the following expression [1]

$$S_{STGR} = \int d^4x \sqrt{-g}\, Q. \quad (8)$$

The non-metricity scalar, Q, and the Ricciscalar $\mathring{R}$ for the Levi–Civita connection $\mathring{\Gamma}^\lambda{}_{\mu\nu}$of the metric tensor $g_{\mu\nu}$ differ by a boundary term B, that is, [4]

$$B = \mathring{R} - Q, \quad (9)$$

where

$$B = -\mathring{\nabla}_\lambda \left(Q^\lambda - Q'^\lambda \right) \quad (10)$$

and $\mathring{\nabla}_\lambda$ denotes covariant derivative with respect to the Levi–Civita connection, $\mathring{\Gamma}^\lambda{}_{\mu\nu}$.

2.1 $f(Q)$-theory

An extension of STGR which has drawn the attention recently is the $f(Q)$-gravity. In this theory, the gravitational Lagrangian is a nonlinear function $f(Q)$, such that the action integral is [9,10]

$$S_{f(Q)} = \int d^4x \sqrt{-g}\, f(Q).$$

The resulting gravitational field equations are

$$f'(Q) G_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \left(f'(Q)Q - f(Q) \right) + 2f''(Q) \left(\nabla_\lambda Q \right) P^\lambda{}_{\mu\nu} = 0, \quad (11)$$

where $G_{\mu\nu}$ is the Einstein-tensor.

Moreover, variation with respect to the connection $\Gamma^\lambda{}_{\mu\nu}$ gives the equation of motion

$$\nabla_\mu \nabla_\nu \left(\sqrt{-g} f'(Q) P^{\mu\nu}{}_{\mu\nu\sigma} \right) = 0. \quad (12)$$

Furthermore, in the limit at which $f(Q)$ becomes linear, the field equations are reduced to those of symmetric teleparallel gravity (STGR).

Last but not least, in the presence of a matter source minimally coupled to gravity, the field equations (11) are modified as follows

$$f'(Q) G_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \left(f'(Q)Q - f(Q) \right) + 2f''(Q) \left(\nabla_\lambda Q \right) P^\lambda{}_{\mu\nu} = T_{\mu\nu}, \quad (13)$$

with the energy–momentum tensor $T_{\mu\nu}$ to give the degrees of freedom for the matter source.

2.2 Symmetric teleparallel scalar–tensor theory

Although the symmetric teleparallel scalar–tensor theory it is not a pure Machian theory [4], it has properties similar to that of a Machian theory [49].

The gravitational action integral is [4]

$$S_{ST\varphi}=\int d^4x\sqrt{-g}\left(\frac{F(\varphi)}{2}Q-\frac{\omega(\varphi)}{2}g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}-V(\varphi)\right), \tag{14}$$

where $V(\phi)$ is the scalar field potential, which drives the dynamics and $F(\phi)$ is the coupling function between the scalar field and the gravitational scalar Q. The function, $\omega(\phi)$, can be eliminated with the introduction of the new scalar field $d\Phi=\sqrt{\omega(\varphi)}d\varphi$. Hence, the action integral (14) becomes

$$S_{ST\Phi}=\int d^4x\sqrt{-g}\left(\frac{F(\Phi)}{2}Q-\frac{1}{2}g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu}-V(\Phi)\right). \tag{15}$$

The field equations which follow from the gravitational action (14) are

$$F(\varphi)G_{\mu\nu}+2F_{,\phi}\varphi_{,\lambda}P^{\lambda}{}_{\mu\nu}+g_{\mu\nu}V(\varphi)+\frac{\omega(\varphi)}{2}\left(g_{\mu\nu}g^{\lambda\kappa}\varphi_{,\lambda}\varphi_{,\kappa}-\varphi_{,\mu}\varphi_{,\nu}\right)=0, \tag{16}$$

$$\nabla_\mu\nabla_\nu\left(\sqrt{-g}F(\varphi)P^{\mu\nu}_{\mu\nu\sigma}\right)=0 \tag{17}$$

and

$$\frac{\omega(\varphi)}{\sqrt{-g}}g^{\mu\nu}\partial_\mu\left(\sqrt{-g}\partial_\nu\varphi\right)+\frac{\omega_{,\varphi}}{2}g^{\lambda\kappa}\varphi_{,\lambda}\varphi_{,\kappa}+\frac{1}{2}F_{,\varphi}Q-V_{,\varphi}=0. \tag{18}$$

It is important to observe that for $\omega(\varphi)=0$, $F(\varphi)=\varphi$, the latter field equations take the functional form of $f(Q)$-theory [4], where now $\varphi=f'(Q)$ and $V(\varphi)=\left(f'(Q)Q-f(Q)\right)$.

3 Conformal transformation

The symmetric teleparallel scalar–tensor theory satisfies Mach's principle, that is, the gravitational theory is defined in the Jordan frame. A similar result holds for the $f(Q)$-theory. The Jordan frame is related to the Einstein frame through a conformal transformation. This transformation relates theories which are conformal equivalent. This equivalence it has to do with the trajection solutions for the field equations, but it is not a physical equivalence; since the physical properties of the theories do not remain invariant under a conformal transformation. Conformal transformations for the four-dimensional manifold were investigated in [5], see also [6,7]. Below we consider a n-dimensional space.

Let $\bar{g}_{\mu\nu}$, $g_{\mu\nu}$ be two conformal equivalent metrics related according to

$$\bar{g}_{\mu\nu}=e^{2\Omega(x^\kappa)}g_{\mu\nu},\quad \bar{g}^{\mu\nu}=e^{-2\Omega(x^\kappa)}g^{\mu\nu}.$$

Therefore, for the nonmetricity tensor we find

$$\bar{Q}_{\lambda\mu\nu}=e^{2\Omega}Q_{\lambda\mu\nu}+2\Omega_{,\lambda}\bar{g}_{\mu\nu}. \tag{19}$$

Moreover,

$$\bar{Q}_\mu=\bar{Q}_\mu{}^\nu{}_\nu=Q_\mu+2n\Omega_{,\mu}, \tag{20}$$

$$\bar{Q}'_\mu=\bar{Q}_{\mu\nu}{}^\nu=Q'_\mu+2\Omega_{,\mu} \tag{21}$$

and

$$\bar{P}^\lambda=\bar{P}^\lambda_{\mu\nu}\bar{g}^{\mu\nu}=e^{-2\Omega}P^\lambda+\frac{(n-2)(n-1)}{2}\Omega^{,\lambda}.$$

Therefore, for the non-metricity scalar we find

$$\bar{Q}=\bar{Q}_{\lambda\mu\nu}\bar{P}^{\lambda\mu\nu}=e^{-2\Omega}Q+\left(2\Omega_{,\lambda}P^\lambda+(n-2)(n-1)\Omega_\lambda\Omega^{,\lambda}\right). \tag{22}$$

Consider now the action integral (14) for the n-dimensional conformally related metric $\bar{g}_{\mu\nu}$, that is,

$$\bar{S}_{ST\varphi}=\int d^nx\sqrt{-\bar{g}}\left(\frac{F(\varphi)}{2}\bar{Q}-\frac{\omega(\varphi)}{2}\bar{g}^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}-V(\varphi)\right). \tag{23}$$

With respect to the metric $g_{\mu\nu}$ and the conformal factor Ω, the latter action integral is

$$\bar{S}_{ST\varphi}=\int d^nx\sqrt{-g}\left(e^{(n-2)\Omega}F(\varphi)\left(\frac{Q}{2}+\Omega_{,\lambda}P^\lambda\right)\right)+d^nx\sqrt{-g}\left(e^{n\Omega}\left(\frac{(n-2)(n-1)}{2}\Omega_\lambda\Omega^{,\lambda}-\frac{\omega(\varphi)}{2}e^{-2\Omega}g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}-V(\varphi)\right)\right),$$

We select $F(\varphi)e^{(n-2)\Omega}=1$, that is $\Omega=\frac{1}{2-n}\ln F(\varphi)$.
Therefore, the latter action reads

$$\bar{S}_{ST\varphi}=\int d^nx\sqrt{-g}\left(\frac{Q}{2}+\Omega_{,\lambda}P^\lambda\right)+\int d^nx\sqrt{-g}\left(\left(\frac{(n-1)F(\varphi)^{\frac{n}{2-n}}}{2(n-2)F(\varphi)}-\frac{\omega(\varphi)}{2}\frac{e^{(n-2)\Omega}}{F(\varphi)^{\frac{n}{2-n}}}\right)g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}-F(\varphi)^{\frac{n}{2-n}}V(\varphi)\right). \tag{24}$$

The second terms become

$$\int d^nx\sqrt{-g}\left(\Omega_{,\lambda}P^\lambda\right)=\int d^nx\sqrt{-g}\left(-\Omega\mathring{\nabla}_\lambda P^\lambda\right)=\int d^nx\sqrt{-g}\left(\frac{(n-2)}{4}\Omega B\right).$$

We end with the gravitational Lagrangian

$$\bar{S}_{ST\varphi}=\int d^nx\sqrt{-g}\left(\frac{Q}{2}-\ln F(\varphi)\frac{B}{4}+\frac{A(\varphi)}{2}g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}-V(\varphi)F(\varphi)^{\frac{n}{2-n}}\right) \tag{25}$$

with

$$A(\varphi) = \left(\frac{(n-1)\left(F_{,\varphi}\right)^2}{(n-2)\,F(\varphi)} - \frac{\omega(\varphi)}{F(\varphi)} \right). \tag{26}$$

Nonmetricity theories with a boundary term introduced before in [61]. The latter Action Integral is a particular case of the models presented in [61].

4 Brans–Dicke cosmology in symmetric teleparallel theory

Similarly to the consideration of the Brans–Dicke field in the scalar-curvature theory, we take into account the following action integral within a four-dimensional manifold in the context of symmetric teleparallel theory. Indeed, in action (14) we assume $F(\varphi) = \varphi$ and $\omega(\varphi) = \frac{\omega}{\varphi}$, $\omega = const.$.

Thus, we arrive at the Lagrangian

$$S_{BD\varphi} = \int d^4x\sqrt{-g}\left(\frac{\varphi}{2}Q - \frac{\omega}{2\varphi}g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} - V(\varphi)\right). \tag{27}$$

Parameter ω play a similar role as that of the Brans–Dicke parameter.

We define the new field $\varphi = e^{\phi}$, in order to write the latter action in the form of the dilaton field

$$S_D = \int d^4x\sqrt{-g}e^{\phi}\left(\frac{Q}{2} - \frac{\omega}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - \hat{V}(\phi)\right),\ \hat{V}(\phi) = V(\phi)\,e^{-\phi}. \tag{28}$$

On the other hand, in the Einstein frame, the equivalent action integral is

$$\bar{S}_D = \int d^nx\sqrt{-g}\left(\frac{Q}{2} - \phi\frac{B}{4} + \frac{\bar{\omega}}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\phi)\,e^{-2\phi}\right),\ \bar{\omega} = \frac{3}{2} + \omega\,,\ \bar{V}(\phi) = V(\phi)\,e^{-2\phi}. \tag{29}$$

The solution trajectories of the field equations for the two gravitational theories described by the action integrals (28), (29) are linked by the conformal transformation. However, no definitive conclusion can be drawn concerning the relationship of the physical properties of the solutions under the application of the conformal transformation.

The objective of this study is to examine how the conformal transformation impacts the physical properties of the trajectory solutions in symmetric teleparallel theory. To conduct such an analysis, we consider the background geometry which describes an isotropic and homogeneous spatially flat FLRW universe, with the line element

$$ds^2 = -N^2(t)\,dt^2 + a(t)^2\left(dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right), \tag{30}$$

in which $a(t)$ is the scale factor and $N(t)$ is the lapse function. We derive the field equations for the two conformally related models, namely S_D and $\bar{S}_D$.

We obtain exact and analytic solutions for one of the models and thoroughly examine the physical properties of these solutions. Subsequently, we apply the conformal transformation to ascertain the corresponding exact and analytic solutions for the second model, delineating the specific physical properties of these solutions. Finally, we conduct a comparative analysis of the physical properties between the solutions of the two conformally related theories.

For the spatially flat FLRW geometry described by the line element (30) there are three families of symmetric connections which describe a flat geometry and inherit the symmetries of the background space [3,22,35]. One family is defined in the coincidence gauge, for this family the nonmetricity scalar Q has the same factional form with the torsion scalar of teleparallelism. Thus, for the connection in the coincidence gauge the symmetric teleparallel scalar–tensor theory is equivalent to the scalar-torsion theory and $f(Q)$-theory is equivalent to $f(T)$-theory. The remaining two families of connections are defined in the non-coincidence gauge where, as it was found in [31], a scalar field is introduced into the gravitational theory which describes the connection and there exist a minisuperspace description.

In this piece of study we select to work in the framework of the connection with nonzero components

$$\Gamma^t_{\ tt} = \frac{\ddot{\psi}(t)}{\dot{\psi}(t)} + \dot{\psi}(t),$$

$$\Gamma^r_{\ tr} = \Gamma^r_{\ rt} = \Gamma^\theta_{\ t\theta} = \Gamma^\theta_{\ \theta t} = \Gamma^\phi_{\ t\phi} = \Gamma^\phi_{\ \phi t} = \dot{\psi}(t), \tag{31}$$

$$\Gamma^r_{\ \theta\theta} = -r,\ \Gamma^r_{\ \phi\phi} = -r\sin^2\theta,\ \Gamma^\theta_{\ r\theta} = \Gamma^\theta_{\ \theta r} = \Gamma^\phi_{\ r\phi} = \Gamma^\phi_{\ \phi r} = \frac{1}{r}, \tag{32}$$

$$\Gamma^\theta_{\ \phi\phi} = -\sin\theta\cos\theta\,,\ \Gamma^\phi_{\ \theta\phi} = \Gamma^\phi_{\ \phi\theta} = \cot\theta, \tag{33}$$

in which $\dot{\psi} = \frac{d\psi}{dt}$, and without loss of generality we have assumed that $N(t) = 1$.

Thus, the non-metricity scalar is calculated

$$Q = -6H^2 + 9\dot{\psi}H + 3\ddot{\psi}\,,\ \gamma = \dot{\psi}. \tag{34}$$

We substitute into (28) and subsequently derive the cosmological field equations in the Jordan frame, yielding:

$$3H^2 + \frac{\omega}{2}\dot{\phi}^2 + \frac{3}{2}\dot{\phi}\dot{\psi} - e^{-\phi}V(\phi) = 0, \tag{35}$$

$$2\dot{H} + 3H^2 + 2H\dot{\phi} - \frac{\omega}{2}\dot{\phi}^2 - \frac{3}{2}\dot{\phi}\dot{\psi} - e^{-\phi}V(\phi) = 0, \tag{36}$$

$$3\ddot{\psi}+2\omega\ddot{\phi}+H\left(6\omega\dot{\phi}+9\dot{\psi}\right)-6H^2+\omega\dot{\phi}^2-e^{-\phi}V_{,\phi}=0, \quad (37)$$

$$\ddot{\phi}+\dot{\phi}^2+3H\dot{\phi}=0, \quad (38)$$

where $H=\frac{\dot{a}}{a}$ is the Hubble function. Recall that we have selected $N(t)=1$. Equations (35), (36) are the modified Friedmann's equations (16). Moreover, Eq. (37) is the equation of motion for the connection (17) and the second-order ordinary differential equation (38) is the equation of motion for the scalar field (18).

Not all equations are independent, indeed, Eq. (35) is a conservation law for the dynamical system. Specifically, the second-order differential equations (36)–(38) form a Hamiltonian dynamical system described by the point-like Lagrangian,

$$L\left(a,\dot{a},\phi,\dot{\phi},\psi,\dot{\psi}\right) = e^{\phi}\left(3a\dot{a}^2+\frac{\omega}{2}a^3\dot{\phi}^2+\frac{3}{2}a^3\dot{\phi}\dot{\psi}\right)+a^3V(\phi), \quad (39)$$

in which Eq. (35) is the constraint equation describing the conservation law of "energy" for the classical Hamiltonian system. Recall that for $\omega=0$, the latter Lagrangian reduces to that of $f(Q)$-gravity for the same connection.

We observe that for $\psi(t)=\psi_0$ and $\phi(t)=\phi_0$, the equations read

$$3H^2-e^{-\phi_0}V(\phi_0)=0, \quad (40)$$

$$2\dot{H}+3H^2-e^{-\phi_0}V(\phi_0)=0, \quad (41)$$

$$3H^2+\frac{1}{2}e^{-\phi_0}V(\phi_0)_{,\phi_0}=0, \quad (42)$$

where the limit of General Relativity is recovered when $-2V(\phi_0)=V(\phi_0)_{,\phi_0}$; that is, $V(\phi)=0$, or $V(\phi)=e^{-2\phi}$.

We consider the conformally related metric,

$$d\bar{s}^2=-\bar{N}^2(\tau)\,d\tau^2+\alpha^2(\tau)\times\left(dr^2+r^2\left(d\theta^2+\sin^2\theta d\phi^2\right)\right), \quad (43)$$

with $a(t)=\alpha(t)\,e^{-\frac{\phi(t)}{2}}$, $N(t)=\bar{N}(t)\,e^{-\frac{\phi(t)}{2}}$ and $d\tau=e^{-\frac{\phi(t)}{2}}dt$.

The field equations for the conformal equivalent theory (29) are

$$3\bar{H}^2-3\bar{H}\phi'+\frac{\bar{\omega}}{2}\phi'^2+\frac{3}{2}\phi'\psi'-e^{-2\phi}V(\phi)=0, \quad (44)$$

$$2\bar{H}'+3\bar{H}^2+3\bar{H}\phi'-\frac{\bar{\omega}}{2}\phi'^2-\frac{3}{2}\phi'\psi'-e^{-2\phi}V(\phi)=0, \quad (45)$$

$$2\bar{H}'+3\bar{H}^2+\frac{2}{3}e^{-2\phi}\left(V_{,\phi}-2V\right)-\left(\psi''+3\bar{H}\psi'\right)=0, \quad (46)$$

$$\phi''+3\bar{H}\phi'=0, \quad (47)$$

where we have assumed $\bar{N}(t)=1$, and $\alpha'=\frac{d\alpha}{d\tau}$, $\bar{H}(\tau)=\frac{\alpha'}{\alpha}$ or $\bar{H}(t)=e^{-\frac{\phi}{2}}\left(H+\frac{\dot{\phi}}{2}\right)$.

Last but not least, the point-like Lagrangian for the field equations is

$$L\left(\alpha,\alpha',\phi,\phi',\psi,\psi'\right)=3\alpha\alpha'^2-3\alpha^2\alpha'\phi'+\frac{\bar{\omega}}{2}\alpha^3\phi'^2+\frac{3}{2}\alpha^3\phi'\psi'+\alpha^3\bar{V}(\phi). \quad (48)$$

At this juncture, it is crucial to highlight that, for $\bar{\omega}=0$, the latter gravitational theory is equivalent to the non-metricity theory with boundary term, specifically with the $f(Q,B)=Q+f(B)$ theory of gravity [41].

5 Exact solutions

In this Section we determine the existence of exact solutions for the field equations that hold special significance. Additionally, we investigate the physical properties of the solutions within the conformal equivalent theory. Our focus is on determining the prerequisites for the existence of singular solutions, corresponding to universes dominated by an ideal gas, as well as identifying the conditions for a de Sitter solution. Subsequently, we utilize the conformal transformation to deduce the exact solution in the second frame, subsequently studying the physical properties and conducting a comparative analysis of the solutions between the two frames.

5.1 Singular solution in the Jordan frame

We assume the scaling solution, $a(t)=a_0t^p$, $H(t)=\frac{p}{t}$, for the cosmological model defined in the Jordan frame. This scale factor describes a universe dominated by an ideal gas with equation of state parameter $w_{eff}=\frac{2-3p}{3p}$. Thus, for $p=\frac{2}{3}$, the solution describes a universe dominated by a pressureless fluid, i.e. dust. For $p=\frac{1}{2}$ it describes a universe dominated by radiation. Moreover, for $p>1$ or $p<0$, the exact solution describes acceleration.

For the power-law singular solution $a(t)=a_0t^p$ and from the equation of motion (38) it follows that

$$\phi(t)=\phi_0+\ln\left(t^{1-3p}+\phi_1\right),\ p\neq\frac{1}{3} \quad (49)$$

and from the remaining of the field equations we derive

$$V(t)=\frac{e^{\phi_0}p(3p-1)\phi_1}{t^2} \quad (50)$$

and

$$\dot{\psi}=\frac{1}{3(3p-1)}\left(\frac{6p^2}{t}+2p\phi_1t^{3p-2}+\frac{(1-3p)^2\omega}{t\left(1+t^{3p-1}\right)\phi_1}\right),\ \phi_1\neq0, \quad (51)$$

$$\dot{\psi} = \frac{1}{3(3p-1)} \frac{\omega(6p-1) - 3p^2(2+3\omega)}{t}, \ \phi_1 = 0. \quad (52)$$

Hence, from (49) and (50), we can write the potential function as follows

$$V(\phi) = e^{\phi_0} p (3p-1) \phi_1 \left(e^{\phi-\phi_0} - \phi_1\right)^{\frac{2}{2p-1}}. \quad (53)$$

For the stiff fluid solution, i.e. $p = \frac{1}{3}$, we calculate

$$\phi(t) = \phi_0 + 2\left(\ln \frac{\sqrt{1+2\phi_1 t^2}}{t}\right), \ p = \frac{1}{3} \quad (54)$$

$$V(t) = \frac{4e^{\phi_0}\phi_1}{t^2}, \ \dot{\psi}(t) = \frac{3 + 2\omega + 4t^2\left(\phi_1^2 t^2 + 2\phi_1\right)}{3t\left(1 + 2\phi_1 t^2\right)}. \quad (55)$$

Therefore the scalar field potential is

$$V(\phi) = 4\phi_1\left(e^{\phi} - 2\phi_1 e^{\phi_0}\right). \quad (56)$$

5.1.1 Einstein frame

We proceed now with the derivation of the exact solution for the conformally related model in the Einstein frame. Therefore, after the application of the conformal transformation we derive

$$\alpha(t) = e^{\frac{\phi_0}{2}} t^p \sqrt{t^{1-3p} + \phi_1}, \ \bar{H}(t) = e^{-\frac{\phi_0}{2}} \left(\frac{1 - p + 2p\phi_1 t^{3p-1}}{2t^{3p}\left(t^{1-3p} + \phi_1\right)^{\frac{3}{2}}}\right), \ p \neq \frac{1}{3}. \quad (57)$$

The equation of state parameter for the effective fluid, $\bar{w}_{eff}(t) = -1 - \frac{2}{3}\frac{\bar{H}'}{\bar{H}^2}$, is determined

$$\bar{w}_{eff}(t) = \frac{3(1-p)t^{2-6p} - 4p\left((9p-5)\phi_1 t^{-3p} + (3p-2)\phi_1^2\right)}{\left(t^{3p+1}(1-p) + 2p\phi_1\right)^2}. \quad (58)$$

In the special limit for which $\phi_1 = 0$ and $V(\phi) = 0$, the latter expression becomes $\bar{w}_{eff} = 1$, and easily we can write the scale factor in terms of the new parameter τ as $\alpha(\tau) \simeq \tau^{\frac{1}{3}}$. Therefore for $\phi_1 = 0$, any scaling solution in the Jordan frame corresponds to the scaling solution which describes a stiff fluid in the Einstein frame.

On the other hand, for $\phi_1 \neq 0$ and for large values of t, it follows that $\bar{w}_{eff}(t) \simeq -1 + \frac{2}{3p}$ for $p > \frac{1}{3}$. This means that, in the asymptotic limit, the solution in the Einstein and in the Jordan frames has the same physical properties. We recall that $\tau(t \to \infty) \to \infty$ for $\phi_1 > 0$ and $p > \frac{1}{3}$. Hence, as far as we move from the singularity the two frames describe the same physical universe. In the contrary, near to the singularity, that is $t \to 0$, $\bar{w}_{eff}(t) \simeq 1$.

For $p = \frac{1}{3}$, the solution at the Einstein frame is

$$\alpha(t) = e^{\frac{\phi_0}{2}} \sqrt{t^2 + 2\phi_1}, \ \bar{H}(t) = 2\phi_1 e^{\frac{\phi_0}{2}} \frac{t^2}{\left(1 + 2t^2\phi_1\right)^{\frac{3}{2}}} \quad (59)$$

and

$$\bar{w}_{eff}(t) = -\frac{1}{3} - \frac{2\phi_1}{3t^2}. \quad (60)$$

Hence

$$\bar{H}(\alpha) = \frac{1}{3}\left(\frac{1}{\alpha} - \frac{e^{\phi_0}}{9\alpha^3}\right), \ \bar{w}_{eff}(\alpha) = 1 + \frac{12\alpha^2}{9a^2 - e^{\phi_0}}. \quad (61)$$

Thus, for large values of time, the asymptotic solution resembles that of a stiff fluid, similar to the scenario in the Jordan frame.

5.2 de Sitter universe in the Jordan frame

Consider now the de Sitter universe with $a(t) = a_0 e^{H_0 t}$, $H(t) = H_0$. Then from the field equations in the Jordan frame we derive

$$e^{\phi(t)} = e^{\phi_0}\left(1 - e^{-3H_0(t-\phi_1)}\right) \quad (62)$$

and

$$V(\phi) = 3e^{\phi_0} H_0^2, \ \dot{\psi} = \frac{H_0\left(2e^{3H_0 t} - e^{3H_0\phi_1}(3\omega + 2)\right)}{3\left(e^{3H_0 t} - e^{3H_0\phi_1}\right)}. \quad (63)$$

This means that the de Sitter solution exists for constant potential function $V(\phi)$.

5.2.1 Einstein frame

Now we transform the solutions in the Einstein frame. Indeed, the scale factor and the Hubble function becomes

$$\alpha(t) = \sqrt{e^{\phi_0}\left(e^{2H_0 t} - e^{-H_0 t} e^{3H_0\phi_1}\right)}, \quad (64)$$

$$\bar{H}(t) = H_0 \frac{e^{-\frac{\phi_0}{2}} e^{\frac{3}{2}H_0 t}\left(2e^{3H_0 t} + e^{3H_0\phi_1}\right)}{4\left(e^{3H_0 t} - e^{3H_0\phi_1}\right)^{\frac{3}{2}}} \quad (65)$$

while the effective equation of state parameter reads

$$\bar{w}_{eff}(t) = \frac{12e^{3H_0(t+\phi_1)} - 4e^{6H_0 t} + e^{6H_0\phi_1}}{2\left(e^{3H_0 t} + e^{3H_0\phi_1}\right)^2}. \quad (66)$$

Hence for large values of $t \to \infty$, it follows that $\bar{w}_{eff}(t) \simeq -1$, while for small values of $t \to 0$, we determine $\bar{w}_{eff}(t) \simeq 1 - \frac{24}{\left(2+e^{3H_0\phi_1}\right)^2} + \frac{8}{2+e^{3H_0\phi_1}}$, where for $e^{3H_0\phi_1} \to 0$, the limit $\bar{w}_{eff}(t) \simeq -1$ follows, and for $e^{3H_0\phi_1} \to \infty$ we derive $\bar{w}_{eff}(t) \simeq 1$.

5.3 Singular solution in the Einstein frame

Consider now the scaling solution $\alpha(\tau) = \alpha_0 \tau^q$, then from equations (44)-(47) we derive

$$\phi(\tau) = \bar{\phi}_0 + \frac{\bar{\phi}_1}{1-3q}\tau^{1-3q}\ ,\ q \neq \frac{1}{3}, \tag{67}$$

$$\psi' = -\frac{1}{3}\bar{\phi}_1\bar{\omega}\tau^{3q} - \frac{2q}{3\bar{\phi}_1\tau^2}\left(\tau^{3q} - 3\bar{\phi}_1\tau\right) \tag{68}$$

and

$$V(\tau) = \frac{q(3q-1)}{\tau^2}\exp\left(\bar{\phi}_0 + \frac{\bar{\phi}_1}{1-3q}\tau^{1-3q}\right), \tag{69}$$

or equivalently

$$V(\phi) = (3q-1)\left(\frac{\bar{\phi}_1}{(1-3q)}\right)^{\frac{2}{1-3q}}(\phi - \bar{\phi}_0)^{\frac{2}{3q-1}}. \tag{70}$$

In the case of $q = \frac{1}{3}$, the exact solution

$$\phi(\tau) = \bar{\phi}_0 + \bar{\phi}_1 \ln(\tau) \tag{71}$$

follows, that is,

$$\psi' = -\frac{2\left(1 - 3\bar{\phi}_1\right) + 3\left(\bar{\phi}_1\right)^2\bar{\omega}}{9\bar{\phi}_1\tau}\ ,\ V(\tau) = 0. \tag{72}$$

5.3.1 Jordan frame

For $q \neq \frac{1}{3}$, the solution at the Jordan frame is

$$a(\tau) = e^{-\phi_0}\exp\left(\frac{\bar{\phi}_1}{3q-1}\tau^{1-3q}\right)\tau^q\ , \tag{73}$$

$$H(\tau) = e^{\frac{\phi_0}{2}}\exp\left(\frac{\bar{\phi}_1}{3q-1}\tau^{1-3q}\right)\tau^{-1-3q}\left(q\tau^{3q} - \bar{\phi}_1\tau\right), \tag{74}$$

$$w_{eff}(\tau) = \frac{q(2-3q)\tau^{6q} + qt^{1+3q}\bar{\phi}_1 + 2\tau^2\left(\bar{\phi}_1\right)^2}{3\left(qt^{3q} - t\bar{\phi}_1\right)^2}. \tag{75}$$

We remark that $w_{eff}(\tau \to 0) \simeq -\frac{2}{3}$ and $w_{eff}(\tau \to \infty) \simeq -1 + \frac{2}{3q}$. Hence, far from the singularity, the physical properties of the solution remain unchanged under the influence of the conformal transformation.

The case $q = \frac{1}{3}$ was studied before. Thus we omit it.

5.4 de Sitter universe in the Einstein frame

For the exponential scale factor $\alpha(\tau) = \alpha_0 e^{\bar{H}_0\tau}$, from the field Eqs. (44)–(47) in the Einstein frame we determine the exact solution

$$\phi(\tau) = \bar{\phi}_0 - \frac{3}{\bar{H}_0}\bar{\phi}_1 e^{-3\bar{H}_0\tau},$$

$$V(\tau) = 3\bar{H}_0^2\exp\left(2\bar{\phi}_0 - \frac{2}{3}\bar{\phi}_1 e^{-3\bar{H}_0\tau}\right), \tag{76}$$

$$\psi' = 2\bar{H}_0 - \frac{\bar{\omega}}{3}\bar{\phi}_1 e^{-3\bar{H}_0\tau}. \tag{77}$$

Therefore, the scalar field potential is

$$V(\phi) = 3\bar{H}_0^2. \tag{78}$$

5.4.1 Jordan frame

Finally, in the Jordan frame the latter solution is

$$a(\tau) = \exp\left(\bar{H}_0 t - \bar{\phi}_0 + \frac{\bar{\phi}_1}{3\bar{H}_0}e^{-3\bar{H}_0 t}\right), \tag{79}$$

$$H(\tau) = \exp\left(-3\bar{H}_0 t + \frac{\bar{\phi}_0}{2} - \frac{\bar{\phi}_1}{6\bar{H}_0}e^{-3\bar{H}_0 t}\right)\left(\bar{H}_0 e^{3\bar{H}_0 t} - \bar{\phi}_1\right) \tag{80}$$

and

$$w_{eff}(\tau) = -\frac{3e^{6\bar{H}_0 t}\bar{H}_0^2 + e^{3\bar{H}_0 t}\bar{H}_0\bar{\phi}_1 + 2\left(\bar{\phi}_1\right)^2}{3\left(\bar{H}_0 e^{3\bar{H}_0 t} - \bar{\phi}_1\right)^2}. \tag{81}$$

From these expressions we have the limits $w_{eff}(\tau \to 0) = -\frac{3\bar{H}_0^2 + \bar{H}_0\bar{\phi}_1 + (\bar{\phi}_1)^2}{3(\bar{H}_0 - \bar{\phi}_1)}$ and $w_{eff}(\tau \to \infty) = -1$. We conclude that the de Sitter universe is the asymptotic solution in the two frames.

The above discussion highlights that the solutions exhibit identical physical properties in both the Jordan and Einstein frames at the asymptotic limits. This observation is significant and sets it apart from the scalar-curvature or scalar-torsion theories of gravity, where such equivalence does not hold true.

6 Analytic solution

In the preceding Section, we explored the existence of exact solutions for the field equations. The derived solutions exhibit fewer degrees of freedom compared to the original dynamical system, rendering them special or asymptotic solutions. Subsequently, we proceed to establish the analytic solution for the field equations. Specifically, for the Brans–Dicke field with the potential function $V(\phi) = V_0 \exp((\lambda - 1)\phi)$, we derive the analytic solution for the field Eqs. (35)–(38). The field equations form a three-dimensional Hamiltonian system with six degrees of freedom, enabling the application of the Hamilton–Jacobi method to simplify the field equations and to construct the analytic solution.

We consider the point transformation

$$\ln a = \frac{1}{6}u\ ,\ \phi = \Phi - \frac{u}{\lambda},\ \psi = \psi, \tag{82}$$

in which the Lagrangian function of the field equations is

$$L\left(N,u,\dot{u},\Phi,\dot{\Phi},\psi,\dot{\psi}\right)=\frac{\exp\left(\frac{\lambda-2}{2\lambda}u+\Phi\right)}{12\lambda^{2}N}\left(\left(\lambda^{2}-6\right)\dot{u}^{2}\right.$$
$$\left.-6\lambda^{2}\dot{\Phi}\left(\omega\dot{\Phi}+3\dot{\psi}\right)+6\lambda\left(2\omega\dot{\Phi}+3\dot{\psi}\right)\right)$$
$$-V_{0}N\exp\left(\exp\left(\frac{\lambda-2}{2\lambda}u+(\lambda-1)\Phi\right)\right). \quad (83)$$

We have considered the lapse function $N(t)$ to be a non-constant function, we see below that this necessary in order to write the closed-form solution of the field equations.

From Lagrangian function (83) we can define the momentum

$$p_{u}=\frac{\partial L}{\partial\dot{u}},\quad p_{\Phi}=\frac{\partial L}{\partial\dot{\Phi}},\quad p_{\psi}=\frac{\partial L}{\partial\dot{\psi}}, \quad (84)$$

that is,

$$\dot{u}=-\frac{3N}{\lambda}\exp\left(\frac{\lambda-2}{2\lambda}u-\Phi\right)\left(\lambda p_{u}+p_{\Phi}\right), \quad (85)$$

$$\dot{\Phi}=-\frac{N}{3\lambda^{2}}\exp\left(\frac{\lambda-2}{2\lambda}u-\Phi\right)\left(9\left(\lambda p_{u}+p_{\Phi}\right)+\lambda^{2}p_{\psi}\right), \quad (86)$$

$$\dot{\psi}=-\frac{N}{9}\exp\left(\frac{\lambda-2}{2\lambda}u-\Phi\right)\left(3p_{\Phi}-2\omega p_{\psi}\right). \quad (87)$$

Therefore, the Hamiltonian function $\mathcal{H}=p_{q}\frac{\partial L}{\partial\dot{q}}-L$ can be written

$$\mathcal{H}\equiv N\exp\left(\frac{\lambda-2}{2\lambda}u-\Phi\right)\left(36V_{0}\lambda^{2}e^{\lambda\Phi}\right.$$
$$\left.-27\left(\lambda p_{u}+p_{\Phi}\right)^{2}-6\lambda^{2}p_{\Phi}p_{\psi}+2\lambda^{2}\omega p_{\Phi}^{2}\right)=0, \quad (88)$$

where $\mathcal{H}=0$, follows from the constraint equation (35).

Consequently, Hamilton's equations are

$$\dot{p}_{u}=0\,,\ \dot{p}_{\psi}=0 \quad (89)$$

and

$$\dot{p}_{\Phi}=2V_{0}\lambda e^{\lambda\Phi}N\exp\left(\frac{\lambda-2}{2\lambda}u-\Phi\right), \quad (90)$$

from which we infer that p_u and p_ψ are constants, that is $p_u=p_u^0$, and $p_\psi=p_\psi^0$.

Let $S=S(u,\Phi,\psi)$ be the Action, then from (88) we can write the Hamilton–Jacobi equation

$$\left(36V_{0}\lambda^{2}e^{\lambda\Phi}-27\left(\lambda\frac{\partial S}{\partial u}+\frac{\partial S}{\partial\Phi}\right)^{2}\right.$$
$$\left.-6\lambda^{2}\frac{\partial S}{\partial\Phi}\frac{\partial S}{\partial\psi}+2\lambda^{2}\omega\left(\frac{\partial S}{\partial\Phi}\right)^{2}\right)=0. \quad (91)$$

Moreover, from (89) it follows that $S(u,\Phi,\psi)=p_u^0u+p_\psi^0\psi+\hat{S}(\Phi)$, that is,

$$\left(36V_{0}\lambda^{2}e^{\lambda\Phi}-27\left(\lambda p_{u}^{0}+\hat{S}_{,\Phi}\right)^{2}\right.$$
$$\left.-6\lambda^{2}p_{\psi}^{0}\hat{S}_{,\Phi}+2\lambda^{2}\omega\left(\hat{S}_{,\Phi}\right)^{2}\right)=0. \quad (92)$$

Therefore

$$p_{\Phi}\equiv\hat{S}_{,\Phi}=-\lambda\left(p_{u}^{0}+\frac{\lambda}{9}p_{\psi}^{0}\right)$$
$$\pm\frac{|\lambda|}{9}\sqrt{108V_{0}e^{\lambda\Phi}+p_{\psi}^{0}\left(18\lambda p_{u}^{0}+p_{\psi}^{0}\left(\lambda^{2}+6\omega\right)\right)}. \quad (93)$$

Using the above mentioned expression, we can derive the action $\hat{S}(\Phi)$. The field Eqs. (85)–(87) are reduced to the following dynamical system

$$\frac{1}{N}\dot{u}=-\frac{3}{\lambda}\exp\left(\frac{\lambda-2}{2\lambda}u-\Phi\right)\left(\lambda p_{u}^{0}+p_{\Phi}\right), \quad (94)$$

$$\frac{1}{N}\dot{\Phi}=-\frac{1}{3\lambda^{2}}\exp\left(\frac{\lambda-2}{2\lambda}u-\Phi\right)\left(9\left(\lambda p_{u}^{0}+\hat{S}_{,\Phi}\right)+\lambda^{2}p_{\psi}^{0}\right), \quad (95)$$

$$\frac{1}{N}\dot{\psi}=-\frac{1}{9}\exp\left(\frac{\lambda-2}{2\lambda}u-\Phi\right)\left(3\hat{S}_{,\Phi}-2\omega p_{\psi}^{0}\right). \quad (96)$$

We consider the new independent variable to the scalar field Φ, such that $u=u(\Phi)$ and $\psi=\psi(\Phi)$. Thus, the analytic solution is expressed in terms of the closed-form functions

$$u(\Phi)=u_{0}+\lambda\Phi+\frac{2\lambda\sqrt{p_{\psi}^{0}}}{\sqrt{18p_{u}^{0}\lambda+p_{\psi}^{0}\lambda^{2}+6p_{\psi}^{0}\omega}}$$
$$\times\arctan h\left(\sqrt{\frac{108V_{0}e^{\lambda\Phi}+p_{\psi}^{0}\left(18\lambda p_{u}^{0}+p_{\psi}^{0}\left(\lambda^{2}+6\omega\right)\right)}{\sqrt{p_{\psi}^{0}\left(18p_{u}^{0}\lambda+18p_{u}^{0}\lambda+p_{\psi}^{0}\lambda^{2}+6p_{\psi}^{0}\omega\right)}}}\right) \quad (97)$$

and

$$\psi(\Phi)=\psi_{0}+\frac{\lambda}{9}\left(\ln\left(108V_{0}e^{\lambda\Phi}\right)\right.$$
$$\left.+\frac{9p_{u}^{0}\lambda+p_{\psi}^{0}\lambda^{2}+6p_{\psi}^{0}\omega}{p_{\psi 0}\lambda^{2}}\left(u(\Phi)-u_{0}-\lambda\Phi\right)\right). \quad (98)$$

The Hubble function and the equation of state parameter w_{eff} are expressed as

$$H(\Phi)=\frac{\dot{\Phi}}{N}\left(\frac{1}{a}\frac{da}{d\Phi}\right),\quad w_{eff}(\Phi)=-1-\frac{2}{3H^{2}}\frac{\dot{\Phi}}{N}\frac{dH}{d\Phi}. \quad (99)$$

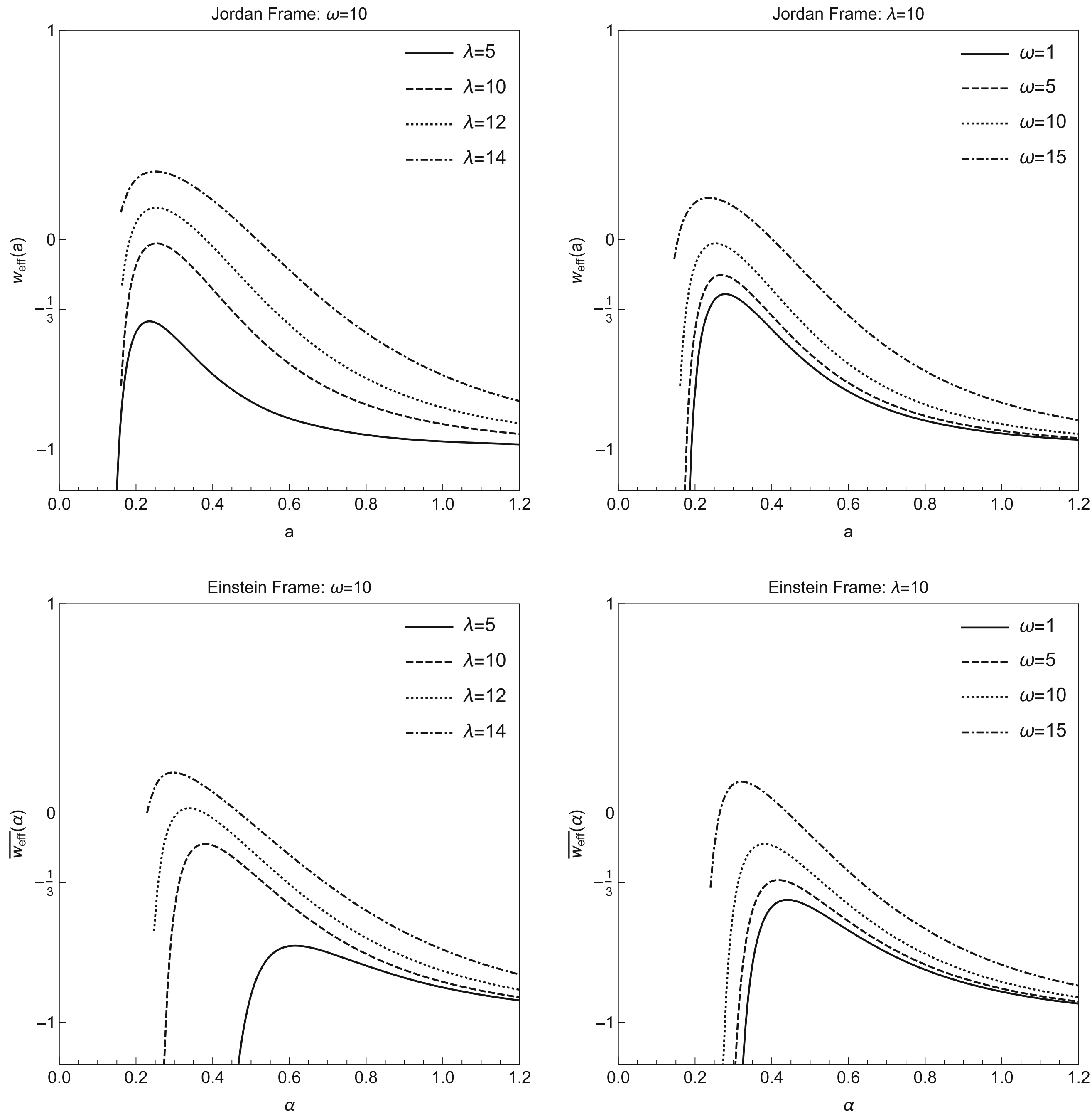

Fig. 1 Qualitative evolution of the effective equation of state parameter in the Jordan frame $w_{eff}\left(\Phi\left(a\right)\right)$ and in the Einstein frame $\bar{w}_{eff}\left(\Phi\left(\alpha\right)\right)$ for different values of the free parameters. For all the plots we consider the initial conditions $\left(p_u^0, p_\psi^0, u_0\right) = (1, 0.8, -10)$, and $V_0 = 1$. We observe that the behaviour for the equation of state parameter is similar in the two frames and the de Sitter solution is a common future solution

From (93) and (95) it follows that

$$\frac{1}{N}\dot{\Phi} = -\frac{\pm|\lambda|}{27\lambda^2}\exp\left(\frac{\lambda-2}{2\lambda}u-\Phi\right)\times\sqrt{108V_0e^{\lambda\Phi}+p_\psi^0\left(18\lambda p_u^0+p_\psi^0\left(\lambda^2+6\omega\right)\right)} \tag{100}$$

which means that Φ is a monotonically function, as long as N does not change sign. Indeed, for $N=1$, Φ is a monotonically function.

Figure 1 illustrates the qualitative evolution of the equation of state parameter, $w_{eff}\left(a\right)$, for the above mentioned analytical solution, considering various values of the free parameters. Additionally, we calculate and display the evolution of the equation of state parameter $\bar{w}_{eff}(\alpha)$ for the conformal equivalent theory as defined in the Einstein frame. The plots in both frames utilize identical values for the free parameters, reflecting corresponding initial conditions.

It is observed that the universe initiates from a big rip singularity, subsequently progresses towards a saddle point characterized by an ideal gas, representing the matter-dominated era and finally transitions to the de Sitter point. This behaviour is consistent across solution trajectories in both frames, mirroring the findings for the asymptotic solutions in the preceding section. While previously, the resemblance in the evolution of physical parameters was noted at the asymptotic limits, Fig. 1 demonstrates that this similarity persists throughout the global evolution of the cosmological solution. The Big Rip behaviour observed before for the case of $f(Q)$-theory in [26] and in scalar-nonmetricity theory [60].

7 Conclusions

We performed an extensive analysis on the influence of the conformal transformation on the physical properties of cosmological solution trajectories within symmetric teleparallel gravity's conformal equivalent theories. To undertake this analysis, we introduced the Brans–Dicke model in the context of non-metricity gravity, alongside an analogue of the Brans–Dicke parameter. Notably, when this parameter approaches zero, the non-metricity scalar–tensor theory is reduced to the $f(Q)$-theory.

Regarding the background geometry, we focused on the isotropic and homogeneous spatially flat FLRW metric. Concerning the theory's connection, we specifically examined a connection defined within the non-coincidence gauge. It is worth recalling that in the coincidence gauge, the cosmological field equations simplify to those of scalar-torsion theory, limiting the new information that could be deduced from this study. For this particular cosmological model, we derived the field equations in both the Jordan and the Einstein frames.

In scalar-curvature theory conformal transformation has been used as an approach to avoid singularity [62] because singular solutions in the one frame can correspond to non-singular solutions for the other frame and vice versa [63–65] and there is not necessary an one-to-one correspondence for the physical properties of the solutions under a conformal transformation. For instance in [66] specific type singularities change type under the conformal transformation. On the other hand in [67] the authors focus on the effects of the conformal transformation on power-law inflationary solutions in scalar-curvature theory, they found that " inflation " depends on the frame selection. By comparing the phase-space of the cosmological equations for the scalar-curvature theory in the Jordan and in the Einstein frame we can see that there are differences on the asymptotic behaviours [68,69]. A similar conclusions follows and by comparing the results of the phase-space analysis for the scalar-torsion gravity [70,71].

In scalar-nonmetricity theory, we derived exact solutions of particular significance in one frame, illustrating both singular and non-singular solutions. Subsequently, we utilized the conformal transformation to reconstruct the exact solutions for the conformal equivalent theory. Our analysis involved a thorough comparison of the physical properties for the two theories, each defined within different frames. Notably, we discovered that the physical properties remained invariant under the influence of the conformal transformation. We focused on power-law and exponential solutions for the scale factor. These two solutions describe the asymptotic behaviour of any potential function. Hence, from the results of this work we can make conclusions about the asymptotic behaviour of the general solutions between the two frames.

Consequently, singular solutions in one frame corresponded to singular solutions in the other frame, displaying similar properties in the asymptotic limit. Furthermore, we observed that the non-singular de Sitter solution remained a de Sitter solution in the alternate frame as well. Furthermore, we constructed for the fist time an analytic solution for the cosmological field equations in non-metricity scalar–tensor theory. This solution describes an cosmological model with Big Rip singularity, which involves to a matter dominated solution and the final state of the universe is that of the de Sitter universe. Surprisingly this specific cosmological history describes the conformal equivalent theory. Hence, the physical equivalence of the physical solutions between the two frames extends the asymptotic limits of the solutions.

In a future study we plan to investigate further such analysis by investigate the case of compact objects.

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