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The equivalence principle for a plane gravitational wave in torsion-based and non-metricity-based teleparallel equivalents of general relativity

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Abstract We study the energy–momentum characteristics of the plane "+"-polarized gravitational wave solution of general relativity in the teleparallel equivalent of general relativity (TEGR) and the symmetric teleparallel equivalent of general relativity (STEGR) using the previously constructed Noether currents. The current components describe energy–momentum locally measured by an observer if the displacement vector ξ is equal to the observer's 4-velocity. To determine the non-dynamical connection in these theories, we use the unified "turning off" gravity principle. For a constructive analysis of the values of Noether currents and superpotentials in TEGR and STEGR, we use the concept of "gauges". The gauge changing can affect the Noether current values. We study under what conditions the Noether current for the freely falling observer is zero. When they are established, the zero result can be interpreted as a correspondence to the equivalence principle, and it is a novelty for gravitational waves in the TEGR and STEGR. We highlight two important cases with positive and zero energy, which reproduce the results of previous works with a different approach for determining gravitational energy–momentum in the TEGR, and give their interpretation.

1 Introduction

Teleparallel theories of gravity have been actively developed in recent years. One of important features of these theories is the use of teleparallel connections with zero Riemann curvature. The notion of a teleparallel connection (flat connection) unites such notions as inertial spin connection, Weitzenböck connection, and symmetric teleparallel affine connection, which are more appropriate in any concrete case. These theories include the teleparallel equivalent of general relativity (TEGR), symmetric teleparallel equivalent of general relativity (STEGR), and modifications of these theories [1–6]. In the TEGR and its modifications, a flat metric-compatible connection with non-zero torsion is used. In the STEGR and its modifications, a flat metric-non-compatible connection with zero torsion is used. The TEGR and STEGR are fully equivalent to general relativity (GR) at the level of field equations; thus, the solutions of the field equations in TEGR and STEGR are exactly the same as those in GR. In many cases, modifications of the TEGR and STEGR have the advantage that their field equations are of the second order, which gives similarities with gauge field theories and potentially links gravity to other theories of fundamental interactions in nature.

Because the field equations in the TEGR and STEGR are equivalent to those in GR, they do not include the teleparallel connections in whole. This follows formally from the fact that the TEGR and STEGR Lagrangians differ from the Hilbert Lagrangian by divergences, which only include teleparallel connections; see [7] and formulae (2.9) and (2.33) below. Thus, varying the TEGR and STEGR Lagrangians with respect to the teleparallel connections, one obtains the identities $0 = 0$, which do not permit one to determine them. In another words, the teleparallel connections in the TEGR and STEGR are not dynamical quantities and do not influence the dynamics of gravitationally interacting objects.

Teleparallel connections (as external fields) can be used in order to represent the TEGR equations in an evidently covariant form with respect to Lorentz rotations of tetrads (where coordinate covariance already holds), or to represent the STEGR equations in an evidently coordinate-covariant

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form with respect to symmetric affine teleparallel connection. As usual, covariantized equations are used to construct conserved quantities; see, for example, the book [3]. However, such a method has considerable difficulties, which are described in detail in the TEGR framework in the conference presentation [8] by Krššák, where the results of previous studies of many authors are summarized. Let us list Krššák's more important requirements for constructing energy–momentum in the TEGR. It must (1) be of the first derivatives only, (2) be covariant with respect to both coordinate transformations and local Lorentz rotations, and (3) permit the construction of global (integral) conserved quantities, or conserved charges. Most of the known variants of energy–momentum in the TEGR satisfy the first requirement. However, requirements (2) and (3) are not consistent as a rule. On the one hand, one has well-defined conserved charges expressed through well-defined surface integrals, but one has only Lorentz non-covariant conserved energy–momentum. On the other hand, one can construct Lorentz-covariant conserved energy–momentum, but then the related conserved charges cannot be defined.

In the framework of the TEGR, in papers [9,10], using Noether's theorem, the aforementioned problem is resolved, and fully covariant conserved quantities are constructed in the formalism of differential forms. However, this formalism is not so popular, and unfortunately, the papers [9,10] did not receive relevant development. In [11,12], we constructed fully covariant conserved quantities in the TEGR (including well-defined charges) in the tensorial formalism by direct application of the Noether theorem. Success was achieved thanks to including the teleparallel connection (inertial spin connection) and preserving displacement vectors in expressions after applying the Noether theorem to the diffeomorphically invariant TEGR action. Choosing displacement vectors as proper vectors of observers, or Killing spacetime vectors, etc., one defines a character of conserved quantities. Our approach differs from others in that the observer is associated with a displacement vector rather than a time-like tetrad vector.

Concerning the STEGR, we do not find an extensive study in construction of conserved quantities. Nevertheless, in [13,14], covariant Noether conserved quantities were constructed for general metric-affine gravity including the TEGR and STEGR. However their correspondence to physically expected values was not studied. In [15], following the ideology of [11,12], we constructed covariant conserved quantities as well.

The new expressions for conserved quantities constructed in [11,12,15] have been used in various applications. It was shown that Noether's current corresponds to the weak equivalence principle for the freely falling observers "frozen" into the Hubble flow in the Friedmann–Lemaître–Robertson–Walker universe and (anti-)de Sitter space, and Noether's charge gives correct mass for the Schwarzschild black hole. In [16,17], in the TEGR, the new Noether's conserved quantities were studied in more detail, and ambiguities of a special character were clarified as follows. These conserved quantities evidently contain teleparallel connections, and they cannot be suppressed as in field equations in whole. But they are not dynamic variables of the theory and cannot be defined inside the TEGR and STEGR themselves. Authors typically use a principle of "turning off" ("switching off") gravity; see, for example, [18,19]. In [11,12], we generalized this principle in the TEGR. In [15], we used it in the STEGR, although it was not formulated there. In fact, we formulate it here.

It turns out that even the generalized principle of "turning off" gravity cannot determine teleparallel connections by a unique way of suppressing ambiguities. In the TEGR, we study such a problem in more detail in [16,17] on the example of the Schwarzschild solution. It was considered (1) a static tetrad with a related teleparallel connection and (2) a freely falling tetrad with another teleparallel connection. In the first case, static observers at spacelike infinity measure the standard mass of the black hole, and in the second case, freely falling observers measure zero energetic characteristics that correspond to the weak equivalence principle. Thus, in both cases, one has acceptable results. However, if one considers a freely falling observer together with the first pair of tetrad and connection, and if one considers a static distant observer together with the second pair of tetrad and connection, one does not obtain acceptable results in either of the cases. Thus, it can be concluded that each of the concrete tasks requires the appropriate pair of tetrad and teleparallel connection.

Møller [20], constructing covariant energy–momentum for a gravitational field in the tetrad form instead of pseudotensors, clarified that, being coordinate-covariant, it is not covariant with respect to local Lorentz rotations. This problem has been resolved when teleparallel connection (inertial spin connection) is taken into consideration; however, the problems accented by Krššák [8] appear. Such problems have been resolved as well [9–12,15], but the problem of ambiguity in determination of teleparallel connection remains. How can it be studied? Each concrete solution and each concrete task require a separate approach. For example, in [21–24], the authors consider solutions with spherical, cylindrical, and cosmological symmetries in modified theories (it can be easily adapted to TEGR and STEGR), and for each of the symmetries, they develop a concrete derivation. Such a situation is analogous to the consideration in the standard metric presentation of GR. For example, Katz, Bichak, and Lynden-Bell [25] suggested a bi-metric representation of GR where conserved quantities become covariant ones instead of classical pseudotensors. A non-covariance of the latter is connected with non-localizability of energy, momentum, etc., in GR [26]. However, covariant quantities in [25] essentially depend on the choice of a background metric that is another manifestation of the non-localizability. As a result, a background metric has to be chosen for concrete solutions separately. A detailed comparison of the approach in [25] with constructions in the TEGR, where a role of the background metric is played by teleparallel connections, is given in [11]. Thus, a reasonable (possible) interpretation of problems appearing with a choice of teleparallel connections in the TEGR and STEGR can be connected with the non-localizability of conserved quantities in GR and cannot be avoided in principle.

In the literature, various approaches for constructing conserved quantities in both TEGR and STEGR and their modifications have already been tested for the Schwarzschild solution and cosmological models; see, for example, [10, 14, 18, 27–32] and references therein. However, to the best of our knowledge, construction of conserved quantities for the gravitational waves in teleparallel gravity has attracted little attention among researchers. We could cite the central papers with such study only in the TEGR [33–37] with a few references therein. In calculations, the energy–momentum tensor of gravitational field [3] is used there. In these studies, the authors sometimes obtained unexpected results. Thus, for example, in [34], non-positive energy for gravitational waves is obtained; on the other hand, in [35], zero energy for gravitational waves is obtained. Such results are not acceptable; indeed, the modern cosmological and astrophysical observable data support the textbook predictions that gravitational waves result in positive energy [38–40]. Lastly, we have not found any works in the literature where the problem of constructing conserved quantities for gravitational waves has been considered in the STEGR.

To define energy and momentum densities of a gravitational wave (or charges for it), one has to introduce observers, whose proper vectors are chosen to be corresponding displacement vectors. However, it is not so clear how one can provide this study in the formalism under consideration. Indeed, the gravitational wave spacetime has no timelike Killing vector, and one cannot construct charges for static distant observers because it is not clear how they have to be defined. Nevertheless, in Sect. 3, we discuss the problem of constructing energy and momentum densities of a gravitational wave in the framework of our fully covariant formalism and provide a related interpretation that differs from the interpretation in [36, 37].

The importance of defining a correspondence to the equivalence principle for various solutions has been noted many times; see, for example, the recent paper [41] and references therein. For black holes and cosmological models, such a correspondence was noted, for example, in [11, 15, 16, 28, 42]. However, up to now, no correspondence to the equivalence principle for gravitational wave solutions has been described (although such attempts have been made [36, 37]).

Unlike determining a timelike Killing vector for the gravitational wave, or determining a distant observer for the gravitational wave filling infinite space, it is easy to take a displacement vector as a freely falling observer's proper vector. Then it remains only to determine the teleparallel connections that give the Noether current corresponding to the equivalence principle. Such a task can be resolved by corresponding efforts. Thus, the more important purpose of this paper is to achieve a correspondence with the equivalence principle. We consider only a plane exact gravitational wave in the TEGR and STEGR with only the one "+" polarization, and making use of the fully covariant formalism that we developed [11, 15, 16].

The paper is organized as follows:

In Sect. 2, we briefly introduce the TEGR and STEGR theories, introduce the Noether currents and superpotentials for these theories, and describe the unified "turning off" gravity principle to define the teleparallel connection.

In Sect. 3, we calculate energy and momentum for the plane "+" polarized gravitational wave measured by the freely falling observers in the TEGR and STEGR in the simplest frames. In this case, we obtain a result that does not correspond to the equivalence principle. In the framework of the formalism [11, 15, 16], such a result can be explained as the fact that the teleparallel connections found do not correspond to the equivalence principle for the chosen freely falling observers. However, at least in linear approximation, the result coincides with the Landau–Lifshitz formula for the energy density and energy density flux. This coincidence is interpreted.

In Sect. 4, we set a task to find teleparallel connections, the substitution of which into the Noether current expression for freely falling observers in TEGR and STEGR gives zero. This goal is achieved, and thus a correspondence with the equivalence principle is stated.

In Sect. 5, using the results of [35], we take other coordinates (not the simplest ones) for the plane "+" polarized gravitational wave, after which the equivalence principle for freely falling observers is established in another way.

In Sect. 6, we discuss the results.

In Appendix A related to Sect. 4.1, we state that in the case of Lorentz rotations depending only on retarded time, there are no compositions of such rotations, which can change the Noether current.

In Appendix B related to Sect. 5, we show that the tetrad considered in [35] is not a freely falling tetrad, and thus the zero result in [35] is not in correspondence with the equivalence principle.

All definitions in the TEGR correspond to [3], and all definitions in the STEGR correspond to [1].

2 Preliminaries

In this section, we briefly introduce the elements of the teleparallel equivalents of GR (TEGR and STEGR) and, following our papers [12,15,16], give totally covariant expressions for conserved quantities, which are necessary for our calculations.

2.1 Elements of the TEGR

The Lagrangian in the TEGR has a form [3]

$$\overset{\bullet}{\mathcal{L}} = \frac{h}{2\kappa}\overset{\bullet}{T} \equiv \frac{h}{2\kappa}\left(\frac{1}{4}\overset{\bullet}{T}{}^{\rho}{}_{\mu\nu}\overset{\bullet}{T}{}_{\rho}{}^{\mu\nu} + \frac{1}{2}\overset{\bullet}{T}{}^{\rho}{}_{\mu\nu}\overset{\bullet}{T}{}^{\nu\mu}{}_{\rho} - \overset{\bullet}{T}{}^{\rho}{}_{\mu\rho}\overset{\bullet}{T}{}^{\nu\mu}{}_{\nu}\right), \tag{2.1}$$

where $\overset{\bullet}{T}$ is referred to as the torsion scalar, and $\kappa = 8\pi$ in $c = G = 1$ units. Since the main goal of the present paper is to consider gravitational waves in vacuum only, we do not add to (2.1) a matter Lagrangian. The torsion tensor $\overset{\bullet}{T}{}^{a}{}_{\mu\nu}$ is defined as

$$\overset{\bullet}{T}{}^{a}{}_{\mu\nu} = \partial_\mu h^a{}_\nu - \partial_\nu h^a{}_\mu + \overset{\bullet}{A}{}^{a}{}_{c\mu}h^c{}_\nu - \overset{\bullet}{A}{}^{a}{}_{c\nu}h^c{}_\mu, \tag{2.2}$$

where $h^a{}_\nu$ are the tetrad components connected to the metric by

$$g_{\mu\nu} = \eta_{ab}h^a{}_\mu h^b{}_\nu \tag{2.3}$$

and $h = \det h^a{}_\nu$. We denote tetrad indexes of a quantity by Latin letters and spacetime indexes by Greek letters. It is useful to recall that the transformation of tetrad indexes into spacetime indexes and vice versa is performed by contraction with tetrad vectors, for example, $\overset{\bullet}{T}{}^{\alpha}{}_{\mu\nu} = h^\alpha{}_a\overset{\bullet}{T}{}^{a}{}_{\mu\nu}$.

Henceforth, we denote by $\bullet$ all teleparallel quantities which are constructed using the teleparallel affine connection (Weitzenböck connection) $\overset{\bullet}{\Gamma}{}^{\alpha}{}_{\kappa\lambda}$. Thus, the inertial spin connection (ISC) $\overset{\bullet}{A}{}^{a}{}_{c\nu}$ is defined as

$$\overset{\bullet}{A}{}^{a}{}_{b\mu} = -h_b{}^\nu\,\overset{\bullet}{\nabla}_\mu\,h^a{}_\nu, \tag{2.4}$$

where the covariant derivative $\overset{\bullet}{\nabla}_\mu$ corresponds to $\overset{\bullet}{\Gamma}{}^{\alpha}{}_{\kappa\lambda}$. The connection $\overset{\bullet}{\Gamma}{}^{\alpha}{}_{\kappa\lambda}$ is flat, i.e., the related curvature is equal to zero:

$$\overset{\bullet}{R}{}^{\alpha}{}_{\beta\mu\nu} \equiv \partial_\mu\,\overset{\bullet}{\Gamma}{}^{\alpha}{}_{\beta\nu} - \partial_\nu\,\overset{\bullet}{\Gamma}{}^{\alpha}{}_{\beta\mu} + \overset{\bullet}{\Gamma}{}^{\alpha}{}_{\kappa\mu}\,\overset{\bullet}{\Gamma}{}^{\kappa}{}_{\beta\nu} - \overset{\bullet}{\Gamma}{}^{\alpha}{}_{\kappa\nu}\,\overset{\bullet}{\Gamma}{}^{\kappa}{}_{\beta\mu} = 0. \tag{2.5}$$

The connection $\overset{\bullet}{\Gamma}{}^{\alpha}{}_{\kappa\lambda}$ is also compatible with the physical metric, that is, the corresponding non-metricity is zero:

$$\overset{\bullet}{Q}{}_{\mu\alpha\beta} \equiv \overset{\bullet}{\nabla}{}_{\mu}g_{\alpha\beta} = 0. \tag{2.6}$$

In our notations, following [3], quantities denoted by a $\circ$ are constructed using the Levi–Civita affine connection $\overset{\circ}{\Gamma}{}^{\alpha}{}_{\kappa\lambda}$. Thus, $\overset{\circ}{A}{}^{a}{}_{b\rho}$ is the usual Levi–Civita spin connection (L-CSC) defined by

$$\overset{\circ}{A}{}^{a}{}_{b\mu} = -h_b{}^\nu\,\overset{\circ}{\nabla}_\mu\,h^a{}_\nu, \tag{2.7}$$

where the covariant derivative $\overset{\circ}{\nabla}_\mu$ is constructed with $\overset{\circ}{\Gamma}{}^{\alpha}{}_{\kappa\lambda}$. Now it is useful to introduce a contortion tensor defined as

$$\overset{\bullet}{K}{}^{a}{}_{b\rho} = \overset{\bullet}{A}{}^{a}{}_{b\rho} - \overset{\circ}{A}{}^{a}{}_{b\rho}. \tag{2.8}$$

Let us discuss the role of the ISC incorporated into the torsion tensor (2.2). From the beginning, let us rewrite the Lagrangian (2.1) in the other form [3]:

$$\overset{\bullet}{\mathcal{L}} = \overset{\circ}{\mathcal{L}} - \frac{1}{\kappa}\partial_\mu\left(h\overset{\bullet}{T}{}^{\nu\mu}{}_{\nu}\right), \tag{2.9}$$

with the Hilbert Lagrangian

$$\overset{\circ}{\mathcal{L}} = -\frac{h}{2\kappa}\,\overset{\circ}{R}, \tag{2.10}$$

where $\overset{\circ}{R}$ is the Riemannian curvature scalar presented as a function of tetrad components in correspondence with (2.3); see [43]. Thus, the TEGR Lagrangian contains the ISC in the divergence only. Then, first, varying the action with the Lagrangian (2.1), the same (2.9), with respect to tetrad components, one obtains the Euler–Lagrange equation

$$E_a{}^\rho \equiv \frac{\delta\,\overset{\bullet}{\mathcal{L}}}{\delta h^a{}_\rho} \equiv \frac{\partial\,\overset{\bullet}{\mathcal{L}}}{\partial h^a{}_\rho} - \partial_\sigma\left(\frac{\partial\,\overset{\bullet}{\mathcal{L}}}{\partial h^a{}_{\rho,\sigma}}\right) = 0, \tag{2.11}$$

where $E_a{}^\rho$ does not actually depend on the ISC because

$$E_a{}^\rho \equiv \frac{\delta\,\overset{\bullet}{\mathcal{L}}}{\delta h^a{}_\rho} \equiv \frac{\delta\,\overset{\circ}{\mathcal{L}}}{\delta h^a{}_\rho} = 0. \tag{2.12}$$

This also means that the TEGR and GR are equivalent. Second, varying the action with the Lagrangian (2.1), the same (2.9), with respect to $\overset{\bullet}{A}{}^{a}{}_{b\rho}$ (that is included in the divergence only), one obtains $0 = 0$. Thus, the ISC cannot be determined in the framework of the TEGR itself. Then, being an external structure, it can be defined by additional requirements only.

However, what is the role of $\overset{\bullet}{A}{}^{a}{}_{b\rho}$? At least its presence allows us to present the torsion tensor in a covariant form with respect to local Lorentz rotations (see (2.2)). This form enables us to represent the following tensors in fully and evidently covariant form. The contortion tensor can be rewritten in the convenient form:

$$\overset{\bullet}{K}{}^{\rho}{}_{\mu\nu} = \frac{1}{2}(\overset{\bullet}{T}{}_{\mu}{}^{\rho}{}_{\nu} + \overset{\bullet}{T}{}_{\nu}{}^{\rho}{}_{\mu} - \overset{\bullet}{T}{}^{\rho}{}_{\mu\nu}). \tag{2.13}$$

The torsion scalar in (2.1) can be rewritten in the form

$$\overset{\bullet}{T} = \frac{1}{2}\overset{\bullet}{S}_a{}^{\rho\sigma}\overset{\bullet}{T}{}^a{}_{\rho\sigma}, \tag{2.14}$$

where the teleparallel superpotential $\overset{\bullet}{S}_a{}^{\rho\sigma}$ defined as

$$\overset{\bullet}{S}_a{}^{\rho\sigma} = \overset{\bullet}{K}{}^{\rho\sigma}{}_a + h_a{}^\sigma \overset{\bullet}{K}{}^{\theta\rho}{}_\theta - h_a{}^\rho \overset{\bullet}{K}{}^{\theta\sigma}{}_\theta \tag{2.15}$$

is an antisymmetric tensor in the last two indexes.

All tensors $\overset{\bullet}{T}{}^a{}_{\mu\nu}$, $\overset{\bullet}{K}{}^{\rho\sigma}{}_a$, and $\overset{\bullet}{S}_a{}^{\rho\sigma}$, are covariant with respect to both coordinate transformations and local Lorentz transformations. (For tensors without Lorentz indexes or scalars, we say "invariant with respect to Lorentz transformations".) Note that, working in the covariant formulation of teleparallel gravity, the local Lorentz covariance means that the tensorial quantities are transformed covariantly under the simultaneous transformation of both the tetrad and the ISC:

$$h'^a{}_\mu = \Lambda^a{}_b(x) h^b{}_\mu, \tag{2.16}$$

$$\overset{\bullet}{A}{}'^a{}_{b\mu} = \Lambda^a{}_c(x)\overset{\bullet}{A}{}^c{}_{d\mu}\Lambda_b{}^d(x) + \Lambda^a{}_c(x)\partial_\mu \Lambda_b{}^c(x), \tag{2.17}$$

where $\Lambda^a{}_c(x)$ is the matrix of a local Lorentz rotation, and $\Lambda_a{}^c(x)$ is an inverse matrix of the latter. The operation on the right-hand side of (2.17) tells us that the ISC can be equalized to zero by an appropriate local Lorentz transformation. Then, by another local Lorentz rotation, it can be represented in the form

$$\overset{\bullet}{A}{}^a{}_{c\nu} = \Lambda^a{}_b \partial_\nu (\Lambda^{-1})^b{}_c. \tag{2.18}$$

Let us return to the field equations (2.11). With the zero matter part, they can be rewritten as

$$\kappa h\ \overset{\bullet}{J}_a{}^\rho = \partial_\sigma\left(h\overset{\bullet}{S}_a{}^{\rho\sigma}\right) = \overset{\circ}{\nabla}_\sigma\left(h\overset{\bullet}{S}_a{}^{\rho\sigma}\right), \tag{2.19}$$

where the gravitational energy–momentum $\overset{\bullet}{J}_a{}^\rho$ is defined as

$$\overset{\bullet}{J}_a{}^\rho = \frac{1}{\kappa}h_a{}^\mu \overset{\bullet}{S}_c{}^{\nu\rho}\overset{\bullet}{T}{}^c{}_{\nu\mu} - \frac{h_a{}^\rho}{h}\overset{\bullet}{\mathcal{L}} + \frac{1}{\kappa}\overset{\bullet}{A}{}^c{}_{a\sigma}\overset{\bullet}{S}_c{}^{\rho\sigma}; \tag{2.20}$$

in [3], it is called as a gravitational current as well. The teleparallel superpotential (2.15) is antisymmetric in the upper indexes; therefore, the current density of (2.20) is conserved $\partial_\rho(h\ \overset{\bullet}{J}_a{}^\rho) = 0$. Because both the l.h.s. and the r.h.s. of (2.19) are coordinate-covariant, one can define well-defined charges. However, the current $\overset{\bullet}{J}_a{}^\rho$ itself is not Lorentz-covariant, which is a problem.

Because (2.19) in whole is Lorentz-covariant, one has a possibility to construct both the l.h.s. and r.h.s. in evidently Lorentz-covariant form as well. Thus, (2.19) can be rewritten as

$$\kappa h \overset{\bullet}{J}{}'_a{}^\rho = \overset{\bullet}{\mathcal{D}}_\sigma\left(h\overset{\bullet}{S}_a{}^{\rho\sigma}\right), \tag{2.21}$$

where the Lorentz-covariant current is defined as [3]

$$\overset{\bullet}{J}{}'_a{}^\rho = \frac{1}{\kappa}h_a{}^\mu \overset{\bullet}{S}_c{}^{\nu\rho}\overset{\bullet}{T}{}^c{}_{\nu\mu} - \frac{h_a{}^\rho}{h}\overset{\bullet}{\mathcal{L}}, \tag{2.22}$$

and $\overset{\bullet}{\mathcal{D}}_\sigma$ is the Lorentz-covariant derivative. For example, applied to a tetrad co-vector, V_a, it is defined as

$$\overset{\bullet}{\mathcal{D}}_\sigma V_a = \partial_\sigma V_a - \overset{\bullet}{A}{}^c{}_{a\sigma} V_c. \tag{2.23}$$

However, the form (2.21) does not allow us to construct well-defined charges. It is just the problem highlighted by Krššák [8] that has been resolved by introducing our formalism [11, 12].

As the conserved quantities suggested in [11,12] are obtained by applying the Noether theorem they can be classified as canonical conserved quantities. This means that the divergences in Lagrangians have to be taken into account. Thus, considering the form of the TEGR Lagrangian (2.9), one concludes that the conserved quantities contain the ISC in an explicit form, so their values are sensitive to changes in the ISC. Therefore, some external procedure is needed to restrict the ISC so that the conserved quantities will have physically meaningful values; see in greater detail Sects. 2.3 and 2.4.

2.2 Elements of the STEGR

The Lagrangian in the STEGR has the following form [1]:

$$\mathcal{L} = \frac{\sqrt{-g}}{2\kappa} g^{\mu\nu}(L^\alpha{}_{\beta\mu}L^\beta{}_{\nu\alpha} - L^\alpha{}_{\beta\alpha}L^\beta{}_{\mu\nu}), \tag{2.24}$$

where again $\kappa = 8\pi$, and $g = \det g_{\mu\nu}$. The disformation tensor $L^\alpha{}_{\mu\nu}$ is defined as

$$L^\alpha{}_{\mu\nu} = \frac{1}{2}Q^\alpha{}_{\mu\nu} - \frac{1}{2}Q_\mu{}^\alpha{}_\nu - \frac{1}{2}Q_\nu{}^\alpha{}_\mu, \tag{2.25}$$

and the non-metricity tensor $Q_{\alpha\mu\nu}$ is defined as follows:

$$Q_{\alpha\mu\nu} \equiv \nabla_\alpha g_{\mu\nu}, \tag{2.26}$$

that is, not zero in general, unlike (2.6). Here, the covariant derivative ∇_α is defined with the use of the flat affine connection $\Gamma^\alpha{}_{\mu\nu}$, which is symmetric in the lower indexes; we denote it below by the abbreviation STC, for symmetric teleparallel connection. The corresponding torsion is zero: $T^\alpha{}_{\mu\nu} \equiv \Gamma^\alpha{}_{\mu\nu} - \Gamma^\alpha{}_{\nu\mu} = 0$. The curvature tensor for STC is zero as well:

$$R^\alpha{}_{\beta\mu\nu}(\Gamma) = \partial_\mu \Gamma^\alpha{}_{\nu\beta} - \partial_\nu \Gamma^\alpha{}_{\mu\beta} + \Gamma^\alpha{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\beta} - \Gamma^\alpha{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\beta} = 0. \tag{2.27}$$

Thus, in the STEGR framework, the metric components are thought of as dynamic variables, and gravitational effects are encoded in the non-metricity (2.26).

One can easily verify that the decomposition of a general connection $\Gamma^{\beta}{}_{\mu\nu}$ into the Levi–Civita connection, contortion, and the disformation terms (see [1]) reduces to

$$L^{\beta}{}_{\mu\nu} \equiv \Gamma^{\beta}{}_{\mu\nu} - \overset{\circ}{\Gamma}{}^{\beta}{}_{\mu\nu}. \tag{2.28}$$

Here, the Levi–Civita connection $\overset{\circ}{\Gamma}{}^{\beta}{}_{\mu\nu}$ is considered as a function of a metric in the usual way [43]. Then, with the use of (2.25)–(2.28), one can rewrite (2.24) as

$$\mathcal{L} = \overset{\circ}{\mathcal{L}} + \frac{\sqrt{-g}g^{\mu\nu}}{2\kappa} R_{\mu\nu} + \mathcal{L}'. \tag{2.29}$$

The first term is the Hilbert Lagrangian (2.10), however presented here as a function of metric

$$\overset{\circ}{\mathcal{L}} = -\frac{\sqrt{-g}}{2\kappa} \overset{\circ}{R}. \tag{2.30}$$

According to [15], we neglect the second term in (2.29).[1] The third term $\mathcal{L}'$ is a total divergence:

$$\mathcal{L}' = -\frac{\sqrt{-g}}{2\kappa} \overset{\circ}{\nabla}_{\alpha} (Q^{\alpha} - \hat{Q}^{\alpha}) = \partial_{\alpha} \left(-\frac{\sqrt{-g}}{2\kappa} (Q^{\alpha} - \hat{Q}^{\alpha}) \right) = \partial_{\alpha} \mathcal{D}^{\alpha}, \tag{2.31}$$

where $Q_{\alpha} = g^{\mu\nu} Q_{\alpha\mu\nu}$ and $\hat{Q}_{\alpha} = g^{\mu\nu} Q_{\mu\alpha\nu}$. Using (2.25), one easily finds that

$$Q^{\alpha} = -2g^{\mu\nu} L_{\mu}{}^{\alpha}{}_{\nu}, \qquad \hat{Q}^{\alpha} = -g^{\mu\nu} \left(L^{\alpha}{}_{\mu\nu} + L_{\mu}{}^{\alpha}{}_{\nu} \right), \tag{2.32}$$

where $L^{\alpha}{}_{\mu\nu}$ is thought of as defined in (2.28). Thus, STC is included in divergence only, and we consider the Lagrangian

$$\mathcal{L} = -\frac{\sqrt{-g}}{2\kappa} \overset{\circ}{R} + \partial_{\alpha} \mathcal{D}^{\alpha}, \tag{2.33}$$

with

$$\mathcal{D}^{\alpha} \equiv -\frac{\sqrt{-g}}{2\kappa} (Q^{\alpha} - \hat{Q}^{\alpha}) \tag{2.34}$$

instead of (2.24) or (2.29).

Varying action with the Lagrangian (2.33) with respect to $g^{\alpha\beta}$, one obtains the vacuum field equations

$$\overset{\circ}{R}_{\alpha\beta} \left(\delta^{\alpha}_{\mu} \delta^{\beta}_{\nu} - \tfrac{1}{2} g_{\mu\nu} g^{\alpha\beta} \right) = 0, \tag{2.35}$$

where

$$\overset{\circ}{R}_{\alpha\beta} = \partial_{\rho} \overset{\circ}{\Gamma}{}^{\rho}{}_{\alpha\beta} - \partial_{\alpha} \overset{\circ}{\Gamma}{}^{\rho}{}_{\rho\beta} + \overset{\circ}{\Gamma}{}^{\rho}{}_{\alpha\beta} \overset{\circ}{\Gamma}{}^{\sigma}{}_{\sigma\rho} - \overset{\circ}{\Gamma}{}^{\rho}{}_{\alpha\sigma} \overset{\circ}{\Gamma}{}^{\sigma}{}_{\beta\rho}. \tag{2.36}$$

[1] The second term in (2.29 is equal to zero due to (2.27). However, if we preserve it in the Lagrangian, we need to vary it. Thus, variation of (2.29) together with the second term means the variation of (2.24) exactly. However, variation of (2.24) with respect to $\Gamma^{\beta}{}_{\mu\nu}$ gives $\Gamma^{\beta}{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\beta}{}_{\mu\nu}$. This means that the flat STC $\Gamma^{\beta}{}_{\mu\nu}$ can be equal to the Levi–Civita connection that is not flat in general. Since this is not permitted, the second term in (2.29) has to be canceled.

Because Eqs. (2.35) are the usual GR equations (which do not contain the STC in whole), we are convinced that GR and STEGR are equivalent. The reason for the absence of the STC in (2.35) lies in the fact that the STC is included in the divergence in (2.33) only. However, the expression for the Ricci tensor (2.36), containing partial derivatives, is not evidently covariant; at least, one has to prove it additionally. Thus, the same expression (2.36) can be rewritten with the use of the STC in evidently covariant form:

$$\overset{\circ}{R}_{\alpha\beta} = \nabla_{\alpha} L^{\rho}{}_{\rho\beta} - \nabla_{\rho} L^{\rho}{}_{\alpha\beta} + L^{\rho}{}_{\alpha\beta} L^{\sigma}{}_{\sigma\rho} - L^{\rho}{}_{\alpha\sigma} L^{\sigma}{}_{\beta\rho}, \tag{2.37}$$

where $L^{\alpha}{}_{\mu\nu}$ defined in (2.28) is a tensor. The situation is analogous to the one in the TEGR, where the field equations (2.22) are evidently and fully covariant after incorporation of the ISC into them.

Let us reiterate that the STEGR Lagrangian contains the STC in the divergence only. By this, first, varying the action with the Lagrangian (2.33) with respect to the metric components, one obtains the Euler–Lagrange equations (2.35). Second, varying the action with the Lagrangian (2.33) with respect to $\Gamma^{\alpha}{}_{\mu\nu}$ (which is included in the divergence only), one obtains $0 = 0$. In summary, the STC cannot be determined in the framework of the STEGR itself. This means that the STC, like the ISC in the TEGR, being an external structure, can be defined by additional requirements only.

On the other hand, the conserved quantities suggested in [15] are canonical ones because they are obtained by applying the Noether theorem. This means that the divergence in the Lagrangian (2.33) has to be taken into account. As a result, conserved quantities contain the STC in evident form that has to be determined by an additional procedure; see in greater detail Sects. 2.3 and 2.4.

2.3 Noether conserved quantities

The Noether conserved quantities were derived for the TEGR Lagrangian (2.1) in our papers [11,12] and for the STEGR Lagrangian (2.33) in another paper [15]. In both theories, Noether current $\mathcal{I}^{\alpha}(\xi)$ is a vector density of the weight +1, and the Noether superpotential $\mathcal{I}^{\alpha\beta}(\xi)$ is an antisymmetric tensor density of the weight +1. For such quantities, $\overset{\circ}{\nabla}_{\mu} \equiv \partial_{\mu}$, and thus the conservation laws have evidently covariant form:

$$\partial_{\alpha} \mathcal{I}^{\alpha}(\xi) \equiv \overset{\circ}{\nabla}_{\alpha} \mathcal{I}^{\alpha}(\xi) = 0, \tag{2.38}$$

$$\mathcal{I}^{\alpha}(\xi) = \partial_{\alpha} \mathcal{I}^{\alpha\beta}(\xi) \equiv \overset{\circ}{\nabla}_{\alpha} \mathcal{I}^{\alpha\beta}(\xi). \tag{2.39}$$

It is important to provide a physical interpretation of the quantities presented above. We follow the prescription in [11,15]. First, one has to choose the displacement vector ξ^{α}, which can be a Killing vector, a proper vector of an observer, etc. Second, setting a time coordinate as $t = x^0$ and choosing a space section as $\Sigma := t = \text{const}$, one can interpret $\mathcal{I}^0(\xi)$ as a density on the section Σ of the quantity related to a chosen ξ^α. For example, if ξ^α is a timelike Killing vector, it is interpreted as the energy density on Σ. On the other hand, if ξ^α is an observer's proper vector, then the components $\mathcal{I}^\alpha(\xi)$ can be interpreted as components of the energy–momentum vector measured by such an observer.

The Noether current $\overset{\bullet}{\mathcal{J}}{}^\alpha(\xi)$ in the TEGR [11,12] in a vacuum and when the field equations (2.11) hold acquires the form

$$\overset{\bullet}{\mathcal{J}}{}^\alpha(\xi) = h\, \overset{\bullet}{\theta}_\sigma{}^\alpha \xi^\sigma + \frac{h}{\kappa} \overset{\bullet}{S}_\sigma{}^{\alpha\rho}\, \overset{\circ}{\nabla}_\rho\, \xi^\sigma, \tag{2.40}$$

where the gravitational Noether energy–momentum tensor $\overset{\bullet}{\theta}_\sigma{}^\alpha$ is

$$\overset{\bullet}{\theta}_\sigma{}^\alpha \equiv \frac{1}{\kappa} \overset{\bullet}{S}_a{}^{\alpha\rho} \overset{\bullet}{K}{}^a{}_{\sigma\rho} - \frac{1}{h} \overset{\bullet}{\mathcal{L}}\, \delta^\alpha_\sigma. \tag{2.41}$$

The related Noether superpotential in the TEGR is

$$\overset{\bullet}{\mathcal{J}}{}^{\alpha\beta}(\xi) = \frac{h}{\kappa} \overset{\bullet}{S}_a{}^{\alpha\beta} h^a{}_\sigma \xi^\sigma. \tag{2.42}$$

In correspondence with (2.39), one can derive the current from the superpotential:

$$\overset{\bullet}{\mathcal{J}}{}^\alpha(\xi) = \partial_\beta \overset{\bullet}{\mathcal{J}}{}^{\alpha\beta}(\xi). \tag{2.43}$$

Both $\overset{\bullet}{\mathcal{J}}{}^\alpha(\xi)$ and $\overset{\bullet}{\mathcal{J}}{}^{\alpha\beta}(\xi)$ essentially depend on tensors $\overset{\bullet}{T}{}^a{}_{\mu\nu}$, $\overset{\bullet}{K}{}^{\rho\sigma}{}_a$, and $\overset{\bullet}{S}_a{}^{\rho\sigma}$ defined in (2.2), (2.13), and (2.15), and which are covariant with respect to both coordinate transformations and local Lorentz rotations. Thus, both $\overset{\bullet}{J}{}^\alpha(\xi)$ and $\overset{\bullet}{J}{}^{\alpha\beta}(\xi)$ are explicitly spacetime-covariant and Lorentz-invariant.

Such an advantage is achieved by the fact that $\overset{\bullet}{T}{}^a{}_{\mu\nu}$, $\overset{\bullet}{K}{}^{\rho\sigma}{}_a$, and $\overset{\bullet}{S}_a{}^{\rho\sigma}$ contain the components of the ISC which are transformed simultaneously with the tetrad components; see (2.16) and (2.17). This means that if the pair comprising the tetrad and ISC is fixed, one has $\overset{\bullet}{J}{}^\alpha(\xi)$ and $\overset{\bullet}{J}{}^{\alpha\beta}(\xi)$ defined respectively to this pair. In [16,17], keeping in mind this situation, we introduce the notion of "gauges" in the TEGR. A gauge is defined as a set of pairs (tetrad and ISC) which can be obtained from a given combination of the tetrad and the ISC by an arbitrary Lorentz rotation, where the tetrad transforms as (2.16), and the inertial spin connection transforms as (2.17) simultaneously, and/or arbitrary coordinate transformations. The case in which the zero ISC corresponds to some tetrad is traditionally called the Wietzenböck gauge [3]. In the above cases, we use the word "gauge" in different senses: when we say "Wietzenböck gauge", we mean only the one pair, tetrad and zero ISC, and when we say "gauge" in our definition, we mean the whole equivalence class of pairs of tetrads and ISCs in which two pairs are equivalent if and only if they are connected as defined above.

In the STEGR with the Lagrangian (2.33), the conserved quantities were derived in [15] for each term separately. The superpotential $\mathcal{J}^{\alpha\beta}_{GR}$ for the Hilbert term (2.10) is the Komar superpotential [44,45]

$$\mathcal{J}^{\alpha\beta}_{GR} = \mathcal{K}^{\alpha\beta} = \frac{\sqrt{-g}}{\kappa} \overset{\circ}{\nabla}{}^{[\alpha} \xi^{\beta]}. \tag{2.44}$$

For the divergent term $\mathcal{L}' = \partial_\alpha \mathcal{D}^\alpha$ (see (2.31) and (2.34)), the Noether superpotential is

$$\mathcal{J}^{\alpha\beta}_{div} = \frac{\sqrt{-g}}{\kappa} \delta^{[\alpha}_\sigma (Q^{\beta]} - \hat{Q}^{\beta]}) \xi^\sigma. \tag{2.45}$$

The total Noether superpotential of the Lagrangian (2.33) in STEGR is

$$\mathcal{J}^{\alpha\beta} = \mathcal{J}^{\alpha\beta}_{GR} + \mathcal{J}^{\alpha\beta}_{div}. \tag{2.46}$$

Taking the divergence of each term of (2.46) in correspondence with (2.39), one gets the total Noether current

$$\mathcal{J}^\alpha = \mathcal{J}^\alpha_{GR} + \mathcal{J}^\alpha_{div}. \tag{2.47}$$

As we can see, $\mathcal{J}^\alpha(\xi)$ and $\mathcal{J}^{\alpha\beta}(\xi)$ are explicitly spacetime-covariant.

In [15], the concept of gauges in the STEGR was not introduced. In the present paper, we want to have the same terminology in the STEGR as in the TEGR. So, formally, we take some specific pair of coordinates $\bar{x}^\mu$ and STC $\bar{\Gamma}^\alpha{}_{\mu\nu}$, and then associate with this pair a class of pairs $(x^\mu, \Gamma^\alpha{}_{\mu\nu})$ which are connected to it by the transformations

$$\begin{aligned} x^\mu &= x^\mu(\bar{x}^\alpha), \\ \Gamma^\alpha{}_{\mu\nu} &= \frac{\partial x^\alpha}{\partial \bar{x}^{\bar{\alpha}}} \frac{\partial \bar{x}^{\bar{\mu}}}{\partial x^\mu} \frac{\partial \bar{x}^{\bar{\nu}}}{\partial x^\nu} \bar{\Gamma}^{\bar{\alpha}}{}_{\bar{\mu}\bar{\nu}} + \frac{\partial x^\alpha}{\partial \bar{x}^{\bar{\lambda}}} \frac{\partial}{\partial x^\mu} \left(\frac{\partial \bar{x}^{\bar{\lambda}}}{\partial x^\nu} \right). \end{aligned} \tag{2.48}$$

We define such a set as a "gauge". Usually, the case of a zero STC is referred to as "coincident gauge" [46,47]. Here, we again note that in the term "coincident gauge", we mean only the case of coordinates in which the STC is zero (the same as "Wietzenböck gauge" in the TEGR), and when we say "gauge" here, we mean the set of all possible coordinates and values of STCs in them, such that the relation (2.48) is satisfied for each of the pairs $(x^\mu, \Gamma^\alpha{}_{\mu\nu})$ of the class.

2.4 Defining the connection: "turning off" gravity principle

Let us reiterate the main claims derived above. Teleparallel connections in the TEGR and STEGR are not dynamical quantities and are left undetermined [1,7] in theories themselves.

To determine the ISC in the TEGR for a *given solution*, we introduced the generalized "turning off" gravity principle in [11,12]. This principle is based on the assumption that Noether's current and superpotential are proportional to contortion components $\overset{\bullet}{K}{}^a{}_{c\mu}$, or alternatively, $\overset{\bullet}{T}{}^\alpha{}_{\mu\nu}$ or $\overset{\bullet}{S}{}_a{}^{\mu\nu}$.

In the absence of gravity, both the current and the superpotential have to vanish, which follows from vanishing $\overset{\bullet}{K}{}^{a}{}_{c\mu}$. Thus, to determine $\overset{\bullet}{A}{}^{a}{}_{c\mu}$ according to this requirement, we turn to the formula (2.8) for a given solution. For a GR solution under consideration, it was suggested

1. to choose a convenient tetrad and define $\overset{\circ}{A}{}^{a}{}_{c\mu} = -h_b{}^{\nu}\, \overset{\circ}{\nabla}_{\mu}\, h^{a}{}_{\nu}$ (see (2.7));
2. to construct a related curvature of a Levi–Civita spin connection

$$\overset{\circ}{R}{}^{i}{}_{j\mu\nu} = \partial_{\mu}\overset{\circ}{A}{}^{i}{}_{j\nu} - \partial_{\nu}\overset{\circ}{A}{}^{i}{}_{j\mu} + \overset{\circ}{A}{}^{i}{}_{k\mu}\overset{\circ}{A}{}^{k}{}_{j\nu} - \overset{\circ}{A}{}^{i}{}_{k\nu}\overset{\circ}{A}{}^{k}{}_{j\mu};$$

3. to "switch off" gravity solving the absent gravity equation $\overset{\circ}{R}{}^{a}{}_{b\gamma\delta} = 0$ for parameters of the chosen GR solution;
4. to take $\overset{\circ}{A}{}^{a}{}_{c\mu} = \overset{\bullet}{A}{}^{a}{}_{c\mu}$ for the found parameter values satisfying $\overset{\circ}{R}{}^{a}{}_{b\gamma\delta} = 0$.

To determine the STC in STEGR, we use the the "turning off" gravity principle adapted for STEGR [15]. This principle is based on the assumption analogical to that in the TEGR, that is, Noether's current (2.47) and superpotential (2.46) have to vanish in the absence of gravity. This goal is achieved when $Q_{\alpha\mu\nu}$ (the same $L^{\alpha}{}_{\mu\nu}$) and $\overset{\circ}{R}{}^{\alpha}{}_{\beta\mu\nu}$ vanish in the absence of gravity as well. To find the STC in the STEGR for a GR solution under consideration, the steps are as follows:

1. Construct a related Riemann curvature tensor of the Levi–Civita connection

$$\overset{\circ}{R}{}^{\alpha}{}_{\beta\mu\nu} = \partial_{\mu}\, \overset{\circ}{\Gamma}{}^{\alpha}{}_{\beta\nu} - \partial_{\nu}\, \overset{\circ}{\Gamma}{}^{\alpha}{}_{\beta\mu} + \overset{\circ}{\Gamma}{}^{\alpha}{}_{\kappa\mu}\, \overset{\circ}{\Gamma}{}^{\kappa}{}_{\beta\nu} - \overset{\circ}{\Gamma}{}^{\alpha}{}_{\kappa\nu}\, \overset{\circ}{\Gamma}{}^{\kappa}{}_{\beta\mu};$$

2. "Switch off" gravity, solving the absent gravity equation $\overset{\circ}{R}{}^{\alpha}{}_{\beta\mu\nu} = 0$ for parameters of the chosen GR solution.
3. Take $\Gamma^{\alpha}{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\alpha}{}_{\mu\nu}$ for the found parameter values which satisfy $\overset{\circ}{R}{}^{\alpha}{}_{\beta\mu\nu} = 0$.

The torsion of the found connection should automatically be zero because we take it from the Levi–Civita connection for some parameter values, and the Levi–Civita connection is always symmetric. The curvature of the found connection should also be zero, because we found it from the equation $\overset{\circ}{R}{}^{\alpha}{}_{\beta\gamma\delta} = 0$.

As noted above, on the level of field equations, in both the TEGR and STEGR, teleparallel connections are arbitrary, and they only have to be flat and nothing more. Our generalized principle of "switching off" gravity formulated above for fixation of a teleparallel connection for a given solution does not determine gauges uniquely in both the TEGR [16,17] and the STEGR [15]. In general, "turning off" gravity in the TEGR, the result depends on the tetrad that we choose initially. In general, for each specific tetrad where we "turn off" gravity, we obtain a different gauge (pairs of a tetrad and ISC). As a rule, these pairs are not connected by (2.16) and (2.17) applied simultaneously. Thus, torsion, contortion, and superpotential expressed in the same tetrad and unified coordinates are completely different in each case. This gives us different values of conserved quantities.[2]

In the same way, "turning off" gravity in the STEGR, the result depends on the coordinates that we choose from the start to construct the STC for a GR solution under consideration. Thus, non-metricity (the same, disformation) components obtained initially in different coordinates, being transformed to unified coordinates, are different, and this gives us different values of conserved quantities in each case.

Such problems have to be resolved separately for each solution under consideration, where for each task one has to determine a related gauge. This situation is analogous to a bi-metric representation of GR in [25], where, for example, to obtain an acceptable value for the mass of a black hole, one has to introduce a background metric in a special appropriate way. Thus, one of the main purposes of this work is to find the gauges in the TEGR and STEGR in which we would have physically meaningful results for the concrete solution—a plane gravitational wave in vacuum.

3 Simplest gauges

3.1 Plane gravitational wave

Because the wave solutions in GR are time-dependent, there are no timelike Killing vectors for them. Therefore, in the framework of the fully covariant formalism, first, one does not expect to obtain energy densities related to a fixed observer presenting a fixed frame. Second, because the wave under consideration fills infinite space, one cannot obtain any conserved charges (total energy); indeed, in this case, one cannot define a fixed observer at infinity or an external observer. Therefore, in this paper, we do not study a general problem of constructing the energy characteristics of the gravitational wave measured by observers of the aforementioned type. However, of course, one can define proper vectors for freely falling observers and, correspondingly, calculate energy–momentum density (current components)

[2] But, sometimes, "turning off" gravity, we can obtain the same gauge [48,49].

measured by such observers. Because such observers should measure zero energy–momentum in correspondence with the equivalence principle, the current components have to vanish. By our fully covariant formalism, one has to find gauges which just give zero components of the current in this case following the main task of the paper.

In this paper, we consider only the plane gravitational wave with only one polarization. The simplest form of the related metric is [36]

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & [f(t-z)]^2 & 0 & 0 \\ 0 & 0 & [g(t-z)]^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.1}$$

Here and below, the numeration of the coordinates is $t = x^0$, $x = x^1$, $y = x^2$, and $z = x^3$.

In this section, in both the TEGR and STEGR, based on the metric (3.1), we construct the simplest gauges. Next, our goal here is to check these simplest gauges in the TEGR and STEGR on the possibility of achieving a correspondence with the equivalence principle. This goal was initiated in particular by the study in [36] based on (3.1) as well, where such a problem was considered in another formalism with a subsequent interpretation. We, obtaining similar results, give a different interpretation corresponding to our full covariant formalism.

3.2 Diagonal polarization in the TEGR

The simplest tetrad that can be introduced for (3.1) is the diagonal tetrad

$$h^A{}_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & f(t-z) & 0 & 0 \\ 0 & 0 & g(t-z) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.2}$$

To shorten the formulae, we will not write the argument $(t-z)$ of functions f and g and their derivatives of each order. Now we follow the "turning off" gravity principle in the TEGR. For this tetrad, we calculate the L-CSC (2.7) which has the non-zero components:

$$\begin{aligned} \overset{\circ}{A}{}^{\hat{0}}{}_{\hat{1}1} = \overset{\circ}{A}{}^{\hat{1}}{}_{\hat{0}1} = -\overset{\circ}{A}{}^{\hat{1}}{}_{\hat{3}1} = \overset{\circ}{A}{}^{\hat{3}}{}_{\hat{1}1} = f', \\ \overset{\circ}{A}{}^{\hat{0}}{}_{\hat{2}2} = \overset{\circ}{A}{}^{\hat{2}}{}_{\hat{0}2} = \overset{\circ}{A}{}^{\hat{3}}{}_{\hat{2}2} = -\overset{\circ}{A}{}^{\hat{2}}{}_{\hat{3}2} = g', \end{aligned} \tag{3.3}$$

where prime indicates a derivative with respect to $u = (t - z)$, and the index with "hat" is the tetrad index. Non-zero components of the Riemann tensor constructed using (3.3) are proportional to f'' or g'', so turning off gravity means that

$$\begin{aligned} f'' = 0, \\ g'' = 0, \end{aligned} \tag{3.4}$$

i.e.,

$$\begin{aligned} f = c_1(t-z) + d_1, \\ g = c_2(t-z) + d_2, \end{aligned} \tag{3.5}$$

where c_1, c_2, d_1, and d_2 are integration constants. Turning off gravity by (3.5), we get the ISC with non-zero components:

$$\begin{aligned} \overset{\bullet}{A}{}^{\hat{0}}{}_{\hat{1}1} = \overset{\bullet}{A}{}^{\hat{1}}{}_{\hat{0}1} = -\overset{\bullet}{A}{}^{\hat{1}}{}_{\hat{3}1} = \overset{\bullet}{A}{}^{\hat{3}}{}_{\hat{1}1} = c_1, \\ \overset{\bullet}{A}{}^{\hat{0}}{}_{\hat{2}2} = \overset{\bullet}{A}{}^{\hat{2}}{}_{\hat{0}2} = \overset{\bullet}{A}{}^{\hat{3}}{}_{\hat{2}2} = -\overset{\bullet}{A}{}^{\hat{2}}{}_{\hat{3}2} = c_2. \end{aligned} \tag{3.6}$$

Then we calculate the superpotential (2.15) with (2.8) definitions which has the non-zero components:

$$\begin{aligned} \overset{\bullet}{S}{}_{\hat{0}}{}^{03} = -\overset{\bullet}{S}{}_{\hat{0}}{}^{30} = -\overset{\bullet}{S}{}_{\hat{3}}{}^{03} = \overset{\bullet}{S}{}_{\hat{3}}{}^{30} = \tfrac{c_1 - f'}{f} + \tfrac{c_2 - g'}{g}, \\ \overset{\bullet}{S}{}_{\hat{1}}{}^{01} = \overset{\bullet}{S}{}_{\hat{1}}{}^{31} = -\overset{\bullet}{S}{}_{\hat{1}}{}^{10} = -\overset{\bullet}{S}{}_{\hat{1}}{}^{13} = \tfrac{g' - c_2}{fg}, \\ \overset{\bullet}{S}{}_{\hat{2}}{}^{02} = \overset{\bullet}{S}{}_{\hat{2}}{}^{32} = -\overset{\bullet}{S}{}_{\hat{2}}{}^{20} = -\overset{\bullet}{S}{}_{\hat{2}}{}^{23} = \tfrac{f' - c_1}{fg}. \end{aligned} \tag{3.7}$$

We assume that superpotential, torsion, etc., should be zero in the absence of waves; thus, $c_1 = c_2 = 0$. Then the ISC (3.6) becomes

$$\overset{\bullet}{A}{}^a{}_{b\mu} = 0. \tag{3.8}$$

One can classify the pair of the tetrad (3.2) and ISC (3.8) as a pair of the equivalence class presenting the simplest gauge for the solution (3.1) in the TEGR.

Now let us construct the Noether superpotential for this gauge. First, we consider the simplest case of a freely falling observer that is static in co-moving coordinates of the metric (3.1). Then components of the observer's proper vector are

$$\xi^\sigma = (-1, 0, 0, 0). \tag{3.9}$$

Then, for the Noether superpotential (2.42) non-zero components, one gets

$$\overset{\bullet}{\mathcal{J}}{}^{03} = -\overset{\bullet}{\mathcal{J}}{}^{30} = \frac{gf' + fg'}{8\pi}. \tag{3.10}$$

By (2.43), taking the divergence of the superpotential and using the Einstein equations

$$\frac{f''}{f} + \frac{g''}{g} = 0, \tag{3.11}$$

we get the Noether current

$$\overset{\bullet}{\mathcal{J}}{}^\mu = \left\{ -\frac{f'g'}{4\pi}, 0, 0, -\frac{f'g'}{4\pi} \right\}. \tag{3.12}$$

Let us generalize (3.9). Solving the geodesic equation directly, we derive the general freely falling observer's 4-velocity ξ^μ:

$$\begin{aligned} \xi^0 = -\tfrac{C_1{}^2 e^{\alpha_0}}{2f^2} - \tfrac{C_2{}^2 e^{\alpha_0}}{2g^2} - \cosh\alpha_0, \\ \xi^1 = -\tfrac{C_1}{f^2}, \\ \xi^2 = -\tfrac{C_2}{g^2}, \\ \xi^3 = -\tfrac{C_1{}^2 e^{\alpha_0}}{2f^2} - \tfrac{C_2{}^2 e^{\alpha_0}}{2g^2} - \sinh\alpha_0, \end{aligned} \tag{3.13}$$

where C_1, C_2, and α_0 are constants of integration. One can see that components (3.13) go to the ones in (3.9) when $C_1 = C_2 = \alpha_0 = 0$.

Taking this general form of the observer's proper vector, one gets Noether superpotential non-zero components:

$$\begin{aligned} \overset{\bullet}{\mathcal{J}}{}^{01} = \overset{\bullet}{\mathcal{J}}{}^{31} = -\overset{\bullet}{\mathcal{J}}{}^{10} = -\overset{\bullet}{\mathcal{J}}{}^{13} = -\tfrac{C_1 g'}{8\pi f}, \\ \overset{\bullet}{\mathcal{J}}{}^{02} = \overset{\bullet}{\mathcal{J}}{}^{32} = -\overset{\bullet}{\mathcal{J}}{}^{20} = -\overset{\bullet}{\mathcal{J}}{}^{23} = -\tfrac{C_2 f'}{8\pi g}, \\ \overset{\bullet}{\mathcal{J}}{}^{03} = -\overset{\bullet}{\mathcal{J}}{}^{30} = \tfrac{e^{-\alpha_0}(gf' + fg')}{8\pi}. \end{aligned} \tag{3.14}$$

Taking the divergence of the superpotential and using the Einstein equations (3.11), we get Noether current

$$\overset{\bullet}{\mathcal{J}}{}^{\mu} = \left\{ -\frac{e^{-\alpha_0} f' g'}{4\pi}, 0, 0, -\frac{e^{-\alpha_0} f' g'}{4\pi} \right\}. \tag{3.15}$$

Note, first, that the components of the current (3.15) do not depend on C_1 and C_2, and second, that for $\alpha_0 = 0$, the components (3.15) coincide with (3.12).

Because we have obtained non-zero components of the current (3.12) and (3.15) measured by a freely falling observer, we did not achieve the settled goal to obtain a correspondence with the equivalence principle. Just the analogous conclusion has been given by other authors; for example, in [36], who studied (3.1) in the framework of other approaches. But, unlike previous approaches, we interpret the failure as an inappropriate gauge, which stimulates a search for appropriate gauges, which we do in the next sections.

3.3 Diagonal polarization in the STEGR

Let us turn to the STEGR, where we follow the "turning off" gravity principle as well. The non-zero components of the Levi–Civita connection for the metric (3.1) are

$$\begin{aligned} \overset{\circ}{\Gamma}{}^{0}{}_{11} = \overset{\circ}{\Gamma}{}^{3}{}_{11} = ff', \\ \overset{\circ}{\Gamma}{}^{0}{}_{22} = \overset{\circ}{\Gamma}{}^{3}{}_{22} = gg', \\ \overset{\circ}{\Gamma}{}^{1}{}_{01} = \overset{\circ}{\Gamma}{}^{1}{}_{10} = -\overset{\circ}{\Gamma}{}^{1}{}_{13} = -\overset{\circ}{\Gamma}{}^{1}{}_{31} = \tfrac{f'}{f}, \\ \overset{\circ}{\Gamma}{}^{2}{}_{02} = \overset{\circ}{\Gamma}{}^{2}{}_{20} = -\overset{\circ}{\Gamma}{}^{2}{}_{23} = -\overset{\circ}{\Gamma}{}^{2}{}_{32} = \tfrac{g'}{g}. \end{aligned} \tag{3.16}$$

In order to switch off gravity, one has to equalize the Riemannian tensor with (3.16) to zero. By (3.5), the Riemann tensor becomes zero, and the Levi–Civita connection has to be taken as the STC (flat and torsionless) which has the non-zero components:

$$\begin{aligned} \Gamma^{0}{}_{11} = \Gamma^{3}{}_{11} = c_1(c_1(t-z) + d_1), \\ \Gamma^{0}{}_{22} = \Gamma^{3}{}_{22} = c_2(c_2(t-z) + d_2), \\ \Gamma^{1}{}_{01} = \Gamma^{1}{}_{10} = -\Gamma^{1}{}_{13} = -\Gamma^{1}{}_{31} = \tfrac{c_1}{c_1(t-z)+d_1}, \\ \Gamma^{2}{}_{02} = -\Gamma^{2}{}_{23} = -\Gamma^{2}{}_{32} = \Gamma^{2}{}_{20} = \tfrac{c_2}{c_2(t-z)+d_2}. \end{aligned} \tag{3.17}$$

Then we can obtain the non-metricity (2.26) and disformation (2.28).

Non-metricity (2.26) in the absence of waves (i.e. for the Minkowski metric) is

$$\begin{aligned} Q_{011} &= -Q_{311} = -\frac{2c_1}{c_1(t-z) + d_1}, \\ Q_{022} &= -Q_{322} = -\frac{2c_2}{c_2(t-z) + d_2}, \\ Q_{101} &= Q_{110} = -Q_{113} = -Q_{131} \\ &= -\frac{c_1}{c_1(t-z) + d_1} + c_1(c_1(t-z) + d_1), \\ Q_{202} &= Q_{220} = -Q_{223} = -Q_{232} \\ &= -\frac{c_2}{c_2(t-z) + d_2} + c_2(c_2(t-z) + d_2). \end{aligned} \tag{3.18}$$

We assume that non-metricity should be zero in the absence of waves. Thus, we have $c_1 = c_2 = 0$, and (3.17) becomes

$$\Gamma^{\alpha}{}_{\mu\nu} = 0. \tag{3.19}$$

Thus, one can classify the pair of coordinates (t, x, y, z) in the metric (3.1) and the STC (3.19) as a pair of the equivalence class which is the simplest gauge for the solution (3.1) in the STEGR.

Proceeding further in this gauge, to get the Noether superpotential, we choose the observer's proper vector as (3.9). The Komar superpotential (2.44) becomes zero. After calculation of the divergent part of the superpotential (2.45), we get the total superpotential with non-zero components:

$$\mathcal{J}^{03} = -\mathcal{J}^{30} = \mathcal{J}^{03}_{div} = -\mathcal{J}^{30}_{div} = \frac{f'g + g'f}{8\pi}. \tag{3.20}$$

Taking the divergence of the Noether superpotential and using the Einstein equations (3.11), we get the Noether current

$$\mathcal{J}^{\mu} = \left\{ -\frac{f'g'}{4\pi}, 0, 0, -\frac{f'g'}{4\pi} \right\}. \tag{3.21}$$

Note that it coincides exactly with (3.12).

Second, to get another Noether superpotential, we choose the observer's proper vector as in (3.13). The Komar superpotential (2.44) is zero again. Calculating the divergent part (2.45), we get the total superpotential

$$\begin{aligned} \mathcal{J}^{01} = \mathcal{J}^{31} = -\mathcal{J}^{10} = -\mathcal{J}^{13} = -C_1 \tfrac{f'g + g'f}{8\pi f^2}, \\ \mathcal{J}^{02} = \mathcal{J}^{32} = -\mathcal{J}^{20} = -\mathcal{J}^{23} = -C_2 \tfrac{f'g + g'f}{8\pi g^2}, \\ \mathcal{J}^{03} = -\mathcal{J}^{30} = \tfrac{e^{-\alpha_0}(f'g + g'f)}{8\pi}. \end{aligned} \tag{3.22}$$

And then taking the divergence and using the Einstein equations (3.11), we get the Noether current

$$\mathcal{J}^{\mu} = \left\{ -\frac{e^{-\alpha_0} f'g'}{4\pi}, 0, 0, -\frac{e^{-\alpha_0} f'g'}{4\pi} \right\}. \tag{3.23}$$

Note that it exactly coincides with (3.15).

Again, obtaining non-zero components of the current (3.21) and (3.23) measured by a freely falling observer, one concludes that the settled goal to obtain a correspondence with the equivalence principle is not achieved in this case. Again, we interpret the failure as an inappropriate gauge of coordinates (t, x, y, z) in the metric (3.1) and the STC (3.19), which stimulates a search for appropriate gauges that we do in the next sections.

3.4 A linear approximation

In this subsection, we consider a linear approximation of the results in the previous two subsections. It is initiated for two reasons. First, a linear approximation has already been suggested in other works in other formalisms (see [33,36,37]), where a correspondence with the equivalence principle is absent as well. Therefore, it is useful to give a comparison of these results with ours. Second, in the works [33,36,37], there is a coincidence of the linear approximation with the Landau–Lifshitz energy–momentum density for the linear gravitational wave [43]. Therefore, it is quite desirable to provide a linear approximation of our results and give an appropriate interpretation.

Now let us represent the Noether currents in both the TEGR (3.12) and the STEGR (3.21), which are the same, in the linear approximation. For the weak wave in Minkowski space we choose the simplest presentation [43]:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{3.24}$$

where $\eta_{\mu\nu} = \text{diag}[-1,\ +1,\ +1,\ +1]$. Thus, in the case of the metric (3.1) in the transverse–traceless-gauge (TT-gauge) [43], one has

$$f = 1 + \frac{1}{2}h_{xx}(t-z), \quad g = 1 - \frac{1}{2}h_{xx}(t-z). \tag{3.25}$$

The main item in calculation of the current is a determined gauge. Turning to the exact consideration, we have defined such gauges; they are the tetrad (3.2) and ISC (3.8) in the TEGR, coordinates (t, x, y, z) in the metric (3.1), and the STC (3.19) in the STEGR. By the logic of construction, to represent these gauges in the linear approximation, one has only to exchange the exact metric (3.1) in expressions (including a tetrad (3.2)) by the linearized metric (3.25). Switching off gravity in the TEGR and STEGR in the linear case directly, we first equate the Riemann tensor of the metric (3.1) with the condition (3.25) to zero and find that $h''_{xx} = 0$. Integrating it, we get the condition $h_{xx} = c_1(t-z) + d_1$. Calculating the superpotential and the non-metricity with the latter, we again find that these tensors are zero when $c_1 = 0$, and thus the corresponding ISC and STC are zero. Thus, the linearization procedure and the "turning off" gravity procedure commute here.

Another necessary item in calculating the current is a determined displaced vector ξ^σ. For a spacetime with the metric (3.24), one obtains for a freely falling observer

$$\xi^\sigma = (-1, 0, 0, 0) \tag{3.26}$$

that coincides with (3.9) for the exact metric. Taking into account all the above, the current (3.12), the same (3.21), becomes in the lower order

$$\mathcal{J}^\mu_{lin} = \left\{ \frac{h'^2_{xx}}{16\pi},\ 0,\ 0,\ \frac{h'^2_{xx}}{16\pi} \right\}. \tag{3.27}$$

Considering a more general case of freely falling observers (3.13) in the linear approximation (3.24) and (3.25), one obtains for the current (3.15), the same (3.23) analogous expression

$$\mathcal{J}^\mu_{lin} = \left\{ \frac{e^{-\alpha_0}h'^2_{xx}}{16\pi},\ 0,\ 0,\ \frac{e^{-\alpha_0}h'^2_{xx}}{16\pi} \right\}. \tag{3.28}$$

Because we see non-zero components for the currents, we also have no correspondence with the equivalence principle in linear approximation, which is not surprising due to the exact consideration. The discrepancy in interpretation of the equivalence principle in the framework of our approach in linear approximation is explained in the same way as in the exact case; namely, the gauges used are not appropriate.

However, let us return to the expression (3.27) and concentrate on it. It exactly coincides with the result in [43] for the case of the weak plane gravitational wave with only one polarization propagating on a flat background. The component $\mathcal{J}^0_{lin}$ represents the energy density, whereas $\mathcal{J}^3_{lin}$ is the energy density flux of such a wave propagating along the z axis. One can see that these quantities are positively defined. The same coincidence has been obtained for the gravitational wave (3.1) in other formalisms, for example, in [36]. This fact is considered there as a criterion that supports the correctness of the result. In the following paper [37], this coincidence is supported by a relation to a so-called ideal frame.

To be more convincing, we need to explain the quite acceptable result (3.27), where components are explained as energetic characteristics of a gravitational wave, in the framework of our formalism. Let us consider more deeply the Landau–Lifshitz prescription, which is based on their pseudotensor. Like all pseudotensors, it is non-covariant—it is not transformed as a tensor under arbitrary coordinate transformations, which makes the physical interpretation of the conserved quantities more difficult. One of the ways to improve the situation is the possibility to covariantize pseudotensors by introducing a fixed Minkowskian background with the Minkowski metric (together with the dynamical spacetime with the metric tensor $g_{\mu\nu}$); see [45]. Then, the Landau–Lifshitz conserved pseudotensor $t^{\mu\nu}_{LL}$, which has the mathematical weight +2, permits us to construct a conserved covariant current[3]:

$$\mathcal{J}^\mu_{LL} = \left(\sqrt{-\det \eta_{\alpha\beta}}\right)^{-1} t^{\mu\nu}_{LL}\bar{\xi}_\nu. \tag{3.29}$$

[3] The analogous method of covariantization can be applied to an arbitrary pseudotensor.

From the beginning, it is assumed that (3.29) is written in the Lorentzian coordinates, of course, $-\det \eta_{\alpha\beta} = 1$. Then, $\mathcal{J}^{\mu}_{LL}$ is thought as a vector density of the weight +1 and can be represented in arbitrary coordinates in the ordinary way if partial derivatives in $t^{\mu\nu}_{LL}$ are replaced by covariant ones. Vector $\bar{\xi}^{\mu}$ is a Killing vector of the Minkowski space.

Let us choose

$$\bar{\xi}^{\mu} = (-1,\ 0,\ 0,\ 0) \tag{3.30}$$

as a timelike Killing vector of the Minkowski space. Then the current (3.29) becomes

$$\mathcal{J}^{\mu}_{LL} = (\, t^{00}_{LL},\ t^{10}_{LL},\ t^{20}_{LL},\ t^{30}_{LL}). \tag{3.31}$$

Thus, the interpretation of $\mathcal{J}^{0}_{LL}$ and $\mathcal{J}^{3}_{LL}$ as energy density and energy density flux coincides with a related interpretation in t^{00}_{LL} and t^{30}_{LL} in the book [43]. In linear approximation, the expression (3.31) coincides with (3.27).

Returning again to the decompositions (3.24) and (3.25), where perturbations are considered on a flat background with the Minkowski metric, now let us exchange the interpretation. Assume that in the total current (3.12), the same (3.21), one uses the timelike Killing vector of Minkowski space (3.30) instead of the freely falling observer proper vector (3.9) (although, formally, they are the same). As a result, one of course again obtains (3.27) after linearization. However, the coincidence of the result (3.27) with the Landau–Lifshitz one becomes clear.

Let us consider (3.29) without an approximation. Thus, $t^{\mu\nu}_{LL}$ for the metric (3.1) is

$$\mathcal{J}^{\mu}_{LL} = \left\{ -\frac{4fgf'g' + g^2f'^2 + f^2g'^2}{8\pi},\ 0,\ 0,\ -\frac{4fgf'g' + g^2f'^2 + f^2g'^2}{8\pi} \right\}. \tag{3.32}$$

One can see that it differs from (3.12) and (3.21), although its linear approximation gives (3.27). We stress that our results (3.12) and (3.21) are obtained in the framework of the initially covariant method, whereas (3.32) is obtained after covariantization (an additional procedure) of the Landau–Lifshitz pseudotensor.

Finally, it is interesting to discuss the results (3.15), (3.23), and (3.28). The vector (3.13) is not a Killing vector of Minkowski space. However, without changing the results (3.15), (3.23), and (3.28), one can set $C_1 = C_2 = 0$. By this, we can claim that we use vector

$$\xi^{\mu}_{\text{boosted}} = \{-\cosh\alpha_0,\ 0,\ 0,\ -\sinh\alpha_0\}, \tag{3.33}$$

which is a timelike Killing vector of Minkowski space. Indeed, boosting (3.30) by global Lorentz transformation in Minkowski space, one gets (3.33). This transformation has the form $\xi^{\mu}_{\text{boosted}} = \lambda^{\mu}{}_{\nu}\bar{\xi}^{\nu}$, where

$$\lambda^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\alpha_0 & 0 & 0 & \sinh\alpha_0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sinh\alpha_0 & 0 & 0 & \cosh\alpha_0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & 0 & 0 & \frac{v}{\sqrt{1-v^2}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{v}{\sqrt{1-v^2}} & 0 & 0 & \frac{1}{\sqrt{1-v^2}} \end{pmatrix} \tag{3.34}$$

with v a three-dimensional constant velocity in Minkowski spacetime. Then, interpretation of (3.15), (3.23), and (3.28) becomes clear: they are obtained by boosting the vector $\bar{\xi}^{\mu}$ by (3.34) in (3.12), (3.21), and (3.27).

The result of linear approximation is an important result not only because it coincides with the Landau–Lifshitz prediction (and other pseudotensor approaches as well), but mainly because it is checked observationally. This may mean that the equivalence principle itself does not necessarily require a zero current *in this concrete case*, and non-zero values can have a physical meaning with an appropriate interpretation. On the other hand, due to lack of observational evidence for a strong gravitational wave regime, we cannot a priori say whether the general form of the current has the same meaning, regarding its difference from the pseudotensor result for a strong wave.

4 Gauges compatible with the equivalence principle

In this section, as noted above, we are searching for gauges which give zero current for a free-falling observer. Zero values of the Noether current components measured by such observers mean that they detect the absence of gravity, that is, a correspondence with the equivalence principle. In other words, we are searching for gauges compatible with the equivalence principle.

4.1 Gauge changing in the TEGR

Let us try to obtain zero Noether current in the TEGR by changing a gauge. To do this, we change the ISC as usual (2.17):

$$\overset{\bullet}{A}{}'^{a}{}_{b\mu} = \Lambda^{a}{}_{c}(x^{\nu})\overset{\bullet}{A}{}^{c}{}_{d\mu}\Lambda_{b}{}^{d}(x^{\nu}) + \Lambda^{a}{}_{c}(x^{\nu})\partial_{\mu}\Lambda_{b}{}^{c}(x^{\nu}). \tag{4.1}$$

Because we had a zero ISC in the previous section (3.8), our new ISC has the form (2.18)

$$\overset{\bullet}{A}{}'^{a}{}_{b\mu} = \Lambda^{a}{}_{c}(x^{\nu})\partial_{\mu}\Lambda_{b}{}^{c}(x^{\nu}), \tag{4.2}$$

while the tetrad remains the same (3.2). Or another way is as follows: the tetrad changes as (2.16)

$$h'^{a}{}_{\mu} = \Lambda^{a}{}_{b}(x^{\nu})h^{b}{}_{\mu}, \tag{4.3}$$

while the ISC (3.8) remains zero. In this section, we also assume that the affine connection $\overset{\bullet}{\Gamma}{}^{\alpha}{}_{\mu\nu}$ should have the same symmetries as a solution (3.1). This means that displacements in the x, y, and $t+z$ directions do not change it. Such a proposal was applied to the affine connection in modified teleparallel theories with other symmetries in [21]. Because the connection is dynamical in modified teleparallel theories, both the metric and the connection should have the same symmetries as a solution. In the TEGR, the ISC and affine connection are non-dynamical, so at the level of field equations, the requirement for them to have the same symmetries as the metric might be too strong. Nevertheless, we consider Noether's current (2.40) and superpotential (2.42) and can require for them the same symmetries as the metric has. For this purpose it is sufficient to require the same symmetries for the teleparallel superpotential, contortion, or torsion. For example, we can require for contortion

$$\pounds_\xi \overset{\bullet}{K}{}^{\alpha}{}_{\mu\nu} = 0, \tag{4.4}$$

where $\pounds_\xi$ is the Lie derivative, and ξ is a Killing vector of the solution. For the metric (3.1), the symmetry holds along the directions Δx: $\xi^\mu = (0,\ 1,\ 0,\ 0)$, Δy: $\xi^\mu = (0,\ 0,\ 1,\ 0)$, and $\Delta(t+z)$: $\xi^\mu = (1,\ 0,\ 0,\ 1)$. The contortion is

$$\overset{\bullet}{K}{}^{\alpha}{}_{\mu\nu} = \overset{\bullet}{\Gamma}{}^{\alpha}{}_{\mu\nu} - \overset{\circ}{\Gamma}{}^{\alpha}{}_{\mu\nu} = 0, \tag{4.5}$$

where the Levi–Civita connection has the same symmetries as the metric (3.1). Now let us go back to definitions (2.4) and (2.18). The tetrad (3.2) is already symmetric as metric (3.1). Therefore, $\overset{\bullet}{\Gamma}{}^{\alpha}{}_{\mu\nu}$ is symmetric when the ISC (2.18) is symmetric. This requirement is fulfilled when $\Lambda^a{}_b(x^\nu) = \Lambda^a{}_b(t-z)$ in (2.18), that is, an arbitrary Lorentz rotation depends on $t-z$ only.

Keeping in mind matrices of local Lorentz rotations obtained by this prescription and given in Appendix A, we have found that for the related gauges (pairs of tetrad (3.2) and ISC (2.18), where $\Lambda^a{}_b(t-z)$ is a composition of matrices given in Appendix A, or transformed tetrad (3.2) by $\Lambda^a{}_b(t-z)h^b{}_\mu$ and zero ISC (3.8)), the Noether current does not change! Thus, one cannot find a gauge for which the current vanishes for a free-moving observer with such a restrictive condition.

Now let us assume that $\Lambda^a{}_b$ can be non-symmetrical, for example, along the direction dx; thus, it can depend on x and $(t-z)$. The motivation for this proposal is that the connection in the TEGR is not dynamical and thus cannot be felt by observers like the symmetrical metric can, and therefore we do not need $\Lambda^a{}_b$ to be symmetrical. When we assume the dependence on x, the Noether superpotential can depend on x. However, making the Noether current equal to zero, such a current will automatically satisfy the symmetries of the solution. One of the simplest Lorentz rotations $\Lambda^a{}_b$ which depends on x and $(t-z)$ and can change the Noether current is

$$\Lambda^a{}_b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(x\psi(t-z)) & 0 & \sin(x\psi(t-z)) \\ 0 & 0 & 1 & 0 \\ 0 & -\sin(x\psi(t-z)) & 0 & \cos(x\psi(t-z)) \end{pmatrix}. \tag{4.6}$$

The ISC (2.18) $\overset{\bullet}{A}{}^a{}_{c\mu} = \Lambda_b{}^c \partial_\mu \Lambda^a{}_b$ calculated with (4.6) is

$$\begin{aligned} \overset{\bullet}{A}{}^{\hat 1}{}_{\hat 3 1} = -\overset{\bullet}{A}{}^{\hat 3}{}_{\hat 1 1} &= \psi(t-z); \\ \overset{\bullet}{A}{}^{\hat 1}{}_{\hat 3 0} = \overset{\bullet}{A}{}^{\hat 3}{}_{\hat 1 3} = -\overset{\bullet}{A}{}^{\hat 1}{}_{\hat 3 3} = -\overset{\bullet}{A}{}^{\hat 3}{}_{\hat 1 0} &= x\psi'(t-z). \end{aligned} \tag{4.7}$$

Then one can calculate the contortion (2.8) with (4.6) and (3.3). Then, the teleparallel superpotential (2.15) is

$$\begin{aligned} \overset{\bullet}{S}_{\hat 0}{}^{01} = \overset{\bullet}{S}_{\hat 0}{}^{31} = -\overset{\bullet}{S}_{\hat 0}{}^{10} = -\overset{\bullet}{S}_{\hat 0}{}^{13} &= -\tfrac{x\psi'(t-z)}{f}; \\ \overset{\bullet}{S}_{\hat 0}{}^{03} = -\overset{\bullet}{S}_{\hat 0}{}^{30} &= -\tfrac{f'+\psi(t-z)}{f} - \tfrac{g'}{g}; \\ \overset{\bullet}{S}_{\hat 1}{}^{01} = \overset{\bullet}{S}_{\hat 1}{}^{31} = -\overset{\bullet}{S}_{\hat 1}{}^{10} = -\overset{\bullet}{S}_{\hat 1}{}^{13} &= \tfrac{g'}{fg}; \\ \overset{\bullet}{S}_{\hat 2}{}^{02} = -\overset{\bullet}{S}_{\hat 2}{}^{20} &= \tfrac{f'}{fg}; \\ \overset{\bullet}{S}_{\hat 2}{}^{12} = -\overset{\bullet}{S}_{\hat 2}{}^{21} &= \tfrac{x\psi'(t-z)}{fg}; \\ \overset{\bullet}{S}_{\hat 2}{}^{23} = -\overset{\bullet}{S}_{\hat 2}{}^{32} &= -\tfrac{f'+\psi(t-z)}{fg}; \\ \overset{\bullet}{S}_{\hat 3}{}^{03} = -\overset{\bullet}{S}_{\hat 3}{}^{30} &= \tfrac{f'}{f} + \tfrac{g'}{g}. \end{aligned} \tag{4.8}$$

Taking (3.9), we have the Noether superpotential in the TEGR (2.42):

$$\begin{aligned} \overset{\bullet}{\mathcal{J}}{}^{01} = \overset{\bullet}{\mathcal{J}}{}^{31} = -\overset{\bullet}{\mathcal{J}}{}^{10} = -\overset{\bullet}{\mathcal{J}}{}^{13} &= \tfrac{xg\psi'(t-z)}{8\pi}; \\ \overset{\bullet}{\mathcal{J}}{}^{03} = -\overset{\bullet}{\mathcal{J}}{}^{30} &= \tfrac{g\left(f'+\psi(t-z)\right)+fg'}{8\pi}. \end{aligned} \tag{4.9}$$

Then, taking the divergence (2.43) of (4.9), we get the Noether current in the TEGR:

$$\overset{\bullet}{\mathcal{J}}{}^{\mu} = \Bigg\{-\frac{gf''+2f'g'+fg''+\psi(t-z)g'}{8\pi},\ 0,\ 0,\ -\frac{gf''+2f'g'+fg''+\psi(t-z)g'}{8\pi}\Bigg\}. \tag{4.10}$$

Applying here the Einstein equation (3.11), we get

$$\overset{\bullet}{\mathcal{J}}{}^{\mu} = \left\{-\frac{2f'g'+\psi(t-z)g'}{8\pi},\ 0,\ 0,\ -\frac{2f'g'+\psi(t-z)g'}{8\pi}\right\}. \tag{4.11}$$

Then, the condition for the zero Noether current is

$$\psi(t-z) = -2f'. \tag{4.12}$$

Thus, a gauge compatible with the equivalence principle is constructed. Analogously, one can permit a dependence on y and $(t-z)$ with the same result. A more complicated gauge constructed with the local Lorentz rotations depending simultaneously on x, y, and $(t-z)$ is considered in the next section on the basis of work in [35].

4.2 Gauge changing in the STEGR

In this subsection, from the start we restrict ourselves by the simplest requirement as well. We assume that changed Noether conserved quantities have to depend on $(t-z)$ only. The Komar superpotential obtained for the metric (3.1) and vector (3.9) is left as zero independently of transformations which change a gauge. Now consider the additional part (2.45). Because the metric (3.1) depends only on $(t-z)$, the STC $\Gamma^\alpha{}_{\mu\nu}$ included in the additional part of the Noether superpotential (2.45) the non-metricity (2.26) should also depend only on $(t-z)$. Then we check whether such Noether superpotentials (depending on $(t-z)$ only) can satisfy the equivalence principle.

We assume that there exist some new coordinates (T, X, Y, Z) in which the STC $\Gamma^\alpha{}_{\mu\nu} = 0$. New coordinates (T, X, Y, Z) depend on the coordinates (t, x, y, z) in a general way as

$$T = t + \Delta T, \quad X = x + \Delta X,$$
$$Y = y + \Delta Y, \quad Z = z + \Delta Z. \tag{4.13}$$

When (4.13) is expanded in a Taylor series, the functions ΔT, ΔX, ΔY, and ΔZ depend on the derivatives of (T, X, Y, Z) with respect to (t, x, y, z). These derivatives are included in the formula of the transformed STEGR flat connection, which (after applying the coordinate transformation from (T, X, Y, Z) to (t, x, y, z)) is calculated as

$$\Gamma^\alpha{}_{\mu\nu} = \frac{\partial x^\alpha}{\partial X^\lambda} \frac{\partial}{\partial x^\mu}\left(\frac{\partial X^\lambda}{\partial x^\nu}\right), \tag{4.14}$$

where $x^\mu \equiv (t, x, y, z)$, $X^\mu \equiv (T, X, Y, Z)$. To make the STC (4.14) dependent only on $(t-z)$, the derivatives of (T, X, Y, Z) with respect to (t, x, y, z) should depend on $t-z$ only, and thus it is sufficient to make the functions ΔT, ΔX, ΔY, ΔZ dependent only on $(t-z)$. Thus, assuming that the functions ΔT, ΔX, ΔY, and ΔZ depend only on $(t-z)$, we get for the STC (4.14) non-zero components

$$\Gamma^0{}_{00} = \Gamma^0{}_{33} = -\Gamma^0{}_{03} = -\Gamma^0{}_{30} = \frac{\Delta T' \Delta Z'' - \Delta T''\left(\Delta Z' - 1\right)}{\Delta T' - \Delta Z' + 1};$$
$$\Gamma^1{}_{00} = \Gamma^1{}_{33} = \frac{\Delta X'\left(\Delta Z'' - \Delta T''\right)}{\Delta T' - \Delta Z' + 1} + \Delta X'';$$
$$\Gamma^1{}_{03} = \Gamma^1{}_{30} = \frac{\Delta X'\left(\Delta T'' - \Delta Z''\right) + \Delta X''\left(-\Delta T' + \Delta Z' - 1\right)}{\Delta T' - \Delta Z' + 1};$$
$$\Gamma^2{}_{00} = \Gamma^2{}_{33} = \frac{\Delta Y'\left(\Delta Z'' - \Delta T''\right)}{\Delta T' - \Delta Z' + 1} + \Delta Y'';$$
$$\Gamma^2{}_{03} = \Gamma^2{}_{30} = \frac{\Delta Y'\left(\Delta T'' - \Delta Z''\right) + \Delta Y''\left(-\Delta T' + \Delta Z' - 1\right)}{\Delta T' - \Delta Z' + 1};$$
$$\Gamma^3{}_{00} = \Gamma^3{}_{33} = -\Gamma^3{}_{03} = -\Gamma^3{}_{30} = \frac{\left(\Delta T' + 1\right)\Delta Z'' - \Delta T'' \Delta Z'}{\Delta T' - \Delta Z' + 1}, \tag{4.15}$$

where the functions ΔT, ΔX, ΔY, and ΔZ depend on $u = (t-z)$ only, and prime again indicates the differentiation with respect to u.

We again take the observer's proper vector (3.9). Because the Komar superpotential (2.44) remains zero, the total Noether superpotential (2.46) is determined only by the additional part (2.45) which has non-zero components:

$$\mathcal{J}^{03} = -\mathcal{J}^{30} = \frac{g\left(2f'\left(\Delta T' - \Delta Z' + 1\right) + f\left(\Delta Z'' - \Delta T''\right)\right) + 2fg'\left(\Delta T' - \Delta Z' + 1\right)}{16\pi\left(\Delta T' - \Delta Z' + 1\right)}. \tag{4.16}$$

To make the Noether current zero, it is sufficient to make the Noether superpotential (following (2.39)) constant $\mathcal{J}^{03} = -\mathcal{J}^{30} = A_0$. Then we easily find that

$$\frac{(\Delta T - \Delta Z)''}{(\Delta T - \Delta Z)'} = 2\left(-\frac{8\pi A_0}{fg} + \frac{f'}{f} + \frac{g'}{g}\right). \tag{4.17}$$

Then

$$(\Delta T - \Delta Z)' = A_1 f^2 g^2 \exp\left(-\int\left(\frac{16\pi A_0}{fg}\right) du\right), \tag{4.18}$$

where A_1 is a constant of integration. If we assume $A_0 = 0$, in this case, the Noether superpotential is zero, and

$$\Delta T = \Delta Z + A_1 \int f^2 g^2 du + A_2, \tag{4.19}$$

where A_1 and A_2 are constants of integration. One can easily see that ΔZ, ΔX, and ΔY can be arbitrary functions. Concerning ΔT, it is connected with ΔZ by (4.18) in the case of a non-zero constant Noether current or by (4.19) in the case of a zero Noether current.

In the STEGR, in contrast to the TEGR, even with a very strong restriction on the STC and Noether superpotential, requiring dependence on $t-z$ only, we have reached the goal, that is, we have found a gauge with zero current for a freely moving observer.

5 Obukhov–Pereira–Rubilar gauge

In this section, we continue searching for gauges which give zero current for a freely falling observer, thus gauges compatible with the equivalence principle. For this, we use the representation of the plane wave solution in a special way. Namely, in [34,35], the authors study the problem of the energy that can be brought by a flat-fronted gravitational wave. They use a different set of coordinates (T, X, Y, Z), for which in $(-,+,+,+)$ signature, the metric for the gravitational wave solution has a form

$$g_{\mu\nu} = \begin{pmatrix} -H(T-Z,X,Y)-1 & 0 & 0 & H(T-Z,X,Y) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ H(T-Z,X,Y) & 0 & 0 & -H(T-Z,X,Y)+1 \end{pmatrix}. \tag{5.1}$$

The Einstein equations in vacuum acquire the simple form

$$\frac{\partial^2 H(T-Z,X,Y)}{\partial X^2} + \frac{\partial^2 H(T-Z,X,Y)}{\partial Y^2} = 0. \tag{5.2}$$

We discuss the results of [35] in Appendix B in detail. Here, we consider the tetrad suggested in [35] in the framework of our formalism. The reason is that the result of [35] with zero energetic characteristics could be interpreted as having a relation to the equivalence principle. Thus, Obukhov et al. [35] use a tetrad for the metric (5.1):

$$h^a{}_\mu = \begin{pmatrix} \frac{1}{2}H(T-Z,X,Y)+1 & 0 & 0 & -\frac{1}{2}H(T-Z,X,Y) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2}H(T-Z,X,Y) & 0 & 0 & \frac{1}{2}H(T-Z,X,Y)-1 \end{pmatrix}. \tag{5.3}$$

Here, we need to use the coordinates of (3.1). To transform the metric (5.1) to the metric (3.1) under consideration, we use Formiga's [37] Eq. (2.32):

$$\begin{aligned} T &= t + \frac{1}{2}\left(x^2 f(U)\frac{df}{dU} + y^2 g(U)\frac{dg}{dU}\right), \\ Z &= z + \frac{1}{2}\left(x^2 f(U)\frac{df}{dU} + y^2 g(U)\frac{dg}{dU}\right), \\ X &= f(U)x, \quad Y = g(U)y \\ U &= T - Z = t - z, \\ H(U,X,Z) &= -\frac{1}{f}\frac{d^2 f}{dU^2}(X^2 - Y^2). \end{aligned} \tag{5.4}$$

5.1 Zero current in the TEGR

After the coordinate transformation (5.4) $(T, X, Y, Z) \to (t, x, y, z)$, metric (5.1) transforms to a diagonal form (3.1). Tetrad (5.3) transforms to the form

$$h^a{}_\mu = \begin{pmatrix} \frac{1}{2}\left(x^2 f'^2 + y^2 g'^2 + 2\right) & xff' & ygg' & \frac{1}{2}\left(-x^2 f'^2 - y^2 g'^2\right) \\ xf' & f & 0 & -xf' \\ yg' & 0 & g & -yg' \\ \frac{1}{2}\left(-x^2 f'^2 - y^2 g'^2\right) & -xff' & -ygg' & \frac{1}{2}\left(x^2 f'^2 + y^2 g'^2 - 2\right) \end{pmatrix}. \tag{5.5}$$

As was considered in [35], the corresponding ISC to the tetrad (5.3) was zero. Because the ISC transforms as a spacetime vector under spacetime transformations, ISC related to (5.5) remains zero. In addition, the tetrad (5.5) is connected to the diagonal tetrad (3.2) by $h^a{}_\mu = \Lambda^a{}_b h^b_{(diag)\mu}$, where the Lorentz rotation $\Lambda^a{}_b$ is

$$\Lambda^a{}_b = \begin{pmatrix} \frac{1}{2}\left(x^2 f'^2 + y^2 g'^2 + 2\right) & xf' & yg' & \frac{1}{2}\left(-x^2 f'^2 - y^2 g'^2\right) \\ xf' & 1 & 0 & -xf' \\ yg' & 0 & 1 & -yg' \\ \frac{1}{2}\left(-x^2 f'^2 - y^2 g'^2\right) & -xf' & -yg' & \frac{1}{2}\left(x^2 f'^2 + y^2 g'^2 - 2\right) \end{pmatrix}. \tag{5.6}$$

Thus, if one preserves the zero ISC (3.8) with the tetrad (5.5), one obtains the pair presenting the new gauge which differs from the gauge presented by the tetrad (3.2) and zero ISC (3.8). We call it the Obukhov–Pereira–Rubilar gauge.

For the new gauge, by (2.8) and (2.15), we get the teleparallel superpotential, which in all coordinate indexes has non-zero components

$$\begin{aligned} &\overset{\bullet}{S}_0{}^{01} = \overset{\bullet}{S}_0{}^{31} = \overset{\bullet}{S}_3{}^{10} = \overset{\bullet}{S}_3{}^{13} = -\overset{\bullet}{S}_0{}^{10} \\ &\quad = -\overset{\bullet}{S}_0{}^{13} = -\overset{\bullet}{S}_3{}^{01} = -\overset{\bullet}{S}_3{}^{31} = \frac{xf''}{f}; \\ &\overset{\bullet}{S}_0{}^{02} = \overset{\bullet}{S}_0{}^{32} = \overset{\bullet}{S}_3{}^{20} = \overset{\bullet}{S}_3{}^{23} = -\overset{\bullet}{S}_0{}^{20} = -\overset{\bullet}{S}_0{}^{23} \\ &\quad = -\overset{\bullet}{S}_3{}^{02} = -\overset{\bullet}{S}_3{}^{32} = \frac{yg''}{g}. \end{aligned} \tag{5.7}$$

Taking the freely falling observer's proper vector (3.9) in (t, x, y, z) coordinates, we get the Noether superpotential (2.42):

$$\begin{aligned} &\overset{\bullet}{\mathcal{J}}{}^{01} = \overset{\bullet}{\mathcal{J}}{}^{31} = -\overset{\bullet}{\mathcal{J}}{}^{10} = -\overset{\bullet}{\mathcal{J}}{}^{13} = \tfrac{xgf''}{8\pi}; \\ &\overset{\bullet}{\mathcal{J}}{}^{02} = \overset{\bullet}{\mathcal{J}}{}^{32} = -\overset{\bullet}{\mathcal{J}}{}^{20} = -\overset{\bullet}{\mathcal{J}}{}^{23} = \tfrac{yfg''}{8\pi}. \end{aligned} \tag{5.8}$$

Taking the divergence of the Noether superpotential, we get the Noether current (2.43):

$$\overset{\bullet}{\mathcal{J}}{}^{\mu} = \left\{\frac{gf'' + fg''}{8\pi}, 0, 0, \frac{gf'' + fg''}{8\pi}\right\}. \tag{5.9}$$

It is zero due to the Einstein equation (3.11).

5.2 Zero current in the STEGR

Switching off gravity for the coordinates in (3.1) in STEGR gives a gauge with the metric (3.1) and zero STC (3.19). This gives non-zero current (3.21), or (3.23). Here, we switch off gravity for the metric (5.1) in the coordinates (T, X, Y, Z). It specifically gives $H(U, X, Y) = 0$ and, correspondingly, a zero STC. The metric (5.1) transformed by (5.4) goes to (3.1), and the zero STC is transformed as well (2.48) and should be calculated as

$$\Gamma^\alpha{}_{\mu\nu} = \frac{\partial x^\alpha}{\partial X^\lambda}\frac{\partial}{\partial x^\mu}\left(\frac{\partial X^\lambda}{\partial x^\nu}\right), \tag{5.10}$$

where $x^\mu \equiv (t, x, y, z)$, and $X^\mu \equiv (T, X, Y, Z)$. The non-zero components of the transformed STC are

$$
\begin{aligned}
\Gamma^0{}_{00} &= \Gamma^0{}_{33} = \Gamma^3{}_{00} = \Gamma^3{}_{33} = -\Gamma^0{}_{03} \\
&= -\Gamma^0{}_{30} = -\Gamma^3{}_{03} = -\Gamma^3{}_{30} \\
&= \frac{1}{2}\left(x^2 f f''' + x^2 f' f'' + y^2 g g''' + y^2 g' g''\right); \\
\Gamma^0{}_{01} &= \Gamma^0{}_{10} = \Gamma^3{}_{01} = \Gamma^3{}_{10} = -\Gamma^0{}_{13} \\
&= -\Gamma^0{}_{31} = -\Gamma^3{}_{13} = -\Gamma^3{}_{31} = x f f''; \\
\Gamma^0{}_{02} &= \Gamma^0{}_{20} = \Gamma^3{}_{02} = \Gamma^3{}_{20} = -\Gamma^0{}_{23} \\
&= -\Gamma^0{}_{32} = -\Gamma^3{}_{23} = -\Gamma^3{}_{32} = y g g''; \\
\Gamma^0{}_{11} &= \Gamma^3{}_{11} = f f'; \\
\Gamma^0{}_{22} &= \Gamma^3{}_{22} = g g'; \\
\Gamma^1{}_{00} &= \Gamma^1{}_{33} = -\Gamma^1{}_{03} = -\Gamma^1{}_{30} = \frac{x f''}{f}; \\
\Gamma^1{}_{01} &= \Gamma^1{}_{10} = -\Gamma^1{}_{13} = -\Gamma^1{}_{31} = \frac{f'}{f}; \\
\Gamma^2{}_{00} &= \Gamma^2{}_{33} = -\Gamma^2{}_{03} = -\Gamma^2{}_{30} = \frac{y g''}{g}; \\
\Gamma^2{}_{02} &= \Gamma^2{}_{20} = -\Gamma^2{}_{23} = -\Gamma^2{}_{32} = \frac{g'}{g}. \qquad (5.11)
\end{aligned}
$$

Thus, the new STEGR gauge is presented by the pair of the coordinates in (3.1) with the connection (5.11). Using (5.11), we get the non-metricity (2.26) as follows:

$$
\begin{aligned}
Q_{000} &= Q_{033} = Q_{303} = Q_{330} = -Q_{003} = -Q_{030} \\
&= -Q_{300} = -Q_{333} = x^2 f f''' \\
&\quad + x^2 f' f'' + y^2 g g''' + y^2 g' g''; \\
Q_{100} &= Q_{133} = -Q_{103} = -Q_{130} = 2x f f''; \\
Q_{200} &= Q_{233} = -Q_{203} = -Q_{230} = 2y g g''. \qquad (5.12)
\end{aligned}
$$

Taking the freely falling observer's proper vector (3.9) in (t, x, y, z) coordinates, we get the zero Komar superpotential (2.44) which was found in previous sections; by (2.45), one has a zero additional (following from the divergence) part of the Noether superpotential. As a result, one has zero total Noether superpotential in the STEGR, which gives a zero Noether current.

6 Concluding remarks

Solutions for gravitational waves are quite important due to the recent developments in the modern epoch of detecting gravitational waves [50–52]. To the best of our knowledge, the energy characteristics of gravitational waves in both the TEGR and STEGR have rarely been studied up to now. In this article, using the previously developed fully covariant formalism [11,12,15–17] in the TEGR and STEGR, we have studied this problem on the example of a flat-fronted exact (strong) gravitational wave of the only one polarization "+".

The crucial property of our formalism in both the TEGR and STEGR is that a displacement vector ξ^α is included. Namely, interpretation of conserved quantities is defined by ξ^α, which can be chosen as a Killing vector of spacetime, a proper vector of an observer, etc. Another important notion in the formalism [11,12,15–17] is a gauge, that is, the equivalence class of pairs (tetrad, ISC) in TEGR, or pairs (coordinates, STC) in STEGR. To construct a physically sensible conserved quantity, one has to find a corresponding gauge [15–17].

Since the gravitational wave solution is not stationary, there are no timelike Killing vectors, which formally means that it is impossible to construct energy density or energy density flux. Moreover, as for the model under consideration, we assume that a gravitational wave fills infinite space (there are no static distant observers); it is impossible to construct conserved charges. However, proper vectors of freely falling observers are determined easily and naturally. In correspondence with the equivalence principle, such observers have to measure zero. In the language of our formalism, this means that components of the related current have to be zero. Such a correspondence has not been stated up to now in teleparallel theories. In Sects. 4 and 5, we have closed this gap in both the TEGR and STEGR. We have found gauges when energetic characteristics of a flat gravitational wave measured by a freely falling observer are zero, which is just in correspondence with the equivalence principle. This is the novelty and main result of the paper. One could set a goal to find all possible gauges (not only particular ones, like here) that are in correspondence with the equivalence principle, but it is a more complicated task than the task studied here. It will possibly be considered in the future.

What is interesting is that the simplest gauge used gives non-zero results, which in the limit of a weak wave coincide with the related pseudotensor formulae. Analogous results obtained in the non-covariant formalism are discussed in [36,37]. We should stress here that although free-falling masses can be used for detecting non-zero energy of gravitational waves, at least two masses separated by non-zero distance are needed. This means that a formalism has to be adapted from the local one to a "two-point" formalism, or something analogous to this. Even for a simpler case of the Friedmann–Lemaître–Robertson–Walker cosmological metric, consideration of distant masses makes energy issues more subtle, as shown, for example, in the Harrison paper with a remarkable title "Mining energy in an expanding Universe" [53]. However, in our studies here, we consider only one point-like moving geodesically in a gravitational wave metric. Nevertheless, the obtained non-zero result has a physical sense at least for a weak wave. This means that the connection between the equivalence principle and the energy–momentum characteristics of a gravitational field studied here warrants a deeper investigation.

The present article can be developed in various directions. One can consider (1) a plane gravitational wave with two polarizations, (2) gravitational waves with cylindrical or spherical symmetry, and/or (3) gravitational waves propagating inside matter. We leave such studies for a future work.

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Appendix A: Arbitrary Lorentz rotations

Here, we formally derive the arbitrary Lorentz rotation dependent on $(t-z)$ only, which can be applied only to the tetrad as (2.16) preserving the ISC or only to the ICS as (2.17) preserving the tetrad. We present the compositions of simple Lorentz rotations dependent on $(t-z)$. It is claimed in subsection 4.1 that there are no compositions such that Lorentz rotations of the tetrad only preserving the ISC (or of ISC only preserving tetrad) can change the Noether current.

Arbitrary Lorentz rotation matrix $\Lambda^a{}_b(t-z)$ of $SO(1,3)$ group can be expressed through $so(1,3)$ algebra as

$$\begin{aligned}\Lambda^a{}_b(t-z) = \exp(J_1\alpha_1(t-z) &+ J_2\alpha_2(t-z) + J_3\alpha_3(t-z) + K_1\beta_1(t-z)\\ &+ K_2\beta_2(t-z) + K_3\beta_3(t-z)),\end{aligned} \tag{A.1}$$

where $\alpha_i(t-z)$, $\beta_i(t-z)$ $(i=1,2,3)$ are arbitrary functions, and the generators of algebra $so(1,3)$ are

$$J_1 = i\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix},\ J_2 = i\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{pmatrix},$$

$$J_3 = i\begin{pmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{pmatrix},$$

$$K_1 = i\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\ K_2 = i\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix},$$

$$K_3 = i\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}. \tag{A.2}$$

It is practically difficult to directly calculate the matrix exponent (A.1). Therefore, we assume that there exist functions $\bar\alpha_1(t-z)$, $\bar\alpha_2(t-z)$, $\bar\alpha_3(t-z)$, $\bar\beta_1(t-z)$, $\bar\beta_2(t-z)$, and $\bar\beta_3(t-z)$ such that

$$\begin{aligned}&\exp(J_1\alpha_1 + J_2\alpha_2 + J_3\alpha_3 + K_1\beta_1 + K_2\beta_2 + K_3\beta_3)\\ &\quad = \exp(J_1\bar\alpha_1)\exp(J_2\bar\alpha_2)\exp(J_3\bar\alpha_3)\exp(K_1\bar\beta_1)\\ &\quad\ \exp(K_2\bar\beta_2)\exp(K_3\bar\beta_3),\end{aligned} \tag{A.3}$$

where $\bar\alpha_i \equiv \bar\alpha_i(t-z)$, $\bar\beta_i \equiv \bar\beta_i(t-z)$, $\alpha_i \equiv \alpha_i(t-z)$, $\beta_i \equiv \beta_i(t-z)$, $i=1,2,3$. $\bar\alpha_i$, $\bar\beta_i$ can be connected with α_i, β_i using the Baker–Campbell–Hausdorff formula [54]. This formula gives direct expression for matrix Z in $\exp(X)\exp(Y)=\exp(Z)$, as a function of $Z=Z(X,Y)=\log(\exp X\exp Y)$. Applying this formula to each term under the exponent in (A.3) sequentially, we can get the exact expressions of matrices $\bar\alpha_i$, $\bar\beta_i$ as functions of matrices α_i, β_i. We do not need to make these calculations directly here. It is enough for us to know that $\bar\alpha_i$, $\bar\beta_i$ can be expressed explicitly through α_i, β_i and then take the Lorentz rotation $\Lambda^a{}_b(t-z)$ as a composition of simple Lorentz rotations:

$$\begin{aligned}\Lambda^a{}_b(t-z) &= Exp(J_1\bar\alpha_1)\exp(J_2\bar\alpha_2)Exp(J_3\bar\alpha_3)\\ &\quad\times Exp(K_1\bar\beta_1)Exp(K_2\bar\beta_2)Exp(K_3\bar\beta_3)\\ &= \Lambda_{(\bar\alpha_1)}(t-z)\Lambda_{(\bar\alpha_2)}(t-z)\Lambda_{(\bar\alpha_3)}(t-z)\\ &\quad\times\Lambda_{(\bar\beta_1)}(t-z)\Lambda_{(\bar\beta_2)}(t-z)\Lambda_{(\bar\beta_3)}(t-z),\end{aligned} \tag{A.4}$$

where $\bar\alpha_i$ and $\bar\beta_i$, $i=1,2,3$ are arbitrary functions of $(t-z)$, and

$$\begin{aligned}\Lambda_{(\bar\alpha_1)}(t-z) &= Exp(J_1\bar\alpha_1(t-z))\\ &= \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&\cos[\bar\alpha_1(t-z)]&-\sin[\bar\alpha_1(t-z)]\\0&0&\sin[\bar\alpha_1(t-z)]&\cos[\bar\alpha_1(t-z)]\end{pmatrix},\\ \Lambda_{(\bar\alpha_2)}(t-z) &= Exp(J_2\bar\alpha_2(t-z))\\ &= \begin{pmatrix}0&0&0&0\\0&\cos[\bar\alpha_2(t-z)]&0&\sin[\bar\alpha_2(t-z)]\\0&0&0&0\\0&-\sin[\bar\alpha_2(t-z)]&0&\cos[\bar\alpha_2(t-z)]\end{pmatrix},\end{aligned}$$

$$\Lambda_{(\bar\alpha_3)}(t-z) = Exp(J_3\bar\alpha_3(t-z)) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos[\bar\alpha_3(t-z)] & -\sin[\bar\alpha_3(t-z)] & 0 \\ 0 & \sin[\bar\alpha_3(t-z)] & \cos[\bar\alpha_3(t-z)] & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda_{(\bar\beta_1)}(t-z) = Exp(K_1\bar\beta_1(t-z)) = \begin{pmatrix} \cosh[\bar\beta_1(t-z)] & \sinh[\bar\beta_1(t-z)] & 0 & 0 \\ \sinh[\bar\beta_1(t-z)] & \cosh[\bar\beta_1(t-z)] & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda_{(\bar\beta_2)}(t-z) = Exp(K_2\bar\beta_2(t-z)) = \begin{pmatrix} \cosh[\bar\beta_2(t-z)] & 0 & \sinh[\bar\beta_2(t-z)] & 0 \\ 0 & 0 & 0 & 0 \\ \sinh[\bar\beta_2(t-z)] & 0 & \cosh[\bar\beta_2(t-z)] & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda_{(\bar\beta_3)}(t-z) = Exp(K_3\bar\beta_3(t-z)) = \begin{pmatrix} \cosh[\bar\beta_3(t-z)] & 0 & 0 & \sinh[\bar\beta_3(t-z)] \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sinh[\bar\beta_3(t-z)] & 0 & 0 & \cosh[\bar\beta_3(t-z)] \end{pmatrix}. \quad \text{(A.5)}$$

Arbitrary compositions of Lorentz rotations $\Lambda_{(\bar\alpha_i)}(t-z)$ and $\Lambda_{(\bar\beta_i)}(t-z)$ $(i = 1, 2, 3)$ are used in the calculations.

Appendix B: Non-freely falling tetrad

Here, we show that the tetrad (5.3) taken in [35] is not a freely falling tetrad, and thus the observer that is at rest in the frame (5.3) is not freely falling.

The inverse tetrad (5.3) is

$$h_a{}^{\mu} = \begin{pmatrix} -1+\frac{1}{2}H(T-Z,X,Y) & 0 & 0 & \frac{1}{2}H(T-Z,X,Y) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2}H(T-Z,X,Y) & 0 & 0 & \frac{1}{2}H(T-Z,X,Y)+1 \end{pmatrix}. \quad \text{(B.1)}$$

The timelike tetrad vector (which is taken as the observer's 4-velocity) is

$$h_{\hat 0}{}^{\mu} = \left\{-1+\frac{1}{2}H(T-Z,X,Y), 0, 0, \frac{1}{2}H(T-Z,X,Y)\right\}, \quad \text{(B.2)}$$

and in [35], it was assumed that (B.2) is equal to the observer's 4-velocity, i.e., the observer is at rest in the frame (5.3).

The Levi–Civita connection for (5.1) is

$$\overset{\circ}{\Gamma}{}^{0}{}_{00} = \overset{\circ}{\Gamma}{}^{3}{}_{33} = \overset{\circ}{\Gamma}{}^{3}{}_{00} = \overset{\circ}{\Gamma}{}^{0}{}_{33} = \frac{1}{2}H^{(1,0,0)}(T-Z,X,Y);$$

$$\overset{\circ}{\Gamma}{}^{3}{}_{03} = \overset{\circ}{\Gamma}{}^{3}{}_{30} = \overset{\circ}{\Gamma}{}^{0}{}_{30} = \overset{\circ}{\Gamma}{}^{0}{}_{03} = -\frac{1}{2}H^{(1,0,0)}(T-Z,X,Y);$$

$$\overset{\circ}{\Gamma}{}^{0}{}_{01} = \overset{\circ}{\Gamma}{}^{1}{}_{00} = \overset{\circ}{\Gamma}{}^{0}{}_{10} = \overset{\circ}{\Gamma}{}^{1}{}_{33} = \overset{\circ}{\Gamma}{}^{3}{}_{01} = \overset{\circ}{\Gamma}{}^{3}{}_{10} = \frac{1}{2}H^{(0,1,0)}(T-Z,X,Y);$$

$$\overset{\circ}{\Gamma}{}^{0}{}_{13} = \overset{\circ}{\Gamma}{}^{0}{}_{31} = \overset{\circ}{\Gamma}{}^{1}{}_{03} = \overset{\circ}{\Gamma}{}^{1}{}_{30} = \overset{\circ}{\Gamma}{}^{3}{}_{13} = \overset{\circ}{\Gamma}{}^{3}{}_{31} = -\frac{1}{2}H^{(0,1,0)}(T-Z,X,Y);$$

$$\overset{\circ}{\Gamma}{}^{0}{}_{02} = \overset{\circ}{\Gamma}{}^{0}{}_{20} = \overset{\circ}{\Gamma}{}^{2}{}_{00} = \overset{\circ}{\Gamma}{}^{2}{}_{33} = \overset{\circ}{\Gamma}{}^{3}{}_{02} = \overset{\circ}{\Gamma}{}^{3}{}_{20} = \frac{1}{2}H^{(0,0,1)}(T-Z,X,Y);$$

$$\overset{\circ}{\Gamma}{}^{0}{}_{32} = \overset{\circ}{\Gamma}{}^{0}{}_{23} = \overset{\circ}{\Gamma}{}^{2}{}_{03} = \overset{\circ}{\Gamma}{}^{2}{}_{30} = \overset{\circ}{\Gamma}{}^{3}{}_{32} = \overset{\circ}{\Gamma}{}^{3}{}_{23} = -\frac{1}{2}H^{(0,0,1)}(T-Z,X,Y), \quad \text{(B.3)}$$

where $U \equiv T - Z$, $H^{(1,0,0)}(U,X,Y) = \frac{\partial H(U,X,Y)}{\partial U}$, $H^{(0,1,0)}(U,X,Y) = \frac{\partial H(U,X,Y)}{\partial X}$, $H^{(0,0,1)}(U,X,Y) = \frac{\partial H(U,X,Y)}{\partial Y}$. The geodesic equation has the form

$$l^{\mu} \equiv \frac{d^2x^{\mu}}{d\tau^2} + \overset{\circ}{\Gamma}{}^{\mu}{}_{\alpha\beta}\frac{dx^{\alpha}}{d\tau}\frac{dx^{\beta}}{d\tau} \equiv \frac{du^{\mu}}{d\tau} + \overset{\circ}{\Gamma}{}^{\mu}{}_{\alpha\beta}u^{\alpha}u^{\beta} \equiv \frac{\partial u^{\mu}}{\partial x^{\kappa}}u^{\kappa} + \overset{\circ}{\Gamma}{}^{\mu}{}_{\alpha\beta}u^{\alpha}u^{\beta} = 0, \quad \text{(B.4)}$$

where u^{α} is the observer's 4-velocity, and the parameter τ can be taken as the observer's proper time because they move along the timelike curves. Let us substitute (B.2) identified with u^{α} and (B.3) into the left-hand side of Eq. (B.4). If we obtain zero, the observer would be freely falling, but for general $H(U, X, Y)$, we obtain a non-zero result:

$$l^{\mu} = \left\{0, \frac{1}{2}H^{(0,1,0)}(U,X,Y), \frac{1}{2}H^{(0,0,1)}(U,X,Y), 0\right\}. \quad \text{(B.5)}$$

Thus, the observer in [35] is not freely falling. Therefore, zero energetic characteristics of the wave obtained in [35] with the use of (2.19) (or (2.20)) cannot be interpreted as a correspondence with the equivalence principle.

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